New conformable fractional ELZAKI transformation: Theory and applications

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Abstract
In this paper, we have introduced new 1 – D and 2 – D conformable fractional Elzaki transformation with the properties and its applications to new fractional derivative with Non – Singular Kernel and conformable differential equation has been solved and results were compared. Also new generalized fractional Elzaki – Tarig Transformation has been defined with existence of inverse and convolution property being proved.

Keywords
Conformable derivative, fractional integral transformation, fractional derivative.

AMS Subject Classification
26A24, 26A33, 26A99.

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1. Introduction

The idea of fractional derivative and fractional integral transformation are going in hand and hand since long ago [14] Fractional integral transform have applications in signal processing, quantum mechanics, fluid dynamics and stochastic processes [2, 29] rediscovered mainly in quantum mechanics, fluid dynamics, Signal Processing and Stochastic Processes. In recent years, many fractional linear and nonlinear [27] initial value problems and boundary value problems are effectively solved by using different integral transforms.

The generalization of integral transforms and fractional integral transforms [3, 22, 24] including Elzaki – Tarig, Laplace, Mellin, L2, Abel’s, Weirstrass, Hilbert, Fourier, Laplace – Carson, Laplace – Stieljes Transformations has being established in well manner. All these Transformations have tremendous applications [14, 27, 29] in fractals, Bio- Mathematics, Computational Fluid Dynamics. There are several types of fractional derivative definitions including Riemann – Liouville, Caputo, Atangana – Baleanu Riemann (ABR), Atangana – Baleanu Caputo(ABC)[1, 2].

The term conformable fractional derivative [4] who has lots of advantageous for getting the solutions of fractional differential equations in analytical form while conformable Laplace [20] were introduced to solve such kind of conformable fractional derivative to get the analytical solutions instead of numerical approximations to the solutions.

The aim of the paper is to solve various kinds of fractional differential equations by using newly defined conformable fractional integral transformation. This paper has been organized as follows. In section 2, we have given some basic definitions which were required for further calculations. In section 3, definition of new generalized fractional integral transformation along with existence of inverse, convolution property has been proved.

In section 4, The 1 -- D conformable fractional Elzaki Transformations has been defined with convolution property and some examples are given along with its rela-
tionship with Conformable Laplace transform has also been shown. In section 5, 2 — D Conformable fractional Elzaki Transformations has been defined with convolution and other property has been proved. In section 6, Analytical solution of the fractional order differential equations with Non – Singular kernel has been obtained by using conformable fractional Elzaki Transformations and also results are compared for various fractional order. Some concluding remarks are given in section 7.

## 2. Preliminaries

In the following, we present some basic definitions needed in proving the main results.

### ABR fractional Derivative:

Atangana - Baleau Riemann fractional derivative of a function \( f \in H^1(a,b) , b > a, \alpha \in [0,1] \) which is of exponential order then the new ABR fractional derivative [2] of \( f(t) \) is defined as,

\[
^{ABR} D_t^{\alpha} (f(t)) = \frac{B(\alpha)}{1-\alpha} \int_{t}^{1} f(x) E_{\alpha} \left(-\alpha \frac{(t-x)^\alpha}{1-\alpha} \right) dx
\]

(2.1)

with \( b > a, \alpha \in [0,1] \) and \( B(\alpha) \) is normalization function obeying \( B(0) = B(1) = 1 \)

### ABC fractional Derivative:

Atangana – Baleau Caputo fractional derivative of a function \( f \in H^1(a,b) , b > a, \alpha \in [0,1] \) which is of exponential order then the new ABC fractional derivative [2] of \( f(t) \) is defined as,

\[
^{ABC} D_t^{\alpha} (f(t)) = \frac{B(\alpha)}{1-\alpha} \int_{t}^{1} f'(x) E_{\alpha} \left(-\alpha \frac{(t-x)^\alpha}{1-\alpha} \right) dx
\]

(2.2)

with \( b > a, \alpha \in [0,1] \) and \( B(\alpha) \) is normalization function obeying \( B(0) = B(1) = 1 \)

### Conformable fractional Derivative

If \( f : (0, \infty) \rightarrow \mathbb{R} \) is a real valued function then its conformable fractional derivative of \( f \) of order \( \alpha \in (0,1] \) at \( t = 0 \) is defined as,

\[
T_{\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon t^{1-\alpha})-f(t)}{\varepsilon}
\]

(2.3)

where at \( \alpha = 0 \) it is defined to be \( T_{\alpha} f(0) = \lim_{\varepsilon \to 0} T_{\alpha} f(t) \)

The simplest relationship between usual and conformable fractional derivative is represented as:

\[
T_{\alpha} f(t) = t^{1-\alpha} f'(t)
\]

(2.4)

where \( f \in C^1, \forall t > 0 \) which is not true at \( \alpha = 0 \) [11] as it does not become identity function.

### Conformable fractional Exponential Derivative

The conformable fractional exponential function [4] is defined as \( E_{\alpha}(c,t) = e^{t^\alpha c} \) where \( 0 < \alpha \leq 1 \) and \( c \in \mathbb{R} \).

### Fractional Laplace Transformation

Let \( 0 < \alpha \leq 1 \) and \( f : [0, \infty) \rightarrow \mathbb{R} \) then the fractional Laplace transform of order \( \alpha \) of \( f(t) \) is defined as [26]:

\[
L_{\alpha}[f(t)](p) = \int_{0}^{\infty} E_{\alpha}(-p,t) f(t)dt
\]

(2.4)

with the property that \( L_{\alpha}[D_t^{\alpha} f(t)] = L_{\alpha}[f(t)] - f(0) \) with \( \text{Re.}(p) > 0 \).

### Definition 2.1. Elzaki – Tarig Transformation

Consider the set \( S \) defined as follows:

\[
f(x) \in S = \{ f(x) : \exists k_1, k_2 > 0, |f(x)| < M e^{k_2 x}, x \in (-1)^j X [0, \infty), M > 0 \}
\]

(2.5)

then for the given function which satisfies the condition of the set \( S \), the Elzaki – Tarig transformation of \( f(t) \) is defined as [10],[14].

\[
\mathcal{E}_x \{ f(x) ; p \} = \int_{0}^{\infty} e^{px} f(t) dt
\]

(2.6)

with \( p \neq 0 \).

### Definition 2.2. Tarig Transformation

Given function which satisfies the condition of the set \( S \), then Tarig transformation of \( f(t) \) is defined as [10],[14].

\[
\mathcal{E}_x \{ f(x) ; p \} = \int_{0}^{\infty} \frac{1}{p} e^{px} f(t) dt
\]

(2.7)

with \( p \neq 0 \).

### Definition 2.3. Derivative of Convolution

Given any real valued integrable function \( f \) and \( g \), then the convolution of \( f \) with \( g \) is denoted by \( f* g \) and is given by \( (f* g)(t) = \int_{0}^{t} f(x-t) g(x) dx \) which have the property related to the derivative given by [27].

\[
\frac{d}{dt} [f* g] = \frac{df}{dt} * g(t) = f(t) * \frac{dg}{dt}
\]

(2.8)

### 3. Generalized one dimensional Elzaki – Tarig transformation

We consider the definition of Generalized Elzaki – Tarig Transformation by using the definition [18].

\[
\mathcal{E}_x \{ f(x) ; p \} = \int_{0}^{\infty} \Phi \left( \frac{1}{p} \right) \Phi_1 (p) e^{x} f(x) dx, p \neq 0,
\]

(3.1)
where
\[
f(x) \in S = \{f(x) : \exists k_1, k_2 > 0, |f(x)| < Me^{k_1}, x \in (-1)^j X [0, \infty), M > 0\}
\]
and \(\Phi\left(\frac{1}{p}\right)\), \(\Phi_1 (p)\) are invertible functions of \(p\) with \(\varepsilon(x) = f e^{-a(x)} dx\) an exponential function and \(a(x)\) as invertible function, thus from the definitions above it can be seen that it is the generalization of Elzaki – Tarig transformation.

**Inversion formula**

The definition of generalized Elzaki – Tarig transformation tells us that
\[
F(p) = \mathcal{Z}_E\{f(x)\}; p = \int_0^{\infty} \Phi \left(\frac{1}{p}\right) \Phi_1 (p) e^{-\Phi(p)\varepsilon(x)} f(x) dx
\]
p\(\neq 0\) which satisfies the given conditions in (3.1) then the inverse transformation to be defined as,
\[
\mathcal{Z}_E^{-1}(F(p)) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(\Phi^{-1}(p)\Phi_1^{-1}(p)) e^{\Phi(p)\varepsilon(x)} dp
\]

Proof: By definition of generalized Elzaki – Tarig transformation (3.1) with defining \(\Phi \left(\frac{1}{p}\right) = \frac{1}{p}\) and \(\Phi_1 (p) = r\).
\[
\Rightarrow F(\Phi^{-1}(\frac{1}{p})\Phi_1^{-1}(r)) = \int_0^{\infty} e^{\varepsilon(x)} e^{-r\varepsilon(x)} f(x) dx
\]

Put \(\varepsilon(x) = t\) in the above equation, we get
\[
F(\Phi^{-1}(\frac{1}{p})\Phi_1^{-1}(r)) = \int_0^{\infty} e^{-t\varepsilon(x)} f(x) dt = \mathcal{Z}_E\{f(-\varepsilon(x))\}; r
\]

Whenever, \(\Phi \left(\frac{1}{p}\right)\), \(\Phi_1 (p)\) are inverses of each other so that by complex inversion formula for the Laplace transform with \(-\varepsilon(x) = x\) and \(r = p\)

\[
\Rightarrow f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(\Phi^{-1}(p)\Phi_1^{-1}(p)) e^{\Phi(p)\varepsilon(x)} dp
\]

**Generalized Elzaki – Tarig transform of derivative**

Let \(f(x)\) and \(\varepsilon(x)\) satisfies all the conditions in equation (3.1) with \(f'(x)\) has piece wise continuous derivative then
\[
\mathcal{Z}_E\{f'(x)\}; p = \Phi(p)[\mathcal{Z}_E\{\varepsilon(x) f(x)\}; p] - \Phi \left(\frac{1}{p}\right) \Phi_1 (p) f(0^+)
\]

**Proof:** By definition of generalized Elzaki – Tarig transformation (3.1), we have
\[
\mathcal{Z}_E\{f'(x)\}; p = \int_0^{\infty} \Phi \left(\frac{1}{p}\right) \Phi_1 (p) e^{\varepsilon(x)} e^{-\Phi(p)\varepsilon(x)} f'(x) dx
\]
\[
= \Phi \left(\frac{1}{p}\right) \Phi_1 (p) \int_0^{\infty} e^{\varepsilon(x)} e^{-\Phi(p)\varepsilon(x)} f'(x) dx
\]

\(p \neq 0\), applying integration by part to the above equation;
\[
\Rightarrow \mathcal{Z}_E f'(x); p = \Phi \left(\frac{1}{p}\right) \Phi_1 (p) \\Phi \left(\frac{1}{p}\right) e^{-\Phi(p)\varepsilon(x)} f(x) dx
\]
\[
\Phi \left(\frac{1}{p}\right) \Phi_1 (p) \\Phi \left(\frac{1}{p}\right) \\Phi_1 (p) \{\varepsilon(x) f(x) ; p\} - \Phi \left(\frac{1}{p}\right) \Phi_1 (p) e(0^+) f(0^+)
\]

As \(f(x)\) and \(\varepsilon(x)\) are of exponential order.

**Convolution Theorem**

If \(F(p)\) and \(G(p)\) are the generalized Elzaki – Tarig transformations of two functions \(f(x)\) and \(g(x)\) respectively then the generalized convolution theorem for definition is calculated as follows,
\[
F(p)G(p) = \mathcal{Z}_E \{f \ast g; p\} = \int_0^{\infty} \Phi \left(\frac{1}{p}\right) \Phi_1 (p) e^{\varepsilon(x)} e^{-\Phi(p)\varepsilon(x)} f(x) g(t) dt
\]
\[
= \int_0^{\infty} \Phi \left(\frac{1}{p}\right) \Phi_1 (p) \varepsilon(x) e^{-\Phi(p)\varepsilon(x)} f(x) g(t) dt
\]

Substitute \(\varepsilon(x) = t\) and changing the order of integration [18] in the double integral we get,
\[
F(p)G(p) = \int_0^{\infty} \Phi \left(\frac{1}{p}\right) \Phi_1 (p) e^{\varepsilon(x)} e^{-\Phi(p)\varepsilon(x)} f(x) g(t) dt
\]

\(= \int_0^{\infty} \Phi \left(\frac{1}{p}\right) \Phi_1 (p) e^{\varepsilon(x)} e^{-\Phi(p)\varepsilon(x)} f(x) g(t) dt
\]

\[= \mathcal{Z}_E \{f(x) \ast g(x); p\} = \mathcal{Z}_E \{f(x) \ast g(x); p\}
\]

### 4.1D Conformable fractional Elzaki Transformation

Let \(0 < \alpha \leq 1\) and \(f : [0, \infty) \rightarrow \mathbb{R}\) satisfying the conditions of (3.1) then the conformable fractional Elzaki transform of order \(\alpha\) of \(f\) is defined as;
\[
E^\alpha_{\mathcal{Z}_E} [f(x)](p) = \int_0^{\infty} pK_{\alpha}(-p,t)f(t) dt d\alpha t
\]
(4.1)

where, \(K_{\alpha}(-p,t) = E_{\mathcal{Z}_E} \{f(x)\}; p \neq 0 \Rightarrow K_{\alpha}(-p,t) = e^{-\frac{\alpha}{p^\alpha}}

**Properties of Conformable fractional Elzaki Transform**

4.1.1) The relationship between fractional Laplace Transformation and Conformable fractional Elzaki Transformation is
\[
E^\alpha_{\mathcal{Z}_E} [f(t)](p) = pE_\alpha \{f(t)\} \left(\frac{1}{p}\right), p > 0, 0 < \alpha \leq 1
\]

**Proof:** By using the definition(2.4),(4.1) and the relation
where, we get, 

Proof: By using the definition (A) and (3.1), one can have

\[ E\alpha[D^\alpha f(t)](p) = \int_0^\infty e^{-pt} f(t) d\alpha t \]

By substituting, \( \frac{\alpha}{\alpha} = s \) we get,

\[ E\alpha[D^\alpha f(t)](p) = \int_0^\infty e^{-\frac{p}{\alpha} t} f((\alpha s)^\frac{1}{\alpha}) d\alpha s \]

Now by using integration by parts and using the property of \( f(t) \) in (3.1), we get

\[ E\alpha[D^\alpha f(t)](p) = \frac{\alpha}{\alpha} E\alpha[f(t)](p) - pf(0) \]

4.1.2) Conformable fractional ELZAKI Transformation of convolution of order- \( \alpha \):

Let \( f(t) \) and \( g(t) \) satisfies the conditions in (3.1) and \( f, g: [0, \infty) \rightarrow \mathbb{R} \) real valued functions with \( 0 < \alpha \leq 1 \).

\[ E\alpha[f(t)], E\alpha[g(t)] \]

are the conformable fractional Elzaki transformation of \( f \ast g \) is given by,

\[ E\alpha[f \ast g](t) = \frac{\alpha}{\alpha} E\alpha[f(t)] E\alpha[g(t)] \]

where, \( (f \ast g)(t) = \int_0^t f(r^\alpha - s^\alpha) g(s) ds \)

Proof: Consider, \( (f \ast g)(t) = \int_0^t f(r^\alpha - s^\alpha) g(s) ds \)

by applying conformable fractional Elzaki transformation on both sides of the above equation,

\[ E\alpha[(f \ast g)(t)](p) = \frac{\alpha}{\alpha} E\alpha[f(t)] E\alpha[g(t)] \]

4.1.3) Conformable fractional ELZAKI Transformation of derivative of convolution

From the definition (6) we have, \( \frac{d}{dt} [f \ast g] = \frac{d}{dt} f \ast g = f \ast \frac{d}{dt} g \)

now by applying conformable fractional Elzaki transformation on both sides and using (4.1.3) we get,

\[ E\alpha[(f \ast g)(t)](p) = \frac{\alpha}{\alpha} E\alpha[f(t)] E\alpha[ \frac{d}{dt} g(t)] \]

4.1.4) Conformable fractional ELZAKI Transformation of derivative of convolution

By using the definition (4.1) and the conditions of (3.1) the \( 1 \rightarrow D \) conformable fractional Elzaki transformation \( E\alpha[f(t)](p) \) will exist and finite provided the following conditions were satisfied:

(I) \( \lim_{t \to \infty} |f(t)| \) exist and finite

(II)Re. \( (p) > 0 \) by using the relation between Laplace and Elzaki Transformation While the inverse existence were by using [30–32].

5. 2-D Conformable fractional Elzaki Transformation

The definition of 1 --- dimensional conformable fractional Elzaki transformation can be extend to 2-- dimensional as follows;

Definition 5.1) 2-D conformable fractional Elzaki Transformation

Given function, \( f(x, t) : \mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R} \) be real valued function satisfying the condition in (3.1) independently then the 2-dimensional conformable fractional Elzaki transformation at a point \( (p, q) \) can be defined as:

\[ E\alpha_2[f(x, t)](p, q) = pq \int_0^\infty \int_0^\infty e^{-\frac{\alpha}{\alpha} x^\alpha \sqrt{\alpha} \alpha y^\alpha \sqrt{\alpha} \alpha} f(x, t) dx dt \]

(5.1)

Definition 5.2) Convolution of order- \( \alpha \) in two dimension

Let \( F_1(x, y) \) and \( F_2(x, y) \) be integral functions then the convolution of \( F_1(x, y) \) and \( F_2(x, y) \) of order - \( \alpha \) is defined as:

\[ [(F_1 \ast F_2)_\alpha](x, y) \]

\[ \int_0^x \int_0^y F_1(x^\alpha - \xi^\alpha, y^\alpha - \eta^\alpha) F_2(\xi, \eta) d\alpha \eta d\alpha \eta, \]

(5.2)

where, \( \ast \alpha \) denotes the 2-dimensional convolution of order \( \alpha \) with respect to \( x \) and \( y \).

Theorem: Let \( f(x, t) \) and \( g(x, t) \) have 2-dimensional conformable fractional Elzaki transformation satisfying all the condition in (3.1) in the variable \( x \) and \( t \) independently. Then,

\[ E\alpha_2[(f \ast g)_\alpha](x, t)](p, q) = \frac{\alpha}{\alpha} E\alpha_2[f(x, t)] E\alpha_2[g(x, t)] \]

Proof: By using the definition of 2-dimensional conformable fractional Elzaki transformation and 2-dimensional convolution of order - \( \alpha \), we have,

\[ E\alpha_2[(f \ast g)_\alpha](x, t)](p, q) \]

\[ = pq \int_0^\infty \int_0^\infty e^{-\frac{\alpha}{\alpha} x^\alpha \sqrt{\alpha} \alpha y^\alpha \sqrt{\alpha} \alpha} [f \ast g]_\alpha(x, t) dx dt \]

\[ = pq \int_0^\infty \int_0^\infty e^{-\frac{\alpha}{\alpha} x^\alpha \sqrt{\alpha} \alpha y^\alpha \sqrt{\alpha} \alpha} \int_0^x \int_0^y F_1(x^\alpha - \xi^\alpha, y^\alpha - \eta^\alpha) d\alpha \eta d\alpha \eta \]

\[ \times d\alpha \eta d\alpha \eta \quad \text{from (5.2)} \]

Define \( \beta^\alpha = x^\alpha - \xi^\alpha \) and \( \gamma^\alpha = t^\alpha - \eta^\alpha \) and adjusting the terms; we get,
Example 1) Consider the conformable fractional differential equation, $D^\alpha_t z(t) = E_a(-1,t) - z(t)$ with initial condition $z(0) = 1, 0 < \alpha \leq 1$

**Solution:** The actual solution by usual method is given by $z(t) = e^{-t} (1 + t)$ for $\alpha = 1$

Now consider, $D^\alpha_t z(t) = E_a(-1,t) - z(t), z(0) = 1$ for $0 < \alpha \leq 1$

Applying conformable fractional Elzaki Transformation on both sides of the equation and by using (4.1.2) we get,

$$E^\alpha_a [(f*g)_a(t)] = \frac{1}{p^\alpha} \mathcal{Z}_a(p) - Z(0) - Z_a(p)$$

$$\Rightarrow \frac{1}{p^\alpha} \mathcal{Z}_a(p) - \frac{1}{p^\alpha} = E^\alpha_a [f(t)]$$

$$\Rightarrow \mathcal{Z}_a(p) = p^{\alpha} \mathcal{Z}_a(p) + Z(0)$$

$$\Rightarrow \mathcal{Z}_a(p) = p^{\alpha} Z(0)$$

By using the relation $E^\alpha_a [f(t)](p) = p Z_a(f(t))(1/p^\alpha)$ we get, $Z_a(p) = p^{\alpha} Z(0) + p^{\alpha} (1/p^\alpha) + p^{\alpha} (1/p^\alpha)$

By applying Inverse conformable Elzaki Transform and convolution property on both sides of the above equation we get the solution as,

$$z(t) = (1 - \alpha) e^{-t} \alpha + \alpha e^{-t} \alpha, \quad 0 < \alpha \leq 1$$

Hence the solution is given by, $z(t) = e^{-t} \alpha [\alpha + 1], 0 < \alpha \leq 1$

for $\alpha = 1$, we get the actual solution $z(t) = e^{-t} (1 + t)$

**Graphically the solution for various value of $\alpha$ is given by,**

![Graph](image.png)

**Example 2) Consider the ABF fractional differential equation**

$$D^\alpha_t z(t) = E_a(-1,t) - z(t)$$

with initial condition $z(0) = 0, 0 < \alpha \leq 1$

**Solution:** Given the ABR fractional differential equation $D^\alpha_t z(t) = E_a(-1,t) - z(t)$ we get $E^\alpha_a [D^\alpha_t z(t)] = E^\alpha_a [E_a(-1,t) - z(t)]$

By using the relation (15), we have,

$$\mathcal{Z}_a(p) = \frac{1}{p^{\alpha}} Z_a(p) + p^{\alpha} Z_a(p) = p^{\alpha} Z_a(p) + p^{\alpha} Z_a(p)$$

Put $B(\alpha) = 1$ and initial condition $Z(0) = 0$

$$\Rightarrow \mathcal{Z}_a(p) = \frac{1}{p^{\alpha}} Z_a(p) + p^{\alpha} Z_a(p) = p^{\alpha} Z_a(p) + p^{\alpha} Z_a(p)$$

**Graphically the solution for various value of $\alpha$ is given by,**

![Graph](image.png)
Example 3) Consider the conformable fractional differential equation given by,
\[ D_1^{\alpha} z(t) - z(t) = 0 \]
with initial condition \( z(0) = 0 \) , \( 0 < \alpha \leq 1 \)

**Solution:** Given conformable fractional differential equation
\[ D_1^{\alpha} z(t) - z(t) = 0 \]
Applying conformable fractional Elzaki Transform on both sides, we get
\[
\frac{1}{p} Z_{\alpha}(p) - pZ(0) - Z_{\alpha}(p) = 0
\]
\[ \Rightarrow \frac{1}{p} Z_{\alpha}(p) - Z_{\alpha}(p) = 0 \]
\[ \Rightarrow Z_{\alpha}(p) = (1 + \frac{1}{p}) \]
Applying inverse conformable fractional Elzaki Transformation on both sides we get
\[ z(t) = e^{\frac{t}{\alpha}} \] which is the required solution.

7. Conclusion

We have define the new 1-dimensional conformable fractional Elzaki Transformation with its properties including to conformable fractional derivative. Also we have, extends its extension to 2-dimensional conformable fractional Elzaki Transformation along with its applications to convolution in two dimension of order \( \alpha \).

As an application, we have solve conformable fractional differential equation and fractional differential equation with Non-Singular Kernel. The solution takes the actual value at \( \alpha = 1 \), which tell us that the conformable fractional Elzaki Transformation is a new tool to solve fractional differential equations. The relationship of conformable fractional derivative with conformal fractional Laplace Transformation also been found.

References


