Logarithmic fractional order Euler functions for solving Caputo type modified Hadamard fractional integro-differential equations

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Abstract
In this paper, a new set of logarithmic fractional order Euler functions (LFEFs) is constructed to obtain a numerical method for solving fractional integro-differential equation with kernel of the form $\log(\frac{x}{t})$. To do this an operational matrix of fractional order integration in the Hadamard sense for the LFEFs is derived. The properties and operational matrix of LFEFs is used to transform the problem of fractional integro-differential equation into a system of linear algebraic equations. The error analysis of approximation in terms of LFEFs is also given.

Keywords
Fractional integro-differential equation, Logarithmic Fractional order Euler functions, Modified Caputo Hadamard differential operator, Operational matrix.

AMS Subject Classification
26A33, 34A08, 11B68, 42C40, 65T60.

1. Introduction
In recent decades, the theories of derivatives and integrals with any non-integer arbitrary order have been intensively studied due to many applications in different areas of real life including Science and engineering\(^2\). For example, it has been applied to model the nonlinear oscillation of earthquakes, fluid dynamic traffic, frequency dependent damping behavior of many viscoelastic materials, continuum and statistical mechanics, colored noise, solid mechanics, economics, signal processing and control theory\(^9\). Some numerical techniques have been developed to solve fractional differential equations (FDE). Adomian decomposition method\(^5\), Homotopy perturbation method\(^17\), Homotopy analysis method\(^8\), Variational iteration method\(^4\), Power series method\(^14\) and Laplace transform method\(^15\) are the most frequently used methods.

Recently, approximation of the solution of FDE as a linear combination of basis polynomials or functions have also received considerable attention. The main idea of using these basis functions is that the problem under consideration reduced to a linear or nonlinear algebraic equations. In 2011, a fractional extension of the classical Legendre polynomials by replacing the integer order derivative in Rodrigues formula by fractional order derivatives is generated by Rida and Yousef\(^16\). Subsequently Kazem generated the orthogonal fractional order Legendre functions based on shifted polynomials to obtain the solution of FDEs more simply and efficiently\(^11\). In 2016 a fractional extension of Euler polynomials is constructed for the solution of generalized Pantograph equations\(^10\) and the same is used for the solution of the class of space fractional diffusion equation\(^12\). Recently in\(^1\) Yanxin Wang used fractional order Euler functions (FEFs) for solving fractional integro-differential equations with weakly singular kernel.

In the present article, we generate Logarithmic fractional order Euler functions (LFEFs) to extend the application of FEFs for solving fractional integro-differential equations with
2. Preliminaries

In this section, we recall some definitions and basic results of fractional calculus which will be used throughout the paper.

Definition 2.1. [6] Let $0 \leq a < b \leq \infty$ be finite or infinite interval of the half-axis $\mathbb{R}^+$. The Hadamard fractional integral of order $\alpha \in C$ is defined by

$$I_{a}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (\log \frac{t}{a})^{\alpha-1} f(t) \frac{dt}{t}$$

Property 1
If $\alpha > 0$, $\beta > 0$ and $0 < a < \infty$, then we have [6]

$$I_{a}^{\alpha}(\log \frac{t}{a})^{\beta} = \frac{\Gamma(\beta)}{\Gamma(\beta + \alpha)} (\log \frac{t}{a})^{\alpha + \beta - 1}$$

Definition 2.2. [6] The Hadamard fractional derivative of order $\alpha \in C$ with $\Re(\alpha) \geq 0$ on $(a,b)$ and $a < x < b$ is defined by

$$D_{a}^{\alpha} f(x) = \frac{d}{dx}^{\alpha} (I_{a}^{\alpha-a} f)(x) = (x \frac{d}{dx})^{\alpha} \frac{1}{\Gamma(n - \alpha)} \int_{a}^{x} (\log \frac{t}{a})^{n-\alpha-1} f(t) \frac{dt}{t}$$

Definition 2.3. [6] Let $\Re(\alpha) \geq 0$ and $n = [\Re(\alpha)] + 1$ and $f \in AC_{\delta}^{n}[a,b]$, where $0 < a < b < \infty$ and $AC_{\delta}^{n}[a,b] = \{ g : [a,b] \to C : \delta^{n-1}[g(x)] \in AC[a,b], \delta = x^{\frac{n}{m}} \}$. The Caputo-type modification of Hadamard fractional derivative is defined as follows:

$$CD_{a}^{\alpha} f(x) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{x} \left( \log \frac{t}{a} \right)^{n-\alpha-1} \left( \frac{d}{dt} \right)^{n} f(t) \frac{dt}{t}$$

Lemma 2.4. [6] Let $y \in AC_{\delta}^{n}[a,b]$ or $C_{\delta}^{n}[a,b]$ and $\alpha \in C$. Then,

$$a^{\alpha} (CD_{a}^{\alpha}) y(x) = y(x) - \sum_{k=0}^{n-1} \frac{\delta^{k} y(a)}{k!} \left( \log \frac{x}{a} \right)^{k}$$

Euler polynomials: Here we recall definition and some properties of Euler polynomials [1]. The Euler polynomials of degree $m$ are constructed from the following relation.

$$\sum_{k=0}^{m} \binom{m}{k} E_{k}(t) + E_{m}(t) = 2^{m}$$

3. Main Results

Logarithmic Fractional order Euler functions

The Logarithmic fractional order Euler functions (LFEFs) are constructed by changing the variable $t$ to $(\log \frac{x}{a})^{\alpha}$ ($\alpha > 0, a > 0$) on the Euler polynomials. Let the logarithmic fractional order Euler functions $E_{m}(\log \frac{x}{a})^{\alpha}$ be denoted by $E_{m}^{\alpha}(\log \frac{x}{a})$. The analytic form of $E_{m}^{\alpha}(\log \frac{x}{a})$ is given by

$$\sum_{k=0}^{m} E_{k}^{\alpha}(\log \frac{x}{a}) + E_{m}^{\alpha}(\log \frac{x}{a}) = 2^{\log \frac{x}{a}} \frac{m!}{m+n+1} E_{m+n+1}(0), m,n \geq 1$$

Let $E^{\alpha}(\log \frac{x}{a}) = [E_{0}^{\alpha}(\log \frac{x}{a}), E_{1}^{\alpha}(\log \frac{x}{a}), \ldots, E_{m}^{\alpha}(\log \frac{x}{a})]^{T}$, then fractional order Euler polynomials can be represented in a matrix form as

$$E^{\alpha}(\log \frac{x}{a}) = B^{\alpha} X^{\alpha}(\log \frac{x}{a})$$

where

$$E^{\alpha}(\log \frac{x}{a}) = \begin{bmatrix} E_{0}^{\alpha}(\log \frac{x}{a}) \\ E_{1}^{\alpha}(\log \frac{x}{a}) \\ \vdots \\ E_{m}^{\alpha}(\log \frac{x}{a}) \end{bmatrix}$$
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\[ B^\alpha = B(t) \]

\[ X^\alpha (\log \frac{x}{a}) = \begin{bmatrix} 1 \\ (\log \frac{x}{a})^\alpha \\ \vdots \\ (\log \frac{x}{a})^{m\alpha} \end{bmatrix} \]

The first four logarithmic fractional order Euler polynomials are

\[ E_0^\alpha (\log \frac{x}{a}) = 1, \]
\[ E_1^\alpha (\log \frac{x}{a}) = (\log \frac{x}{a})^\alpha - \frac{1}{2}, \]
\[ E_2^\alpha (\log \frac{x}{a}) = (\log \frac{x}{a})^{2\alpha} - (\log \frac{x}{a})^\alpha \]
\[ E_3^\alpha (\log \frac{x}{a}) = (\log \frac{x}{a})^{3\alpha} - \frac{3}{2}(\log \frac{x}{a})^{2\alpha} + \frac{1}{4}. \]

**Approximation of functions**

Let \( V = \text{span} \{ E_0^\alpha, E_1^\alpha, ..., E_m^\alpha \} \) such that any function \( u(x) \in V \) can be written as a linear combination of the basis functions.

\[ u(x) \approx \sum_{j=0}^{m} c_j E_j^\alpha (\log \frac{x}{a}) \]

Given a function \( f(x) \). Its best approximation \( u(x) \in V \) is determined by using least square method [13], which minimizes the distance between \( u \) and \( f \).

ie, we minimize the squared norm of the distance

\[ E = \langle f - u, f - u \rangle = \langle f - \sum c_j E_j^\alpha, f - \sum c_j E_j^\alpha \rangle \]

Rewriting the above equation,

\[ E(c_0, c_1, ..., c_m) = \langle f, f \rangle - 2 \sum_{j=0}^{m} c_j \langle f, E_j^\alpha \rangle + \sum_{i=0}^{m} \sum_{j=0}^{m} c_i c_j \langle E_i^\alpha, E_j^\alpha \rangle \]

Minimizing a function of \( m+1 \) scalar variables \( \{c_i\}_{i=0}^{m} \) requires differentiation with respect to all \( c_i, j = 0, 1, ..., m \). The resulting equations end up with a linear system of the form

\[ \sum_{j=0}^{m} A_{i,j} c_j = b_i, \quad i = 0, 1, ..., m \]

where

\[ A_{i,j} = \langle E_i^\alpha, E_j^\alpha \rangle, \quad b_i = \langle f, E_i^\alpha \rangle \]

**Operational matrix for Fractional integration for LFEFs**

Let \( M^{(v,\alpha)} \) is the \((m+1) \times (m+1)\) Hadamard fractional operational matrix of integration of order \( v \) for LFEFs of order \( m\alpha \). The Hadamard fractional integration of order \( v \) of the vector \( E^\alpha (\log \frac{x}{a}) \) can be expressed by

\[ I^v E^\alpha (\log \frac{x}{a}) = M^{(v,\alpha)} E^\alpha (\log \frac{x}{a}) \]

By using fractional integration operator \( I^v \) and the matrix representation of LFEFs (3.2), We have

\[ I^v E^\alpha (\log \frac{x}{a}) = I^v B^\alpha X^\alpha (\log \frac{x}{a}) = B^\alpha I^v X^\alpha (\log \frac{x}{a}) \]

\[ = B^\alpha N^{(v,\alpha)} X^\alpha (\log \frac{x}{a}) \]

\[ = B^\alpha N^{(v,\alpha)} (B^\alpha)^{-1} E^\alpha (\log \frac{x}{a}) \]

where \( N^{(v,\alpha)} \) is the Hadamard operation matrix of fractional integration of the vector \( X^\alpha (\log \frac{x}{a}) \).

Using property 1 of \( I^v, \) for \( r = 0, 1, ..., m, \) we have

\[ I^v X^\alpha (\log \frac{x}{a}) = (\log \frac{x}{a})^r \]

Assume that \( (\log \frac{x}{a})^r \) can be expanded in terms of LFEFs as

\[ (\log \frac{x}{a})^r = \sum_{j=0}^{m} k_{(r,j)} \alpha \]

Then

\[ I^v X^\alpha (\log \frac{x}{a}) = (\log \frac{x}{a})^r = \frac{\Gamma(r\alpha + 1)}{\Gamma(r\alpha + 1 + v)} (\log \frac{x}{a})^r \]

Therefore we get

\[ N^{(v,\alpha)} = [a_{r,j}] = \frac{\Gamma(r\alpha + 1)}{\Gamma(r\alpha + 1 + v)} k_{(r,j)} \]

\[ r = 0, 1, ..., m, \quad j = 0, 1, ..., m \]

Hence, combining equation (3.3) and equation (3.4), we get

\[ M^{(v,\alpha)} = B^\alpha N^{(v,\alpha)} (B^\alpha)^{-1} \]  

**Application of the operation matrix of fractional integration**

Consider the following fractional integro differential equation with \( \log(x/t) \) as kernel:

\[ D^\alpha y(x) = f(x) + \lambda \int_{a}^{x} \frac{1}{t \log \frac{x}{t}} d^\beta y(t) \]

\[ y^{(i)}(a) = \gamma_a^{(i)}, \quad i = 0, 1, 2, ..., n-1 \]

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where $f(x)$ is known continuous function on $I = [a, x]$, $y(x)$ is the unknown function, $\beta > 0$, $\nu$ and $\lambda$ are real constants, $n$ is the smallest integer greater than or equal to $\nu$, $y_n^{(i)}$, $i = 0, 1, 2...n - 1$ are given real numbers and $D^\nu$ denote the Caputo-type modification of Hadamard fractional derivative of order $\nu$.

We approximate $D^\nu y(x)$ and $f(x)$ in terms of LFEFs.

$$D^\nu y(x) \approx C^T E^\alpha (\log^\frac{x}{a})$$ \hspace{1cm} (3.8)

$$f(x) \approx F^T E^\alpha (\log^\frac{x}{a})$$ \hspace{1cm} (3.9)

where

$$C^T = \begin{bmatrix} c_0 & c_1 & \ldots & c_m \end{bmatrix}^T$$

$$F^T = \begin{bmatrix} f_0 & f_1 & \ldots & f_m \end{bmatrix}^T.$$

Then,

$$I^\nu D^\nu y(x) = I^\nu C^T E^\alpha (\log^\frac{x}{a})$$

By using lemma(2.4) and equation (3.3),

$$y(x) = I^\nu C^T E^\alpha (\log^\frac{x}{a}) + \sum_{k=0}^{m} \frac{\delta^k y(a)}{k!} (\log^\frac{x}{a})^k$$

$$= C^T M^{(\nu, \alpha)} E^\alpha (\log^\frac{x}{a}) + \sum_{k=0}^{m} \frac{\delta^k y(a)}{k!} (\log^\frac{x}{a})^k$$

We substitute the initial conditions in the above equation and approximate in terms of LFEFs, we obtain,

$$y(x) = (C^T M^{(\nu, \alpha)} + C^T) E^\alpha (\log^\frac{x}{a}) = A^T E^\alpha (\log^\frac{x}{a})$$ \hspace{1cm} (3.10)

Now

$$\int_a^x \text{log}^\frac{x}{t} y(t) \frac{dt}{t} = A^T \int_a^x \text{log}^\frac{x}{t} E^\alpha (\log^\frac{x}{a}) \frac{dt}{t}$$

$$= A^T \Gamma(\beta + 1) A^T M^{(\beta + 1, \alpha)} E^\alpha (\log^\frac{x}{a})$$

$$= \Gamma(\beta + 1) A^T M^{(\beta + 1, \alpha)} E^\alpha (\log^\frac{x}{a})$$ \hspace{1cm} (3.11)

By substituting equations (3.8), (3.9) and (3.11) into equation (3.6), we get

$$C^T E^\alpha (\log^\frac{x}{a}) = F^T E^\alpha (\log^\frac{x}{a})$$

$$+ \lambda \Gamma(\beta + 1) A^T M^{(\beta + 1, \alpha)} E^\alpha (\log^\frac{x}{a})$$

Therefore we have,

$$C^T = F^T + \lambda \Gamma(\beta + 1) A^T M^{(\beta + 1, \alpha)}$$ \hspace{1cm} (3.12)

which is a system of linear algebraic equations in term of the unknown elements of the vector $C$.

**Error analysis**

**Generalized Taylor’s formula in terms of LFEFs:**[3]

Let $0 < a < b < \infty$. Let $0 < \alpha \leq 1$ and let $m$ be an arbitrary non-negative integer. Suppose $H_{a^+}^{j\alpha} f \in C[a,b]$, $j = 0, 1, 2,...m + 1$, Then for $x > a$

$$f(x) = \sum_{j=0}^{m} \left( \log^\frac{x}{a} \right)^{j\alpha} H_{a^+}^{j\alpha} f(a)$$

$$+ \left( \log^\frac{x}{a} \right)^{m+1} \frac{H_{a^+}^{m+1\alpha} f(a)}{\Gamma((m+1)\alpha+1)}$$ \hspace{1cm} (3.13)

where $\xi \in [a,x]$ and $H_{a^+}^{j\alpha}$ is the Caputo modified Hadamard fractional derivative of order $j\alpha$.

A function $f(x) \in L^2[a,b]$ can be approximated as

$$f(x) = \sum_{i=0}^{m} c_i E^\alpha (\log^\frac{x}{a}) = f_m(x)$$

The error analysis is given by the following theorem.

**Theorem 3.1.** Suppose that $D^{j\alpha} f(x) \in C[a,b]$, $j = 1, 2,...m$ and $Y^\alpha_m = \text{span}\{E^\alpha_0 (\log^\frac{x}{a}), E^\alpha_1 (\log^\frac{x}{a}),...E^\alpha_m (\log^\frac{x}{a})\}$ is a vector space. If $f_m(x)$ is the best approximation of $f$ out of $Y^\alpha_m$, then the mean error bound is presented as follows:

$$||f - f_m|| \leq M_{\alpha} \sqrt{b-a} \frac{\Gamma((m+1)\alpha+1)}{\Gamma((m+1)\alpha+1)} \frac{(b-a)^{(m+1)\alpha}}{a}$$ \hspace{1cm} (3.14)

where $M_{\alpha} \geq \sup_{\xi \in [a,b]} |D_{a^+}^{(m+1)\alpha} f(\xi)|$.

**Proof.** We have the Generalized Taylor’s formula in terms of LFEFs,

$$f(x) = \sum_{j=0}^{m} \left( \log^\frac{x}{a} \right)^{j\alpha} D_{a^+}^{j\alpha} f(a)$$

$$+ \left( \log^\frac{x}{a} \right)^{m+1} \frac{D_{a^+}^{m+1\alpha} f(a)}{\Gamma((m+1)\alpha+1)}$$

where $D_{a^+}^{j\alpha}$ is the Caputo modified Hadamard fractional derivative.

Let $y(x) = \sum_{j=0}^{m} \left( \log^\frac{x}{a} \right)^{j\alpha} D_{a^+}^{j\alpha} f(a)$, Then,

$$|f(x) - y(x)| \leq M_{\alpha} \frac{\left( \log^\frac{x}{a} \right)^{(m+1)\alpha}}{\Gamma((m+1)\alpha+1)}$$

since $M_{\alpha} \geq \sup_{\xi \in [a,b]} |D_{a^+}^{(m+1)\alpha} f(\xi)|$. Now $f_m$ is the best approximation to $f$ from $Y^\alpha_m$. So we have,
\[ \|f - f_m\|_2^2 \leq \|f - y\|_2^2 \]
\[ = \int_a^b (f(x) - y(x))^2 \, dx \]
\[ = \int_a^b M_a \frac{(\log \frac{x}{a})^{(2m+2)\alpha}}{(\Gamma((m+1)\alpha + 1))^2} \, dx \]
\[ \leq \frac{M_a(b-a)}{(\Gamma((m+1)\alpha + 1))^2} \sup_{x \in [a,b]} \{ (\log \frac{x}{a})^{(2m+2)\alpha} \} \]
\[ \leq \frac{M_a^2(b-a)}{(\Gamma((m+1)\alpha + 1))^2} \sup_{x \in [a,b]} \{ (\frac{x}{a} - 1)^{(2m+2)\alpha} \} \]
\[ \leq \frac{M_a^2(b-a)}{(\Gamma((m+1)\alpha + 1))^2} \left( \frac{b-a}{a} \right)^{(2m+2)\alpha} \]

By taking square roots, we get LFEFs approximation of \( f(x) \) is convergent since Gamma function grows faster than any exponential function with fixed base.

\[ \square \]

### 4. Conclusion

We constructed a new set of functions, Logarithmic fractional order Euler functions (LFEFs), and also the Operational matrix for Hadamard fractional integration of LFEFs is derived. We use LFEFs to approximate the numerical solution of fractional integro-differential equation whose kernel is of the form \( \log \frac{x}{\alpha} \) in Caputo modified Hadamard sense. By using the matrix and LFEFs, the fractional integro-differential equation with initial conditions are reduced to a system linear algebraic equations. We have seen in the error analysis that the LFEFs approximation is convergent.

### References


