On controllability of mixed iterative integrodifferential equations of fractional order

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Abstract
The purpose of the present paper is to introduce the sufficient conditions for the controllability of iterative mixed integrodifferential equations of fractional orders lying between $(0, 1]$ and $(1, 2]$ by applying Banach contraction principle.

Keywords
Controllability, iterative fractional integrodifferential equation, Banach contraction principle, existence of solution.

AMS Subject Classification
93B05, 34A08, 34A40, 34B05, 34B15

1 Introduction
The study of iterative differential and integrodifferential equations are also linked to the wide applications of calculus, which boost mathematical sciences altogether. The importance of these equations is felt when one studies the infection models. It relates to the investigation of the motion of charged particles with retarded interaction see[5, 32]. The work of E. Eder [10] in the year 1984, helped in the expansion of the theory of iterative differential equations. He investigated a solution of the functional differential equation $x'(t) = x(x(t))$ is a function $x : A \rightarrow \mathbb{R}$ from an interval $A \in \mathbb{R}$ (i.e., a connected subset of $\mathbb{R}$ ) into $\mathbb{R}$ such that

$$x'(t) = x(x(t)), \quad x(t_0) = x_0, \quad \forall \ t_0, x_0 \in A$$

and proved the existence, uniqueness and analytic dependence of solutions on initial data.

Qualitative behaviors like the observability, controllability, stability, stabilizability of fractional dynamical systems are the recent trends tackled by researchers. Mainly, the controllability of dynamical systems is broadly applied in the analysis and design of control systems. Any control system is controllable if every state corresponding to this process is affected or controlled in respective time by some control signals. As in the case of first and second-order systems, the investigation of various types of controllability is significant for fractional-order systems. For instance, any system should be controllable to find out optimal control of the system. Some outcomes on controllability for fractional order systems can be observed in ([2]-[4], [6]-[9], [11]-[13], [23]-[29], [31], [34], [35]). In these papers, the techniques used by the authors is to convert the controllability problem into a fixed point problem with the assumption that the controllability operator has an induced inverse on a quotient space.

The investigation of fractional differential and integral equations is associated with the broad applications of fractional calculus in physics, mechanics, signal processing, electromagnetics, biology, economics, and other fields. The study of the theory of fractional differential and integral equations is a recent trend. One can refer to the monographs of Kilbas et.al. [22], Bajlekova [1], Hale [16]. Fractional Cauchy problems are useful in physics.

Many recent papers have handled the existence, uniqueness and other properties of solutions of unique forms of the equations (2.4) - (2.5) and (3.1) - (3.3), see ([14], [17], [19]-
Definition 1. The fractional integral of order \( \gamma > 0 \) of a function \( f(t) \in L^1(J,R^+) \) is defined as
\[
I_0^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_{0}^{t} \frac{f(s)}{(t-s)^{1-\gamma}} ds. \tag{2.1}
\]

Definition 2. The Riemann Liouville fractional derivative of order \( \gamma > 0 \), \( R(\gamma) > 0 \), \( n-1 < \gamma \leq n, n \in N \), is defined by
\[
D_0^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \left( \frac{d^n}{dt^n} \right) \int_{0}^{t} \frac{f(s)}{(t-s)^{n-\gamma+1}} ds. \tag{2.2}
\]

Here the function \( f(t) \) has absolutely continuous derivatives up to order \( n-1 \).

Definition 3. The Caputo fractional derivative of order \( \gamma > 0 \), \( R(\gamma) > 0 \), \( n-1 < \gamma \leq n, n \in N \), is defined by
\[
^C D_0^\gamma f(t) = I_0^{n-\gamma} D^n f(t) = \frac{1}{\Gamma(n-\gamma)} \left( \frac{d^n}{dt^n} \right) \int_{0}^{t} \frac{f^{(n)}(s)}{(t-s)^{n-\gamma+1}} ds, \tag{2.3}
\]
where the function \( f(t) \) has absolutely continuous derivatives up to order \( n-1 \).

Let \( X_T = C(J,X) \), the set of all continuous functions from \( J = [0,T] \) into \( X \) is a Banach space endowed with the norm
\[
\|x\|_T = \sup_{t \in J} \{ \|x(t)\|_X \}, \ x \in X_T.
\]

In the present section, we reflect on the class of iterative mixed integrodifferential equations of fractional order \( 0 < q \leq 1 \) of the type
\[
^C D_T^q x(t) = Ax(t) + \int_{0}^{t} k_2(t,s)x(x(s)) ds + \int_{0}^{T} h_2(t,s)x(x(s)) ds + Bv(t), \tag{2.4}
\]
\[
x(0) = x_0, \tag{2.5}
\]
where \( A \) is the infinitesimal generator of a strongly continuous semi-group \( \{T_t \ | \ t \geq 0\} \) in a Banach space \( X \). For every \( x_0 \), there exists \( K > 0 \) such that \( \sup_{t \in J} \|T_t\| \leq K \), and the state \( x(t) \) takes values in \( X \). The control function \( v(.) \) is given in \( L^2(J,V) \), a Banach space of admissible control functions and the map \( B : V \to X \) is a bounded linear operator; \( k_2(t,s) \) and \( h_2(t,s) \) are given continuous functions.

Definition 4. A continuous function \( x \in C(J,X) \) is said to be a mild solution of the problem \( (2.4) \) - \( (2.5) \) if \( x \) is the solution of the following integral equation
\[
x(t) = T(t)x_0 + \int_{0}^{t} (t-s)^{q-1} S(t-s) \left[ \int_{0}^{s} k_2(s,\xi)x(x(\xi)) d\xi + \int_{0}^{T} h_2(s,\xi)x(x(\xi)) d\xi + Bv(\xi) \right] ds,
\]
where
\[
T(t) = \int_{0}^{\infty} \phi_\theta(t)T_1(t\theta^q)x d\theta,
\]
\[
S(t) = q \int_{0}^{\infty} \theta \phi_\theta(t)T_1(t\theta^q)x d\theta,
\]
where \( \phi_\theta(t) = \frac{1}{\theta} \theta^{1-\frac{1}{q}} \psi_{\theta^{-\frac{1}{q}}} \) satisfies the conditions of probability density function defined on \( (0,\infty) \), that is, \( \phi_\theta(t) \geq 0 \) and \( \int_{0}^{\infty} \phi_\theta(t) d\theta = 1 \). Also the term \( \psi_\theta(\theta) \) is defined as
\[
\psi_\theta(\theta) = \frac{1}{\theta} \sum_{n=1}^{\infty} (-1)^{n-1} \theta^{-nq-1} \frac{\Gamma(nq+1)}{n!} \sin(n\pi q), \theta \in (0,\infty).
\]

Lemma 2.1. [33] The operator \( T(t) \) and \( S(t) \) have the following properties:
(i) \( \|T(t)x\| \leq K \|x\| \) and \( \|S(t)x\| \leq \frac{K}{q} \|x\| \), where \( \|S(t)\| \leq K \) for all \( t \in J \).
(ii) The operator \( T,S : [0,\infty) \to L(x) \) are continuous, where \( L(x) \) is set of all linear operators from \( X \) to \( X \).
(iii) The operator \( T(t) \) and \( S(t) \) are strongly continuous on \( X \) for \( t > 0 \).

Definition 5. The system \( (2.4) - (2.5) \) is said to be exact controllable on \( J \), if for every \( x_0, x_1 \in X \), there exists a control \( v(.) \in L^2(J,V) \) such that the mild solution \( x(.) \) of \( (2.4) - (2.5) \) together with initial state \( x(0) = x_0 \), also satisfies \( x(T) = x_1 \).
For the convenience, we list the following hypotheses which are used in further discussion.

\((G_1)\) A generates a strongly continuous semigroup \(\{T_{t}(t)\}_{t \geq 0}\) in the Banach space \(X\).

\((G_2)\) There exists constants \(k_T\) and \(h_T\) such that
\[ k_T = \sup \{k_2(s, \xi) : 0 \leq \xi \leq s \leq T\} \quad \text{and} \quad h_T = \sup \{h_2(s, \xi) : (\xi, \xi) \in J \times J\}. \]

\((G_3)\) There exists a constant \(\mu > 0\) such that \(\|x(t_1) - x(t_2)\| \leq \mu |t_1 - t_2|\) for \(x \in C(J, X), t_1, t_2 \in J, t_1 \leq t_2\).

\((G_4)\) The linear operator \(M: L_2(J, V) \rightarrow X\) defined by
\[ MV = \int_0^T S(T-s) B v(s) d s \]
has an invertible operator \(M^{-1}\) (induced by \(M\)) defined on \(L_2(J, V)/\text{Ker} M\).

\((G_5)\) There exists positive constants \(n_1, n_2 > 0\) such that \(\|M^{-1}\| \leq n_1, \|B\| \leq n_2\).

**Theorem 2.2.** If hypotheses \((G_1) - (G_5)\) are satisfied and
\[ \frac{K T^{q+1}}{\Gamma(q+1)} (k_T + h_T) (\mu + 1) \left[ 1 + \frac{K n_2 T}{\Gamma(q)} \right] \leq 1. \]
Then the system \((2.4) - (2.5)\) is controllable on \(J\).

**Proof.** Let \(C_{\mu} = \{x \in C(J, X) : \|x(t_1) - x(t_2)\| \leq \mu |t_1 - t_2|\}, \) for all \(t_1, t_2 \in J\) and \(C_{\mu, \omega} = \{x \in C_{\mu} : \|x(t)\| \leq \omega\}\) where
\[ K \|x_0\| + \frac{K n_2 T}{\Gamma(q)} (\|x_1\| + \|x_0\|) + \frac{K T^{q+1} (k_T + h_T)}{\Gamma(q+1)} \left[ 1 + \frac{K n_2 T}{\Gamma(q)} \right] \leq \omega. \]

For an arbitrary \(x \in C_{\mu, \omega}\), we define the control
\[ v(t) = (T - t)^{1-q} M^{-1} \left[ x_1 - T(T)x_0 - \int_0^T (T-s)^{q-1} S(T-s) \left( \int_0^s k_2(s, \xi) x(\xi) d \xi \right) d s + \int_0^T h_2(s, \xi) x(\xi) d \xi \right] d s. \]
Applying this control, the nonlinear operator \(Q: X \rightarrow X\) defined by
\[ (Qx)(t) = T(t)x_0 + \int_0^t (t-s)^{q-1} S(t-s) \left( \int_0^s k_2(s, \xi) x(\xi) d \xi \right) d s + \int_0^T h_2(s, \xi) x(\xi) d \xi + Bv(s) d s \]
has a fixed point. This fixed point is then a solution of \((2.4) - (2.5)\), putting the control \(v(t)\) in the above equation, we get
\[ (Qx)(t) = T(t)x_0 + \int_0^t (t-s)^{q-1} S(t-s) \left( \int_0^s k_2(s, \xi) x(\xi) d \xi \right) d s + \int_0^T h_2(s, \xi) x(\xi) d \xi + \frac{BM^{-1}}{\Gamma(q)} \left[ x_1 - T(T)x_0 - \int_0^T (T-s)^{q-1} S(T-s) \right] \times \int_0^T (t-s)^{q-1} S(t-s) \left( \int_0^s k_2(s, \xi) x(\xi) d \xi \right) d s \]
Clearly, \((Qx)(T) = x_1\), which states that the control \(v(t)\) steers the system from the initial state \(x_0\) to \(x_1\) in time \(T\) provided we obtain a fixed point of the nonlinear operator \(Q\).

We prove that \(C_{\mu, \omega}\) is an invariant set with respect to \(Q\). That is, we have \(Q(C_{\mu, \omega}) \subset C_{\mu, \omega}\). If conditions \((G_1) - (G_4)\) holds, then for any \(x \in C_{\mu, \omega}\) and \(t \in J\), we have
\[ \|Qx(t)\| \leq \|T(t)x_0\| + \int_0^t (t-s)^{q-1} S(t-s) \left( \int_0^s k_2(s, \xi) x(\xi) d \xi \right) d s + \int_0^t h_2(s, \xi) x(\xi) d \xi + \frac{BM^{-1}}{\Gamma(q)} \left[ x_1 - T(T)x_0 - \int_0^T (T-s)^{q-1} S(T-s) \right] \times \int_0^T (t-s)^{q-1} S(t-s) \left( \int_0^s k_2(s, \xi) x(\xi) d \xi \right) d s \]
\[ + \left[ \int_0^t (t-s)^{q-1} (T-s)^{q-1} S(t-s) \right] \times \int_0^T (T-s)^{q-1} S(T-s) \left( \int_0^s k_2(s, \xi) x(\xi) d \xi \right) d s \]
\[ + \left\| \int_0^t (t-s)^{q-1} (T-s)^{q-1} S(t-s) \right\| \|B\| \|M^{-1}\| \left( \|x_1\| + \|T(t)x_0\| + \int_0^T (T-s)^{q-1} \|S(T-s)\| \right) \]
\[ \left\{ \int_0^t (k_2(s, \xi) \|x(\xi)\| d \xi + \int_0^T (h_2(s, \xi) \|x(\xi)\| d \xi) d s \right\} \right[ \|x_1\| + K \|x_0\| \]
\[ + \frac{K T^{q+1}}{\Gamma(q)} \int_0^T (T-s)^{q-1} (k_T T^3 + h_T T^3) d s \]
Then $Qx \in C_{\mu, \omega}$ for any $x \in C_{\mu, \omega}$. Therefore $Q \colon C_{\mu, \omega} \to C_{\mu, \omega}$ is a contraction map. Let $x, y \in C_{\mu, \omega}$. Then

$$
\|Qx(t) - (Qy)(t)\| \\
\leq \left(\int_0^t (t-s)^{q-1}S(t-s) \left( \left( \int_0^s k(s, \tau)x(x(\tau))d\tau \right)^2 + \int_0^s h(s, \tau)x(x(\tau))d\tau \right) \right) \left( \int_0^s k(s, \tau)y(y(\tau))d\tau \right) \right) + \left( \int_0^s h(s, \tau)y(y(\tau))d\tau \right)ds + \left( \int_0^s (t-s)^{q-1} \right) \left( \int_0^T (T-s)^{q-1}S(T-s) \left( \left( \int_0^T k(s, \tau)x(x(\tau))d\tau \right)^2 + \int_0^T h(s, \tau)x(x(\tau))d\tau \right) \right) \\
- \left( \int_0^s k(s, \tau)y(y(\tau))d\tau \right) + \left( \int_0^s h(s, \tau)y(y(\tau))d\tau \right)ds \right) T \left( T - s \right)^{q-1} \left( T - s \right)^{q-1} ||S(t - \xi)|| \right) ||B|| \left( ||M^{-1}|| T \right)
$$
The unique fixed point of $A$ exists a unique fixed point $x$ where $\frac{\partial^q}{\partial t^q} \left(\int_0^t (t-s)^{q-1} ||S(t-s)|| \left[ kT \int_0^T (\mu + 1) ||x(\tau) - y(\tau)|| d\tau + hT \int_0^T (\mu + 1) ||x(\tau) - y(\tau)|| d\tau \right] ds + \int_0^T (t-\xi)^{q-1} (T-\xi)^{1-q} ||S(t-\xi)|| ||B|| ||M^{-1}|| \int_0^T (T-s)^{-q} ||S(T-s)|| \left[ kT \int_0^T (\mu + 1) ||x(\tau) - y(\tau)|| d\tau + hT \int_0^T (\mu + 1) ||x(\tau) - y(\tau)|| d\tau \right] dsd\xi \right) \leq K \frac{\Gamma(q)}{1 + \Gamma(q)} \left[ T(kT + hT)(\mu + 1) ||x - y|| \int_0^T (T-s)^{-q} ||x - y|| ds \right] d\xi \leq \frac{KT^{q+1}}{\Gamma(q)} \left[ (kT + hT)(\mu + 1) ||x - y|| \frac{K^2n_1n_2T^{q+2}}{\Gamma(q)(\Gamma(q+1))} \right] \left[ (kT + hT)(\mu + 1) ||x - y|| \right] \leq \frac{KT^{q+1}}{\Gamma(q+1)} (kT + hT)(\mu + 1) \left[ 1 + Kn_1n_2T^{-q} \right] ||x - y||.

We find that $Q$ is a contraction mapping on $C_{\mu,\omega}$. Hence there exists a unique fixed point $x \in C_{\mu,\omega}$ such that $(Qx)(t) = x(t)$. The unique fixed point of $Q$ is a mild solution of (2.4) - (2.5) on $J$, which satisfies $x(T) = x_1$. Hence the system is controllable on $J$.

**Example 2.3.** Consider the following nonlinear mixed iterative fractional integrodifferential system of order $0 < q < 1$

\[
\frac{\partial^q}{\partial t^q} x(t, w) = \frac{\partial^2 x}{\partial t^2}(t, w) + \chi(t, w) + \int_0^T \frac{1}{(1+t^2)(1+s)} x(s, w) ds + \int_0^T \frac{1}{(1+t^2)(1+s)} x(s, w) ds \quad (2.6)
\]

where $\chi : J \times (0, \frac{T}{2}) \to J$.

Let $X = L^2 \left(0, \frac{T}{2}\right)$ and $A : D(A) \subset X \to X$ be defined by $A = \frac{d^2}{dw^2}$ with $D(A)$ consisting of all $x \in X$ with $\frac{d^2 x}{dw^2}$ and $x(t,0) = x(t, \frac{\pi}{2}) = 0$, $t \in J$, $w \in \left(0, \frac{\pi}{2}\right)$, $x(0) = x \left(\frac{T}{2}\right)$. Let $e_n$ be the eigenfunction corresponding to the eigenvalue $\lambda_n = -n^2$ of the operator $A$. Then the $C_0-$semigroup $T(t)$ generated by $A$ has $\exp(\lambda_n t)$ as the eigenvalues and $e_n$ as the corresponding eigenfunctions.

Define

\[
x(t, w) = x(t)w \quad \frac{\partial^q x}{\partial t^q}(t, w) = C D^q x(t, w) \quad k_2(t, s)x(s, w) = \frac{1}{8} \sin x(s, w) \quad h_2(t, s)x(s, w) = \frac{1}{(1+t^2)(1+s)} x(s, w) \quad \left( \int_0^T k_2(t, s)x(s) ds + \int_0^T h_2(t, s)x(s) ds \right) = \frac{1}{8} \int_0^T \sin x(s, w) ds + \int_0^T \frac{1}{(1+t^2)(1+s)} x((s, w)) ds
\]
Let $B : J \to X$ be defined by
\[(Bv)(t)(w) = \chi(t, w), \ w \in \left(0, \frac{\pi}{2}\right).\]

Now the linear operator $M$ is given by
\[(Mv)(w) = \int_0^T S(T - s)\chi(s, w)ds, \ w \in \left(0, \frac{\pi}{2}\right).\]

Assume that this operator has a bounded operator $M^{-1}$ in $L^2(J)/\text{Ker}M$. With the choice of $A$ and $B$, the system (2.6)–(2.7) can be written as
\[
C_D^\beta x(t) = Ax(t) + \int_0^t k_1(t, s)x(s))ds \\
+ \int_0^t h_2(t, s)x(s))ds + Bv(t),
\]
where $0 \leq t \leq T$.

Then, by Theorem 2.2, the iterative fractional integrodifferential system (2.6) – (2.7) is controllable on $J$ for any value of $q \in (0, 1]$.

### 3. Controllability of Mixed Iterative Fractional Integrodifferential Equations of order $\beta \in (1, 2]$

In this section, we state and prove the controllability results for mixed iterative integrodifferential equations of fractional order $1 < \beta \leq 2$ of the type
\[
C_D^\beta y(t) = Ay(t) + \int_0^t k_1(t, s)y(s))ds \quad (3.1)
\]
\[
+ \int_0^t h_1(t, s)y(s))ds + Bu(t), \quad (3.2)
\]
y(0) = y_0, y'(0) = 0 \quad (3.3)

where the operator $A$ is an infinitesimal generator of a strongly continuous $\beta$–order cosine family $\{C^\beta(t)\}_{t \geq 0}$ on a Banach space $Y$, the state $y(\cdot)$ takes values in $Y$, the control function $u(\cdot)$ is given in $L^2(J; U)$, a Banach space of admissible control functions and the map $B : U \to Y$ is a bounded linear operator; $k_1(t, s)$ and $h_1(t, s)$ are given continuous functions.

Consider the following problem
\[
C_D^\beta y(t) = Ay(t), \ y(0) = y_0, \ y'(0) = 0, \quad (3.4)
\]
where $1 < \beta \leq 2$, the operator $A : D(A) \subset Y \to Y$ is a closed densely defined linear operator on the Banach space $Y$.

Here we present the basic definitions of $\beta$–order cosine family which are used throughout this paper (see, for instance, [24]).

**Definition 6.** [24] Let $\beta \in (1, 2]$. A strongly continuous $\beta$–order fractional cosine family $\{C^\beta(t)\}_{t \geq 0}$ is a solution for (3.1)–(3.3) if the following conditions are satisfied:

(a) $C^\beta(t)$ is strongly continuous for $t \geq 0$ and $C^\beta(0) = I$.

(b) $C^\beta(t)D(A) \subset D(A)$ and $AC^\beta(t)y_0 = C^\beta(t)Ay_0$, for all $y_0 \in D(A), t \geq 0$.

(c) $C^\beta(t)y_0$ is a solution of $y(t) = y_0 + \frac{1}{\Gamma(\beta)} \int_0^t (t - s)^{\beta - 1}Ay(s)ds$, for all $y_0 \in D(A), t \geq 0$.

Therefore, the operator $A$ is called the infinitesimal generator of $C^\beta(t)$.

**Definition 7.** The fractional sine family $S^\beta : R^+ \to BL(Y)$ associated with $C^\beta$ is defined by
\[
S^\beta(t) = \int_0^t C^\beta(s)ds, \quad t \geq 0. \quad (3.5)
\]

**Definition 8.** The fractional Riemann–Liouville family $P^\beta : R^+ \to BL(Y)$ associated with $C^\beta$ is defined by
\[
P^\beta(t) = \frac{d^\beta}{dt^\beta}C^\beta(t), \quad t \geq 0. \quad (3.6)
\]

where $\frac{d^\beta}{dt^\beta}$ is defined as in Definition 1.

**Definition 9.** A mild solution of (3.1)–(3.3) is defined as the function $y : J \to Y$ such that
\[
y(t) = C^\beta(t)y_0 + S^\beta(t)z_0 + \int_0^t P^\beta(t - s)\left[\int_0^s k_1(s, \xi)y(y(\xi))d\xi + \int_0^T h_1(s, \xi)y(y(\xi))d\xi + Bu(s)\right]ds. \quad (3.7)
\]

**Definition 10.** The system (3.1)–(3.3) is said to be exactly controllable on $J$ if, for every $y_0, y_1 \in Y$, there exists a control $u(\cdot) \in L^2(J; U)$ such that the mild solution $y(\cdot)$ of (3.1)–(3.3) satisfies $y(T) = y_1$.

An operator $A$ is said to belong to $C^\beta(Y; M, \omega)$, if the problem (3.4) has a solution operator $C^\beta(t)$ satisfying (3.4). Denote $C^\beta(\omega) = \cup\{C^\beta(Y; M, \omega) : M \geq 1\}, C^\beta = \cup\{C^\beta(\omega) : \omega \geq 0\}$. In these notations, $C^1$ and $C^2$ are the sets of all infinitesimal generators of $C_0$–semi group and cosine operator families respectively. To establish our results, we need the following lemma:

**Lemma 3.1.** [1] Let $0 < \beta < q \leq 2$, $\gamma = \frac{\beta}{q}$, $\omega \geq 0$. If $A \in C^\gamma(\omega^\gamma)$, then $A \in C^\gamma(\omega^\gamma)$ and the following representation holds
\[
C^\beta(t) = \int_0^\infty \phi_\gamma(s)C^\gamma(s)ds, \quad t > 0,
\]

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where $\phi, \gamma = t^{-\gamma} \sum_{n=0}^{\infty} \frac{(-st^{-\gamma})^n}{n! (-\gamma n + 1 - \gamma)^n}, \ 0 < \gamma < 1$.

For the convenience, we list the following hypotheses used in our further discussion.

\((H_1)\) A generates a strongly continuous $\beta$-order fractional cosine family \(\{C_{\beta}(t)\}_{t \geq 0}\) in the Banach space \(Y\) and there exists constants \(M \geq 1\) and \(\omega \geq 0\) such that
\[
\|C_{\beta}(t)\| \leq M e^{\omega t}, \quad t \geq 0.
\]

\((H_2)\) There exists constants \(k_3\) and \(h_3\) such that \(k_3 = \sup\{ |k_1(s, \xi)| : 0 \leq \xi \leq s \leq T \}\) and \(h_3 = \sup\{ |h_1(s, \xi) : (\xi, s) \in J \times J \}\).

\((H_3)\) There exists a constant \(\lambda > 0\) such that \(\|y(t_1) - y(t_2)\| \leq \lambda \|t_1 - t_2\|\) for \(y \in C(J, Y), t_1, t_2 \in J, \quad t_1 \leq t_2\).

\((H_4)\) The linear operator \(W : L_2(J, U) \to Y\) defined by
\[
W_u = \int_0^T P_{\beta}(T - s) Bu(s) \, ds,
\]
has an invertible operator \(W^{-1}\) induced by \(W\) defined on \(L_2(J, U) / \text{Ker} W\) and there exists positive constants \(l_1, l_2 > 0\) such that \(\|W^{-1}\| \leq l_1\) and \(\|B\| \leq l_2\).

\((H_5)\) There exists positive constants \(M_1, M_2, M_3 > 0\) such that
\[
\|C_{\beta}(t)\| \leq M_1, \quad \|S_{\beta}(t)\| \leq M_2, \quad \|P_{\beta}(t)\| \leq M_3.
\]

Now, we state and prove result related to mixed iterative integrodifferential equations of fractional order.

**Theorem 3.2.** If hypotheses \((H_1) - (H_5)\) are satisfied and
\[
M_3 T^2 (k_3 + h_3) (\lambda + 1) [1 + M_3 l_1 l_2 T] < 1.
\]
Then the system \((3.1) - (3.3)\) is controllable on \(J\).

**Proof.** Let \(C_\lambda = \{ y \in C(J, Y) : \|y(t) - y(s)\| \leq \lambda \|t - s\|\} , \) for all \(s \in J\) and \(C_{\lambda, l} = \{ y \in C_\lambda : \|y(t)\| \leq l\} \) where
\[
(1 + M_3 l_1 l_2 T) [M_1 \|y_0\| + M_2 \|z_0\| + T^4 M_3 (k_3 + h_3)] + M_3 l_1 l_2 T \|y_1\| \leq l.
\]

For \(y \in C_{\lambda, l}\), we define the control
\[
u(t) = W^{-1} \left[ y_1 - C_{\beta}(T)y_0 - S_{\beta}(T)z_0 - \int_0^T P_{\beta}(T - s) \left( \int_0^s k_1(s, \xi) y(\xi) \, d\xi \right) + \int_0^T h_1(s, \xi) y(\xi) \, d\xi \right] \, ds.
\]
(3.8)

Using this control and Banach contraction principle, the operator defined by
\[
(F_{\lambda}) (t) = C_{\beta}(t)y_0 + S_{\beta}(t)z_0 + \int_0^t P_{\beta}(t-s) \left( \int_0^s k_1(s, \xi) y(\xi) \, d\xi \right) + \int_0^T h_1(s, \xi) y(\xi) \, d\xi + Bu(s) \, ds
\]
is proved to have a fixed point: that is, \((F_{\lambda})(t) = y(t)\). This fixed point is then a solution of (3.1) - (3.3). This implies that there exists the control \(u\) which steers the system from the initial state \(y_0\) to the final state \(y_1\) in time \(T\) provided we can obtain a fixed point of the nonlinear operator \(F\). First we prove that \(F\) maps \(C_{\lambda, l}\) into itself. For this, applying conditions \((H_1) - (H_5)\), (3.8) and triangle inequality, we have

\[
\|y(t)\| \leq \|y_1\| + \|y_0\| + \|z_0\| + \int_0^T \|P_{\beta}(s)\| \left( \int_0^s \|k_1(s, \xi) y(\xi)\| \, d\xi \right) + \int_0^T \|h_1(s, \xi) y(\xi)\| \, d\xi \, ds
\]

\[
\|y(t)\| \leq \|y_1\| + \|y_0\| + \|z_0\| + \int_0^T \|P_{\beta}(s)\| \left( \int_0^s \|k_1(s, \xi) y(\xi)\| \, d\xi \right) + \int_0^T \|h_1(s, \xi) y(\xi)\| \, d\xi \, ds
\]

\[
\|y(t)\| \leq \|y_1\| + \|y_0\| + \|z_0\| + \int_0^T \|P_{\beta}(s)\| \left( \int_0^s \|k_1(s, \xi) y(\xi)\| \, d\xi \right) + \int_0^T \|h_1(s, \xi) y(\xi)\| \, d\xi \, ds
\]

Hence \(F\) \(\in C_{\lambda, l}\), whenever \(y \in C_{\lambda, l}\). Therefore \(F : C_{\lambda, l} \to C_{\lambda, l}\). That is, \(F\) is a self mapping. Next \(F\) is proved to be a contraction map. Let \(y_1, y_2 \in C_{\lambda, l}\). Then
Let $\nu_n(x) = \sqrt{\left(\frac{\pi}{2}\right)^n} \sin nx$, $0 \leq x \leq \left(\frac{\pi}{2}\right), n = 1, 2, 3, \ldots$ Here $\nu_n$ is the eigenfunction corresponding to the eigenvalue $-n^2$ of the operator $A$ and $\{\nu_n\}$ is an orthonormal basis for $Y$. Then

$$A\sigma = \sum_{n=1}^{\infty} -n^2(\sigma, \nu_n)\nu_n, \quad \sigma \in D(A).$$

It is simple to show that $A$ is the infinitesimal generator of a strongly continuous cosine family $C(t)$ and

$$C(t)\sigma = \sum_{n=1}^{\infty} \cos nt(\sigma, \nu_n)\nu_n, \quad \sigma \in Y, t \in R.$$

Set

$$\nu(t, z) = y(t)(z)$$

$$k_1(t, s)y(y(s, z)) = \frac{1}{2} \cos y(y(s, z))$$

$$h_1(t, s)y(y(s, z)) = \frac{1}{(1 + r(t))(1 + s)}y(y(s, z))$$

$$\left(\int_0^t k_1(t, s)y(y(s))ds + \int_0^T h_1(t, s)y(y(s))ds\right)\cos y(y(s, z))ds$$

$$= \frac{1}{2} \int_0^t \cos y(y(s, z))ds + \int_0^T \frac{1}{(1 + r(t))(1 + s)}y(y(s))ds$$

Let $B : J \rightarrow Y$ be defined by

$$(Bu)(t)(z) = \nu(t, z), \quad z \in \left(0, \frac{\pi}{4}\right).$$

With the choice of $A, B$ the linear operator $W$ is specified by

$$(Wu)(z) = \int_0^T P_2 \left(\frac{\pi}{4} - s\right) \nu(s, z)ds, \quad z \in \left(0, \frac{\pi}{4}\right).$$

Assume that this operator has a bounded operator $W^{-1}$ in $L^2(J)/\ker W$. Then the system (3.9)-(3.11) can be written as

$$C_D^\beta y(t) = Ay(t) + \int_0^T k_1(t, s)y(y(s))ds$$

$$+ \int_0^T h_1(t, s)y(y(s))ds + Bu(t),$$

$$y(0) = y_0, \quad y'(0) = z_0.$$
Assume that this operator has a bounded operator $W^{-1}$ in $L^2(J)/\ker W$. All the conditions of Theorem 3.2 are satisfied. Therefore, the iterative fractional integrodifferential system (3.9)-(3.11) is controllable on $J$ for any value of $\beta \in (1, 2]$ and $\gamma_0, z_0 \in Y$.

4. Conclusion

In this paper, we investigated the sufficient conditions for the controllability of mixed iterative integrodifferential equations of fractional orders lying between $(0, 1]$ and $(1, 2]$. The fractional derivatives are considered in the Caputo sense. We also utilized the Banach contraction fixed-point theorem.

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References


