On domination in soft graph of some special graphs

S Venkatraman$^1$* and R Helen$^2$

Abstract
In this paper, we compute soft domination number of the soft graph of some special graphs like Wheel graph, Helm graph. Also, we find some results on soft domination number for soft graphs obtained through graphical operations like Join of two graphs, Subdivision, Central, Middle and Total graph of a graph.

Keywords
Soft domination number, central graph, subdivision graph, middle graph, total graph.

AMS Subject Classification
05Cxx, 05C69, 13A15, 06D72, 05C75, 05C99.

1 Introduction
The study of Domination in graphs [7] was further developed in the late 1950’s, it still contributes to solve many critical problems in decision making techniques and also has many applications in several fields like modelling biological networks, social networks, facility location problems, coding theory etc. Theory of Soft Sets [1] was initiated by Molodstov in 1999, after that it has been developed and well studied in [2–6]. S.Venkatraman and et.al introduced a new approach to domination in soft graphs[8] by computing the new parameter called soft domination number. In this paper, we are interested in discussing the concept of domination over soft graphs. We organize this paper as follows. Section 2 is dealt with preliminaries. In section 3, theorems on soft domination number of special graphs are dealt. Section 4 contains results on soft domination number of a graph obtained through some graph operations.

2 Preliminaries

Definition 2.1. (See[1, 4]) Let U be an universal set and A be a set of parameters. A Pair (F, A) is called a soft set over U if and only if F is a mapping of A into the set of all subsets of the set U, i.e., F: A → P(U) where P(U) is the power set of U.

Definition 2.2. Let G(V, E) be a simple graph with n vertices and m edges and A be any non-empty set. Let R be an arbitrary relation between elements of A and elements of V. That is \( R \subseteq A \times V \). A set valued function \( F: A \rightarrow P(V) \) can be defined as \( F(x) = \{ y \in V : xRy \} \). The pair \((F, A)\) is a soft set over V. Then \((F, A)\) is said to be a soft graph \( F(A) \) of G if the subgraph induced by \( F(x) \) in \( G \), \( < F(x) > \) is a connected subgraph of \( G \) for all \( x \in A \).

Definition 2.3. (See [8]) A dominating set \( D \subseteq V(G) \) of a soft graph \( F(A) \) of G is defined as \( D = \bigcup_{x \in A} D_x \), where \( D_x \) is a minimal dominating set of \( F(x) \) for every \( x \in A \).
The central graph of G, denoted by C(G), is obtained by subdividing each edge of G or one is a vertex of G and the other is an edge adjacent vertices of G in C(G).

Definition 2.8. Let G be a simple and undirected graph and let its vertex set and edge set be denoted by V(G) and E(G). The central graph of G, denoted by C(G), is obtained by subdividing each edge of G exactly once and joining all the non-adjacent vertices of G in C(G).

Definition 2.9. The middle graph M(G) of a graph G is the graph whose vertex set is V(G) ∪ E(G) and in which two vertices are adjacent if and only if either they are adjacent edges of G or one is a vertex of G and the other is an edge incident with it.

Definition 2.10. (See [9]) The total graph T(G) of G is that graph whose vertex set is V(G) ∪ E(G), and in which two vertices are adjacent if and only if they are adjacent or incident in G.

3. Theorems on soft domination number of special graphs

Theorem 3.1. If G = W_{1,n} then S_γ[F_A(G)] = 1 where A ⊆ V(G) and F_A(x) = {y ∈ V(G) : d(x, y) ≤ k}, k = 1, 2.

Proof. Let G be a Wheel with a cycle having n vertices v_1, v_2, ..., v_n and a universal vertex v.

Case (i.) Let k = 1, A = V(G).

All induced connected subgraphs contain the universal vertex v of the wheel graph. Therefore D = {v} is the S_γ-set of F_A(G) and hence S_γ[F_A(G)] = 1.

Case (ii.) Let k = 2, A = V(G).

Every < F_A(x) > becomes W_{1,n} and hence dominated by the universal vertex v. Therefore D = {v} and hence S_γ[F_A(G)] = 1.

Hence the theorem.

Theorem 3.2. If G = W_{1,n}, A ⊆ V(G), F_A(x) = {y ∈ V(G) : d(x, y) = k}, then S_γ[F_A(G)] = \begin{cases} \left\lceil \frac{n}{2} \right\rceil + 1 & \text{if } k = 1 \\ \left\lceil \frac{n}{3} \right\rceil & \text{if } k = 2 \text{ and } n > 4 \\ 4 & \text{if } k = 2 \text{ and } n = 4 \end{cases}.

Proof. Let k = 1.

For a, b, c dominates each < F_A(v_i) > where i = 1 to n and < F_A(v) > is a cycle of length n. So that < F_A(v) > can be dominated by \left\lceil \frac{n}{3} \right\rceil vertices of v_i. Hence S_γ[F_A(G)] = \left\lceil \frac{n}{2} \right\rceil + 1.

Let k = 2. A = V(G) - {v}.

It is obvious that S_γ[F_A(W_{1,4})] = 4.

Let n > 4. A = V(G) - {v}, vertex set of a cycle with n vertices.

< F_A(v_i) > = < v_1, v_2, ..., v_n >, < v_{i-1}, v_{i-1}, v_i, v_{i+1} > for 2 ≤ i ≤ n - 1.

< F(v_i) > = < v_3, v_4, ..., v_{n-1} > and

< F(v_n) > = < v_2, v_3, ..., v_{n-2} >. One can verify that S_γ[F_A(W_{1,n})] = \left\lceil \frac{n}{2} \right\rceil for n > 4.

Theorem 3.3. If G = W_{1,n}, A' ⊆ V(L(G)), F'_A(x) = {y ∈ V(L(G)) : d(x, y) ≤ k}, where k = 1, 2 for n = 3, 4, 5 and k = 1, 2, 3 for n > 5 then

S_γ[F'_A(L(G))] = \begin{cases} \left\lceil \frac{n}{2} \right\rceil + 1 & \text{if } k = 1 \text{ and } n \in N \\ \left\lceil \frac{n}{3} \right\rceil & \text{if } k = 2 \text{ and } n \in N \\ \left\lceil \frac{n^2 + 2}{3} \right\rceil + 1 & \text{if } k = 3 \text{ and } n > 5. \end{cases}

Proof. Consider V(G) = {v} ∪ {v_i}, where i = 1, 2, ..., n. Let v_1v_{i+1} = e_i where t = 1, 2, ..., (n - 1) and v_1e_n = e_n. Also let v_nP = e_{(P-1)}P, P = 2, 3, ..., n and v_1 = e_{n1}.

By definition of L(G), we have

(a.) V(L(G)) = E(G).

(b.) e_i's form a cycle of length n, i = 1, 2, ..., n.

(c.) e_{(P-1)}P together with e_{n1} form a clique of order n where P = 2, 3, ..., n and each e_{(P-1)}P is adjacent to e_{(P-1)} and e_{P}, e_{n1} is adjacent to e_n and e_1.

We can construct the minimal dominating set D of F_A(L(G)) as follows:

Case (i.) Let k = 1, n ∈ N and n is even.

Each < F(e_i) > is dominated either by e_i if \('1'\) is odd or by e_{i-1} and e_{i+1} if \('i'\) is even.
Theorem 3.4. \( < F(e_k) > \) is dominated by \( e_1 \) and \( e_{n-1} \). Also, \( < F(e_{(i+1)}) > \) is dominated by \( e_{n} \) and \( e_k \), where \( k \in \{i, i+1\} \) such that \( k \equiv 1 \pmod{2} \). So, \( D = \{e_1, e_2, \ldots, e_{n-1}, e_n\} \).

Case (ii.) Let \( k = 1, n \in N \) and \( n \) is odd.

For \( i = 1, 2, \ldots, (n-2) \), domination in each \( < F(e_i) > \) is similar to that of case (i). Now \( < F(e_{n-1}) > \) is dominated by \( e_{n-2} \) and \( e_{(n-1)n} \). Moreover \( e_1 \) and \( e_{(n-1)n} \) dominated \( < F(e_n) > \).

So that \( D = \{e_1, e_3, \ldots, e_{n-2}, e_{(n-1)n}\} \).

From both the cases we conclude that \( SF(\gamma(F(A)(L(G)))) = \left[ \frac{n}{2} \right] + 1 \).

Case (iii.) Let \( k = 2, n \in N \) and \( n > 2 \).

A clique of order \( n \) formed by the vertices \( e_{12}, e_{23}, e_{n1} \) is appearing in every induced subgraphs \( < F(e_i) > \), where \( i = 1 \) to \( n \). Together with this clique each \( < F(e_i) > \) contains induced set of five other vertices \( \{e_{j+2}, e_{j+1}, e_j, e_{j-1}, e_{j-2}\} \) where \( 3 \leq j \leq n-2 \).

Also, for \( F(< e_1 >), F(< e_2 >), F(< e_n >) \) and \( F(< e_{n-1} >) \) they were \( \{e_{n-1}, e_1, e_2, e_3\}, \{e_n, e_1, e_2, e_3\}, \{e_{n-2}, e_{n-1}, e_n, e_1\} \) and \( \{e_{n-3}, e_{n-2}, e_{n-1}, e_n, e_1\} \) respectively.

Therefore the minimum dominating set
\[
D = \{e_{12}, e_{34}, \ldots, e_{(n-1)n}\} \text{ or } D = \{e_{12}, e_{34}, \ldots, e_{(n-2)(n-1)}, e_{n1}\} \text{ according as } n \text{ is even or odd. Hence } SF(\gamma(F(A)(L(G)))) = \left[ \frac{n}{2} \right].
\]

Case (iv.) Let \( k = 3 \) and \( n > 5 \).

Since \( e_{12} \) dominates the clique of order \( n \) and also \( e_1 \) and \( e_2 \), the remaining \( (n-2) \) vertices \( e_3, e_4, \ldots, e_{n-1}, e_n \) form a path. This path can be dominated by \( \left[ \frac{n-2}{2} \right] \) vertices.

\[
SF(\gamma(F(A)(L(G)))) = \left[ \frac{n-2}{2} \right] + 1. \text{ Hence the theorem.}
\]

Theorem 3.4. If \( G = H_n, (n \geq 4) \) then \( SF(\gamma(F(A)(H_n))) = n \) where \( A \subseteq V(H_n); F(x) = \{y \in V(H_n) : d(x,y) \leq k\}, k = 1, 2, 3, 4. \)

Proof. Let \( V(H_n) = \{v\} \cup \{v_i\}_{i=1}^n \cup \{v'_i\}_{i=1}^n \) where \( v \) is the universal vertex of the \( n \)-wheel graph in \( H_n \), \( v'_i \) s are the vertices of the cycle of length \( n \) in \( H_n \), and \( v_i' \) s are corresponding pendant vertices are adjacent to \( v_i' \) s in \( H_n \).

Clearly \( A = V(H_n) \) for \( k = 1 \) and \( k = 2 \).

\[
A = V(H_n) - \{v\} \text{ for } k = 3.
\]

\[
A = V(v_i') \text{ for } k = 4.
\]

Case (i.) For \( k = 1 \) and \( k = 2 \), \( < F(v) > \) is dominated by all \( v_i' \) s.

Case (ii.)

- For \( k = 1 \)

\( < F(v_i) > \) is dominated by \( v_i \) itself.

- For \( k = 2 \) and \( k = 3 \)

\( < F(v_i) > \) is dominated by all \( v_i' \) s.

Case (iii.)

- For \( k = 1 \) and \( k = 2 \)

\( < F(v_i') > \) is dominated by all \( v_i \), the corresponding vertex adjacent to it.

- For \( k = 3 \) and \( k = 4 \)

\( < F(v_i') > \) is dominated by \( v_i, \forall i \). Since \( D \) is the minimal dominating set, it is evident from all the three cases that \( D = \{v_i\}_{i=1}^n \).

Corollary 3.5. \( SF(\gamma(F(A)(H_n))) = 3 \) where \( A \subseteq V(H_k) \) and \( F(x) = \{y \in V(H_k) : d(x,y) \leq k\}, k = 1, 2, 3, 4. \)

Proposition 3.6. \( SF(\gamma(F(A)(H_k))) \) does not exist for \( k = 4 \) and \( n > 4 \) since no soft graph exists for these values.

Theorem 3.7. If \( G = H_n, A' \subseteq V(L(G)) \) and \( F_{\gamma}(x) = \{y \in V(L(G)) : d(x,y) \leq k\}, k = 1, 2, 3, 4 \) and \( k = 1, 2, 3 \) for \( n \geq 4 \), then \( SF(\gamma(F(A)(L(G)))) = \left[ \frac{n-2}{2} \right] + \left[ \frac{n}{2} \right]. \)

Proof. Let \( V(H_n) = \{v\} \cup \{v_i\}_{i=1}^n \cup \{v'_i\}_{i=1}^n \) where \( i = 1, 2, \ldots, n \). Define edges \( e_1, e_2, \ldots, e_{(n-1)p} \) as in theorem 3.3.

\[
\gamma = e'_1 e'_2 e'_3 \ldots e'_{n-1}.
\]

From the definition of \( L(G) \), we have the same properties \( (a), (b), (c) \) as in theorem 3.3.

Together with these properties we have one more property which is, \( e_{(j-1)} \) is adjacent to \( e_{(j-1)} \), \( e_{j-1} \) and \( e_j \), \( e_j' \) is adjacent to \( e_{n1}, e_n \) and \( e_1 \). Now for all values of \( k \), we construct the minimal dominating set \( D \) as

\[
D = \{e_1, e_{(i+1)(i+2)} \in V(L(H_n)) : l \equiv 1 \pmod{3}\}.
\]

Thus, \( SF(\gamma(F(A)(L(G)))) = \left[ \frac{n-2}{2} \right] + \left[ \frac{n}{2} \right]. \)
4. Results on soft domination number of a graph obtained through some graph operations

4.1 Soft domination number on join of two graphs

Theorem 4.1. If $G = G_1 + G_2$, $A \subseteq V(G_1) \cup V(G_2)$ and $F_A(x) = \{y \in V(G_1 + G_2) : d(x,y) \leq k\}$ where $k = 1, 2$, then

$$S\gamma[F_A(G)] = \begin{cases} \min\{S\gamma[(F_A(G_1))], \min\{S\gamma[(F_A(G_2))]\} & \text{if } k = 1 \\ 2 & \text{if } k = 2 \end{cases}$$

Proof. (i.) Let $k = 1$.

Let $D_1$ and $D_2$ be the minimal dominating sets of the soft graphs $F_A(G_1)$ and $F_A(G_2)$ respectively. In $G = G_1 + G_2$, each $< F_A(v_i) >$ where $v_i \in V(G_1)$ is the join of the graph $G_2$ and the subgraph induced by the vertices $v_i \cup N(v_i)$ in $G_1$. Also, each $< F_A(v_j) >$ where $v_j \in V(G_2)$ is induced in the same manner.

Now we construct the minimal dominating set $D$ of the soft graph $G$ as $D = \begin{cases} D_1 & \text{if } |D_1| \leq |D_2| \\ D_2 & \text{otherwise} \end{cases}$ and therefore

$$S\gamma[F_A(G)] = |D| = \min\{|D_1|, |D_2|\}.$$ 

(ii.) Let $k = 2$.

Now each $< F_A(v) >$ where $v \in V(G_1 + G_2)$ is the same as that of the graph $G$.

So clearly $S\gamma[F_A(G)] = \gamma(G) = 2$. \(\square\)

Proposition 4.2. If $G = P_n + P_m$, $A \subseteq V(P_n) \cup V(P_m)$ and $F_A(x) = \{y \in V(G) : d(x,y) = k\}$ where $k = 1, 2$, then $S\gamma[F_A(G)] = \lceil \frac{(m-1)}{2} \rceil + \lceil \frac{(n-1)}{2} \rceil$.

Proposition 4.3. We arrive the same results as in proposition 4.2 for $G = P_n + C_m$.

4.2 Soft domination number of subdivision graph of a graph $G$

Proposition 4.4. Let $G = P_n$. The subdivision graph of $P_n$ has $(2n-1)$ vertices, which is nothing but $P_{2(n-1)}$.

Since $S\gamma[F_A(P_n)] = \lceil \frac{(n-1)}{2} \rceil$ if $n \geq k + 3$ where $A \subseteq V(P_n)$ and $F_A(x) = \{y \in V(P_n) : d(x,y) \leq k\}$, we have $S\gamma[F_A(P_n)] = n - 1$.

Proposition 4.5. Let $G = C_n$. The subdivision of a star graph $K_{1,n}$ is a spider having $(2n+1)$ vertices. $S\gamma[F_A(S(G))] = n$, where $A \subseteq V(S(G))$ and $F_A(x) = \{y \in V(S(G)) : d(x,y) \leq l\}$, $l \in \{1, 2\}$.

4.3 Soft domination number of central graph $G$

Theorem 4.6. If $G = P_n$, $A \subseteq V(P_n)$, $F_A(x) = \{y \in V(C(G)) : d(x,y) \leq l\}$, $l \in \{1, 2\}$ for $n = 2, 3$ and $k = 1, 2, 3$ for $n \geq 4$ then

$$S\gamma[F_A(C(G))] = \begin{cases} n & \text{if } k = 1 \\ \lceil \frac{n}{2} \rceil + 1 & \text{if } k = 2, 3. \end{cases}$$

Proof. Let $G = P_n$, a path with $n$ vertices $v_1, v_2, \ldots, v_n$.

$C(G)$ has $(2n-1)$ vertices which are

$v_1, u_1, v_2, u_2, \ldots, v_{n-1}, u_{n-1}, v_n$. Here each $u_i$ is adjacent to $v_j$ and $v_{i+1}$ where $i = 1, 2, \ldots, (n-1)$ and each $v_j$ is adjacent with $V(P_n) - N(v_j)$ where $j = 1, 2, \ldots, n$.

(i.) Let $k = 1$.

Each $< F_A(v_j) >$ is dominated by $v_j$ and each $< F_A(u_i) >$ is dominated by $v_i$.

Hence $D = \{v_j \in V(G) : 1 \leq j \leq n\}$.

\(\therefore S\gamma[F_A(C(G))] = |D| = n.\)

(ii.) Let $k = 2$.

Each $< F_A(v_j) >$ and each $< F_A(u_i) >$ can be dominated by the set $\{v_1, v_1\}$ where $l \equiv 0 \pmod{2}$ and $v_l \in V(P_n)$.

\(\therefore S\gamma[F_A(C(G))] = |D| = \lceil \frac{n}{2} \rceil + 1.\)

(iii.) Let $k = 3$.

$A = \{u_i \in C(G)\}$ where $i = 1, 2, \ldots, (n-1)$. Each $< F_A(u_i) >$ is same as that of $C(G)$.

Construct the minimal dominating set $D$ as in (ii.).

Hence $S\gamma[F_A(C(G))] = |D| = \lceil \frac{n}{2} \rceil + 1$. \(\square\)

Proposition 4.7. If $G = C_n$, $A \subseteq V(C(G))$ and $F_A(x) = \{y \in V(C(G)) : d(x,y) \leq k\}$ where $k = 1, 2, 3$ then $S\gamma[F_A(C(G))] = \begin{cases} n & \text{if } k = 1 \\ (n-1) & \text{if } k = 3. \end{cases}$

Proposition 4.8. If $G = K_{1,n}$, $A \subseteq V(C(G))$ and $F_A(x) = \{y \in V(C(G)) : d(x,y) \leq l\}$ where $k = 1, 2$ then $S\gamma[F_A(C(G))] = \begin{cases} n + 1 & \text{if } l = 1 \\ 2 & \text{if } l = 2. \end{cases}$

4.4 Soft domination number of middle graph of a graph $G$

Throughout this section $A \subseteq V(M(G))$ and $F_A(x) = \{y \in V(M(G)) : d(x,y) \leq k\}$.

Proposition 4.9. If $G = P_n$ or $C_n$ then $S\gamma[F_A(M(G))] = \lceil \frac{n}{2} \rceil$.

Proposition 4.10. If $G = K_{1,n}$ then $S\gamma[F_A(M(G))] = n = \gamma(M(K_{1,n}))$. 

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4.5 Soft domination number of a total graph of a graph G

Throughout this section $A \subseteq V(T(G))$ and $F_A(x) = \{ y \in V(T(G)) : d(x,y) \leq k \}$.

**Proposition 4.11.** If $G = P_n$ then $\Sigma'[F_A(T(G))] = \lceil \frac{n+1}{3} \rceil + \lceil \frac{n-1}{3} \rceil$.

**Proof.** Let $G = P_n = < v_1,v_2,\ldots,v_n >$. Let $v_i,v_{i+1} = e_i$ where $i = 1,2,\ldots,(n-1)$. Construct $D = \{ v_i,e_j \in V(T(G)) \}$ such that $i \equiv 2(mod3)$ and $j \equiv 0(mod3)$.

$\Sigma'[F_A(T(G))] = \lceil \frac{n+1}{3} \rceil + \lceil \frac{n-1}{3} \rceil$. \qed

**Proposition 4.12.** If $G = C_n$ then $\Sigma'[F_A(T(G))] = \lceil \frac{n}{2} \rceil$.

**Proposition 4.13.** If $G = K_{1,n}$ then $\Sigma'[F_A(T(G))] = n$.

5. Open Problem

Let $G_1$ and $G_2$ be any two graphs. Define $G = G_1 + G_2$, $A \subseteq V(G_1) \cup V(G_2)$ and $F_A(x) = \{ y \in V(G) : d(x,y) = k \}$ where $k \in \{ 1,2 \}$. Determine the soft domination number for $G$ where $G_1$ and $G_2$ are other than paths and cycles.

6. Conclusion

In this paper, we established some important results on soft domination number of a graph obtained through graph operations like join of two graphs, subdivision, central, middle, total graph of a graph. We also proved some theorems on soft domination number of special graphs. We will study how soft domination number influence on decision making problems and power domination problems in our future work.

References