Layer topology on vertex chain simplicial complexes

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Abstract

In order to study the topological properties of complex shapes, it is convenient to view them as the union of simplex gluing together in a specific manner to form simplicial complexes. Simplex are the simple geometric objects like points, line segments, triangles, tetrahedra etc. Layer Topology on simplicial complexes helps us to understand the topological properties of complex shapes by comparing the topological properties of closed unit interval in the usual topology.

Keywords

Simplicial complex; Barycentric maps; Layer Topology.

AMS Subject Classification

55U10, 55U15.

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Article History: Received 24 January 2019; Accepted 24 May 2019

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1. Introduction

Barycentric Maps and Layer Topology

Let $a_0, a_1, \dots, a_n$ be geometrically independent set in R^N . We define the n -simplex σ spanned by $a_0, a_1, \dots, a_n$ be the set of all points x of R^N such that $x = \sum_{i=0}^n t_i a_i$, where $\sum_{i=0}^n t_i = 1$ and $t_i \geq 0$ for all i . The numbers t_i are uniquely determined by x ; they are called the barycentric coordinates of the point x of σ with respect to $a_0, a_1, \dots, a_n$.

If x is a point of the polyhedron $|K|$, then x is the point of one simplex K whose vertices are $a_0, a_1, \dots, a_n$. Then $x = \sum_{i=0}^n t_i a_i$, where $\sum_{i=0}^n t_i = 1$ and $t_i \geq 0$ for all i . If v is an arbitrary vertex of K , then the barycentric coordinates $t_v(x)$ of x with respect to v is $t_v(x) = 0$ if $v \neq a_i$ and $t_v(x) = t_i$, if $v = a_i$.

By [1], Let $v = a_i$ be any vertex of a simplex σ of dimension $n \geq 1$ with vertices $a_0, a_1, \dots, a_n$. Then $t_v(a_j) = 1$ for

$i = j$ and $t_v(a_j) = 0$ for $i \neq j$.

Let $v = a_0$ be a vertex of a simplex σ of dimension $n \geq 1$ with vertices $a_0, a_1, \dots, a_n$, then the map $t_v : \sigma \rightarrow [0, 1]$ is called barycentric map with respect to v .

Let $v = a_0$ be a vertex of a simplex σ of dimension $n \geq 1$ with vertices $a_0, a_1, \dots, a_n$ and let $t \in [0, 1]$, then the set $l_t(\sigma) = \{x \in \sigma : t_v(x) = t\}$ is called the t^{th} layer of σ .

Let U be a subset of $I = [0, 1]$, then $l_U = \{l_t : t \in U\}$. Whenever U is open we can call l_U as open layer. The collection L of open layers l_U form a basis for a topology on σ

Let $v = a_0$ be a vertex of a simplex σ of dimension $n \geq 1$ with vertices $a_0, a_1, \dots, a_n$. Then the topology on σ in which open layers form a basis is called layer topology.

2. Types of Simplicial Complexes

A Simplicial Complex on R^m is a set K consisting of a finite or infinite set of simplices in R^m having the following conditions

1. Every face of a simplex is also belongs to K
2. For any two simplex σ_1 and σ_2 in K , If $\sigma_1 \cap \sigma_2 \neq \Phi$, then $\sigma_1 \cap \sigma_2$ is a common face of both σ_1 and σ_2

or

For any two distinct simplex σ_1 and σ_2 , $\text{int } \sigma_1 \cap \sigma_2 = \emptyset$,

Every k -simplex in K is called a k -face of K . Zero face is called a vertex and one face is called an edge.

The dimension of a simplicial complex K is the maximum of the dimension of all simplices in K . If $\dim K = d$, then every face of dimension d is called a cell and every face of dimension $d - 1$ is called a facet.

The union $|K|$ of all simplices in K is a subset of R^m . We can define a topology on $|K|$ by defining a subset F of $|K|$ to be closed if and only if $F \cap \sigma$ is closed in σ for every face $\sigma \in K$. The resultant topological space is called the geometric realization of K .

For any simplex σ , we have $|\sigma| = \sigma$.

A polytope is the geometric realization of some simplicial complex. A polytop of dimension 1 is usually called a polygon and a polytop of dimension 2 is called polyhedron.

When K consists of infinitely many simplices, we usually require that K be locally finite, which means that every vertex belongs to finitely many faces. If K is locally finite, then its geometric realization $|K|$ is locally compact. In this paper, we will consider only finite simplicial complexes, that is the complex having finite number of simplices. In finite simplicial complex, the topology of $|K|$ defined above is same as the topology induced from R^m . In this case, for any simplex σ in K , $\text{Int } \sigma$ coincide with interior σ^0 of σ in topological sense and the boundary $\partial \sigma$ coincide with boundary of σ in the topological sense.

Let K_2 be a complex and let $K_1 \subseteq K_2$. Then K_1 is said to be a subcomplex of K_2 if and only if K_1 is a complex.

For any complex K of dimension d , for any i with $0 \leq i \leq d$, the subset $K^{(i)} = \{\sigma \in K : \dim \sigma \leq i\}$ is called the i -skeleton of K . then K^i is a subcomplex of K .

$K^i = \{\sigma \in K / \dim \sigma = i\}$ is not a complex. K^0 is the set of all vertices of K . A simplicial complex K_1 is a subdivision of a complex K_2 if and only if $|K_1| = |K_2|$ and if every face of K_1 is a subset of some face K_2 . A complex K of dimension d is pure (or homogeneous) if and only if every face of K is some face of some d -simplex (or some cell K).

A Complex is connected if and only if $|K|$ is connected.

Let K be any complex and σ be any face of K . Then star of σ denoted by $St(\sigma)$ or $St(\sigma, K)$ is the subcomplex of K consisting of all faces τ containing σ and all faces of τ .

$St(\sigma) = \{\sigma \in K : \exists \tau \in K \text{ such that } \sigma \subseteq \tau \text{ and } \sigma \subseteq \tau\}$

The link of σ denoted by $Lk(\sigma)$ or $Lk(\sigma, K)$ is the subcomplex of K consisting of all faces in $St(\sigma)$ and do not intersect σ .

$Lk(\sigma) = \{\tau \in K : \tau \in St(\sigma) \text{ and } \sigma \cap \tau = \emptyset\}$

If K is a pure complex of dimension d , then star of σ is also pure with dimension d . If dimension of σ is k then $Lk(\sigma)$ is pure of dimension $d - k - 1$.

Open star of σ , $st(\sigma)$, is defined as the subspace of $|K|$ con-

sists of interiors of all faces τ containing σ .

$st(\sigma) = \{\tau - \partial \tau : \sigma \subseteq \tau \text{ and } \tau \in K\}$

Open star is not a complex, but it is a subset of $|K|$.

$$\overline{st(\sigma)} = |st(\sigma)|$$

If σ is a vertex, we have $lk(v) = \overline{st(v)} - st(v)$

$st(\sigma)$ is open for all σ .

For any point $a \in |K|$, there is a unique smallest simplex $\sigma = (a_0, a_1, \dots, a_k)$ such that $a \in \text{int}(\sigma)$

ie $= \sum_{i=0}^n t_i a_i$, where $\sum_{i=0}^n t_i = 1$

and $t_i \geq 0$ for all i with $0 \leq i \leq k$. When $k = 0$, $t_0 = 1$ and $a_0 = a$.

Hence $st(v) = \{a \in |K| : t_v(a) > 0\}$

Thus $|K| - st(v)$ is the union of all faces of K that do not contain v as a vertex. Then clearly it is a closed set. Thus $st(v)$ is open in $|K|$. $st(v)$ is path connected also.

3. Pasting Lemma

Let A and B be closed subsets of a topological space X and $f : A \rightarrow Y$ and $g : B \rightarrow Y$ are continuous functions such that $f(x) = g(x)$ for all $x \in A \cap B$. Then there exist a unique continuous map from $A \cup B$ to Y whose restriction to A is f and to B is g . The result can be extended to finitely many closed sets which cover X .

Definition 3.1. A simplex $\sigma \in K$ is said to be a component simplex of K if there is no simplex $\tau \in K$ such that σ is a proper face of τ . That is σ is a component simplex if σ is a maximal simplex in K containing all faces of σ .

Definition 3.2. A simplicial complex in which exactly one vertex in each component simplex (except the first one), is a vertex on the facet of the preceding component simplex is called a vertex chain complex.

4. Layer Topology on Vertex Chain Complex

Let S be a vertex chain simplicial complex with components $\sigma_1, \sigma_2, \dots, \sigma_n$ (taken in order).

Let $\sigma_k = (a_{k1}, a_{k2}, \dots, a_{km_k})$ be the k^{th} simplex and a_{k1} be the vertex common to σ_k and σ_{k-1} for all $k \neq 1$. Suppose the Layer topology is defined on each σ_k with a_{k1} as the fixed vertex to define the corresponding barycentric map.

Let $s_{a_{k1}} : \sigma_k \rightarrow [0, 1]$ be the barycentric map with respect to a_{k1} , and let $s : [0, 1] \rightarrow [0, 1]$ be defined by $s(t) = 1 - t$. Define $t_{a_{k1}} : \sigma_k \rightarrow [0, 1]$ by $t_{a_{k1}} = s \circ s_{a_{k1}}$ then we can define layer topology on σ_k by keeping open layers as a basis. $(t_{a_{k1}})^{-1} : [0, 1] \rightarrow \sigma_k$ is continuous and open by considering $\sigma_k = \bigcup_{t \in I} l_t(\sigma_k)$ where $l_t(\sigma_k) = \{x \in \sigma : t_{a_{k1}}(x) = t\}$

Define $f_k : I_k = [(k-1)/n, k/n] \rightarrow [0, 1]$ by $f_k(t) = nt - k + 1$, as it is a polynomial function f_k is continuous. Also, we can see that f_k is bijective.

Let $g_k : [(k-1)/n, k/n] \rightarrow \sigma_k$ be defined by

$$g_k(t) = ((t_{a_{k1}}))^{-1} \circ f_k(t).$$

Since $(t_{a_{k_1}})^{-1}$ and f_k are continuous and open, g_k is also continuous and open.

Consider $g_k : [(k-1)/n, k/n] \rightarrow \sigma_k$ and

$$g_{k+1} : [k/n, (k+1)/n] \rightarrow \sigma_{k+1}$$

By definition $g_k(k/n)$ on σ_k and $g_{k+1}(k/n)$ on σ_{k+1} are coincide. By pasting lemma the function $f : [0, 1] \rightarrow S$ where f restricted to $[(k-1)/n, k/n]$ is g_k and it is continuous.

As in [1], we can form layers on vertex chain complex S which will form a basis for a topology on S and we can define it as Layer topology on S .

As per [2], $[0, 1]$ and a simplex S are homeomorphic with respect to Layer Topology. Constituting all results together we can conclude that a simplex σ and a vertex chain complex S are homeomorphic.

5. Conclusion

Simplex of any dimension is homeomorphic to any common vertex simplicial complexes with respect to the layer topology defined on both.

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ISSN(P):2319 – 3786

Malaya Journal of Matematik

ISSN(O):2321 – 5666