A study on fuzzy Chordal graphs

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Abstract

The definition of fuzzy chordal graph is given by John N Moderson and Premchand S Nair in [5]. In this paper we study some properties of fuzzy chordal graphs. We compare fuzzy chordal graphs with chordal graphs in crisp case and some counter examples were constructed.

Keywords

Fuzzy graph, fuzzy chordal graph, cycle, t-level set, fuzzy tree.

AMS Subject Classification

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Contents

1	Introduction	505
2	Preliminaries	505
3	Fuzzy Chordal Graphs	506
	References	507

1. Introduction

It is well known that graphs are used to model relations. The relationships between objects can be interpreted using graphs, the vertices representing objects and the edges representing the relations. When there is vagueness in the description of objects or in the relations we use a fuzzy graph model. In 1965, LA Zadeh [1] introduced the concept of a fuzzy subset of a set as a way of representing uncertainty. It was Rosenfeld [6] gave a mathematical definition for a fuzzy graph in 1975. Yeh and Bang [7] introduced many connectivity concepts including vertex and edge connectivity in fuzzy graph and applied fuzzy graph for the first time in clustering of data. Several authors including Bhutani, Moderson, Vijaya kumar, Sunitha and Sunil Mathew introduced many connectivity concept in fuzzy graph following the job of Zadeh and Rosenfeld.

The concept of chordal graph is widely studied in crisp graph theory. In crisp graph theory chordal graphs are an important class of graphs. All chordal graphs have a treelike structure composed of complete graphs. So many NP-hard problems become polynomial in chordal graphs [2]. A chord of a cycle is an edge with end nodes as non consecutive nodes of the

cycle. A chordal graph is a simple graph in which every cycle of length greater than three has a chord.

2. Preliminaries

In this section we recall some basic definitions which is used in the following sections.

Definition 2.1. [8–11] A fuzzy graph is a pair $G = (\sigma, \mu)$ where σ is a fuzzy subset of a set S and μ is a fuzzy relation on σ , such that $\mu(u, v) \leq \sigma(u) \wedge \sigma(v)$. We assume that S is finite and nonempty, μ is reflexive and symmetric.

Definition 2.2. [8, 9, 11] A path P of length n is a sequence of distinct nodes $x_0, x_1, x_2, \dots, x_n$ such that $\mu(x_{i-1}, x_i) > 0, i = 1, 2, \dots, n$ and the degree of membership of weakest arc is defined as its strength. If $x_0 = x_n$ and $n \geq 3$ then P is called a cycle.

Definition 2.3. [5, 8, 10, 11] The strength of connectedness between two nodes u and v is defined as the maximum of strengths of all paths between u and v and is denoted by $CONN - G(u, v)$.

Definition 2.4. [5, 8, 10, 11] An $u - v$ path P is called a strongest $u - v$ path if its strength equals $CONN - G(u, v)$. A fuzzy graph $G = (\sigma, \mu)$ is connected if for every u, v in $\sigma^* (\sigma^* = \{u \in S : \sigma(u) > 0\})$, $CONN - G(u, v) > 0$.

Definition 2.5. [5, 8–10] An arc is a strong arc if its weight is atleast as great as the strength of connectedness of its end nodes when it is deleted and a strong $u - v$ path is that path which contains only strong arcs.

Definition 2.6. [5, 10, 11] The geodesic distance(g -distance) between two nodes of a connected fuzzy graph, is the length of the shortest strong $u-v$ path in G .

Definition 2.7. [5, 10, 11] The eccentricity of a node is the distance to the farthest node. The nodes having minimum eccentricity are called central nodes. The center of a fuzzy graph is the fuzzy subgraph induced by the central nodes.

The definition of fuzzy chordal graph is given by John N Moderson and Premchand S Nair in the text Fuzzy graphs and Fuzzy Hypergraphs [5].

3. Fuzzy Chordal Graphs

Definition 3.1. [5] :cycle of length n is a sequence of distinct vertices $(x_0, x_1, x_2, \dots, x_n)$ such that $\mu(x_0, x_n) > 0$ and if $1 \leq i \leq n$, then $\mu(x_i, x_{i+1}) > 0$. A fuzzy graph $G = (\sigma, \mu)$ is a chordal if for each cycle with $n \geq 4$, there exist integers $j \neq 0$ or $k \neq n$ such that $0 \leq j < k-1 \leq n$ and $\mu(x_j, x_k) \leq \wedge \mu(x_{j+1}, x_{i+1}) | i = 1, 2, \dots, n \wedge \mu(x_0, x_n)$

G is chordal if and only if for each $t \in (0, 1]$ the t level graph is chordal [5]. In crisp graph tree graphs and complete graphs are trivial examples of chordal graphs. [2]

Remark 3.2. In the case of fuzzy graphs trees need not be a chordal graph. The following example is fuzzy tree but not a fuzzy chordal graph.

Theorem 3.3. Every complete fuzzy graph is fuzzy chordal.

Proof. Let $G = (\sigma, \mu)$ be a fuzzy graph.

Let $C_n(u_1, u_2, \dots, u_n), n \geq 4$ be a cycle in G . The definition of cycle implies

$$\mu(u_i, u_{i+1}) > 0 \text{ for } i = 1, 2, 3, \dots, n-1 \text{ and } \mu(u_1, u_n) > 0 \quad (3.1)$$

Choose one node $u_j \in C$ such that

$$\sigma(u_j) = \wedge \sigma(u_i) | i = 1, 2, \dots, n \quad (3.2)$$

Consider any nonconsecutive nodes in C_n say u_j, u_k where $j < k-1$. G being a fuzzy complete graph

$$\mu(u_j, u_k) = \sigma(u_j) \wedge \sigma(u_k) > 0 \quad (3.3)$$

Then

$$\mu(u_{j-1}, u_j) \leq \sigma(u_{j-1}) \wedge \sigma(u_j) \leq \sigma(u_j) \quad (3.4)$$

So

$$\mu(u_j, u_k) = \sigma(u_j) \geq \wedge_{i=1}^n \sigma(u_i) \quad (3.5)$$

$$\wedge_{i=1}^n \mu(u_{i-1}, u_i) = \wedge_{i=1}^n (\sigma(u_{i-1}), \sigma(u_i)) = \sigma(u_j) \quad (3.6)$$

Therefore

$$\mu(u_j, u_k) \geq \wedge_{i=1}^n \mu(u_{i-1}, u_i) \quad (3.7)$$

So the edge $u_j u_k$ is a chord in C_n .

Therefore every cycle $C_n, n \geq 4$ contains a chord in G . Hence G is a chordal graph. $\square$

Lemma 3.4. [3]: If C is a cycle in a chordal graph, then for every edge (u, v) of C there is a vertex w of C which is adjacent to both u and v .

Theorem 3.5. Let $G = (\sigma, \mu)$ be a fuzzy chordal graph and C be a cycle in G . Then for any two adjacent nodes u and v in C there exists another node w in C which is adjacent to both u and v .

Proof. If C is a cycle of length three the above theorem is trivially true. Suppose C is a cycle of length greater than three. Choose $t > 0$ such that t is the minimum of weight of edges in G . Then G^t , the t -level graph of G is a crisp chordal graph. Then by above lemma [3] applied to the cycle C in G^t implies for each adjacent nodes u and v in C there exists a node w in C which is adjacent to both u and v . Now $\mu(u, w) \geq t$ and $\mu(v, w) \geq t$ being edges in G^t . Therefore the edges uw and vw are in the fuzzy graph G and the node w is in C , adjacent to both u and v . This is the required node in C . $\square$

Theorem 3.6. If $G = (\sigma, \mu)$ is a fuzzy chordal graph and C_n is a cycle in G with $n \geq 4$. Then all the chords in the cycle have weight greater than or equal to minimum weight of the edges in C_n .

Proof. Let $C_n = u_1, u_2, \dots, u_n, n \geq 4$ be a cycle in the fuzzy chordal graph $G = (\sigma, \mu)$. Then by the definition of fuzzy chordal graph there exists chords in C_n . Let

$$t = \wedge \mu(u_i, u_{i+1}) | i = 1, 2, \dots, n-1 \wedge \mu(u_1, u_n) \quad (3.8)$$

Let u_j, u_k be two non-consecutive nodes in C_n such that

$$\mu(u_j, u_k) > 0 \quad (3.9)$$

and if $\mu(u_j, u_k) \geq t$ then (u_j, u_k) is a chord of C_n in G . If $\mu(u_j, u_k) < t$, then by the definition of chord in fuzzy graph (u_j, u_k) is not a chord. Hence every chord in the cycle have weight greater than or equal to minimum weight of the edges in C_n . $\square$

Theorem 3.7. [4] If G is a connected chordal graph, then the induced subgraph $\langle C(G) \rangle$ is connected.

Remark 3.8. By above theorem centre of connected chordal graph is connected in crisp graph[4]. But centre of a connected fuzzy chordal graph need not be connected with respect to g -distance. The following is a connected fuzzy chordal graph whose centre is not connected.

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