Operation-extremally disconnected spaces

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Abstract
In this paper, we have study the characterizations of submaximal spaces and extremally disconnected spaces via operation in soft topological spaces.

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Topological spaces, \(\gamma\)-soft open set, \(\gamma\)-soft regular open set, \(\gamma\)-\(\delta\)-open set.

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1. Introduction
The concept of soft sets was first introduced by Molodtsov [11]. After the introduction of the definition of a soft sets by Molodtsov, a large number of topologists have turned their attention to the generalization of different concepts of a classical sets in this sets. Recently, the concept of soft topological spaces was introduced and studied by Shabir and Naz [17]. A Good number of results are studied in this paper. The study of topological properties via operations was introduced and studied by Biswas and Prasannan in [2]. In this paper, we introduce and study the concept of submaximal spaces and extremally disconnected spaces via operation in soft topological spaces.

2. Preliminaries
Let \(U\) be an initial universe set and \(E_U\) be a collection of all possible parameters with respect to \(U\), where parameters are the characteristics or properties of objects in \(U\). We will call \(E_U\) the universe set of parameters with respect to \(U\).

Definition 2.1. [11] A pair \((F,A)\) is called a soft set over \(U\) if \(A \subset E_U\) and \(F : A \rightarrow P(U)\), where \(P(U)\) is the set of all subsets of \(U\).

Definition 2.2. [6] Let \(U\) be an initial universe set and \(E_U\) be a universe set of parameters. Let \((F,A)\) and \((G,B)\) be soft sets over a common universe set \(U\) and \(A, B \subset E\). Then \((F,A)\) is a subset of \((G,B)\), denoted by \((F,A) \subset (G,B)\), if \(A \subset B\) and for all \(e \in A\), \(F(e) \subset G(e)\). Also \((F,A)\) equals \((G,B)\), denoted by \((F,A) = (G,B)\), if \((F,A) \subset (G,B)\) and \((G,B) \subset (F,A)\).

Definition 2.3. [12] A soft set \((F,A)\) over \(U\) is called a null soft set, denoted by \(\emptyset\), if \(e \in A\), \(F(e) = \emptyset\).

Definition 2.4. [12] A soft set \((F,A)\) over \(U\) is called an absolute soft set, denoted by \(A\), if \(e \in A\), \(F(e) = U\).

Definition 2.5. [12] The union of two soft sets \((F,A)\) and \((G,B)\) over a common universe is the soft set \((H,C)\), where \(C = A \cup B\), and for all \(e \in C\),

\[
H(e) = \begin{cases} 
F(e) & \text{if } e \in A \setminus B, \\
G(e) & \text{if } e \in B \setminus A, \\
F(e) \cup G(e) & \text{if } e \in B \cap A. 
\end{cases}
\]

We write \((F,A) \cup (G,B) = (H,C)\).

Definition 2.6. [6] The intersection of two soft sets \((F,A)\) and \((G,B)\) over a common universe \(U\) is the soft set \((H,C)\), where \(C = A \cap B\), and for all \(e \in C\), \(H(e) = F(e) \cap G(e)\). We write \((F,A) \cap (G,B) = (H,C)\).

Now we recall some definitions and results defined and discussed in [16, 17]. Henceforth, let \(X\) be an initial universe set and \(E\) be the fixed nonempty set of parameter with respect to \(X\) unless otherwise specified.

Definition 2.7. For a soft set \((F,A)\) over \(U\), the relative complement of \((F,A)\) is denoted by \((F,A)\) and is defined by
\[(F,A)' = (F',A),\text{ where } F' : A \to P(U) \text{ is a mapping given by } F'(e) = U \setminus F(e) \text{ for all } e \in A.\]

**Definition 2.8.** Let \(\tau\) be the collection of soft sets over \(X\), then \(\tau\) is called a soft topology on \(X\) if \(\tau\) satisfies the following axioms.

1. \(\emptyset, X\) belong to \(\tau\).
2. The union of any number of soft sets in \(\tau\) belongs to \(\tau\).
3. The intersection of any two soft sets in \(\tau\) belongs to \(\tau\).

The triplet \((X, \tau, E)\) is called a soft topological space over \(X\).

**Definition 2.9.** Let \((X, \tau, E)\) be a soft topological space over \(X\), then the members of \(\tau\) are said to be soft open sets in \(X\).

**Definition 2.10.** Let \((X, \tau, E)\) be a soft topological space over \(X\). A soft set \((F, E)\) over \(X\) is said to be a soft closed set in \(X\), if its relative complement \((F, E)'\) belongs to \(\tau\).

**Definition 2.11.** Let \((X, \tau, E)\) be a soft topological space and \((A, E)\) a soft set over \(X\).

1. The soft interior of \((A, E)\) is the soft set \(\text{Int}(A, E) = \bigcup\{(O, E) : (O, E) \text{ is soft open and } (O, E) \subseteq (A, E)\}\).
2. The soft closure of \((A, E)\) is the soft set \(\text{Cl}(A, E) = \bigcap\{(F, E) : (F, E) \text{ is soft closed and } (A, E) \subseteq (F, E)\}\).

**Definition 2.12.** Let \((F, E)\) be a soft set over \(X\) and \(x \in X\). We say that \(x \in \overline{F}(E)\) read as \(x\) belongs to the soft set \(F(E)\), whenever \(x \in F(\alpha)\) for all \(\alpha \in E\). Note that for \(x \in X, x \notin \overline{F}(E)\) if \(x \notin F(\alpha)\) for \(\alpha \in E\).

**Definition 2.13.** Let \(x \in X\), then \((x, E)\) denotes the soft set over \(X\) for which \(x(\alpha) = \{x\}\) for all \(\alpha \in E\).

**Definition 2.14.** [2] An operation on a soft topology \(\tau\) over \(X\) is called a \(\gamma\)-operation if a mapping from \(\tau\) to the set \(P(X)^{\gamma}\) and defined by \(\gamma : \tau \to P(X)^{\gamma}\) such that for each \((V, E) \in \tau\), \((V, E) \subset \gamma(V, E)\).

**Definition 2.15.** [2] A soft set \((P, E)\) is said to be \(\gamma\)-soft open set if for each \(x \in \overline{P}(E),\) there exists a soft open set \((V, E)\) such that \(x \in \overline{P}(E) \subset \gamma(V, E) \subseteq (P, E)\). The complement of a \(\gamma\)-soft open set is called a \(\gamma\)-soft closed set. The family of all \(\gamma\)-soft open sets of \((X, \tau, E, \gamma)\) is denoted by \(\tau_{\gamma}\).

**Definition 2.16.** [2] Let \((X, \tau, E, \gamma)\) be an operation-soft topological space and \((A, E)\) a soft set over \(X\). Then

1. the \(\tau_{\gamma}\)-soft interior of \((A, E)\) is the soft set \(\text{Int}_{\gamma}(A, E) = \bigcup\{(O, E) : (O, E) \text{ is } \gamma\text{-soft open and } (O, E) \subseteq (A, E)\}\).
2. the \(\tau_{\gamma}\)-soft closure of \((A, E)\) is the soft set \(\text{Cl}_{\gamma}(A, E) = \bigcap\{(F, E) : (F, E) \text{ is } \gamma\text{-soft closed and } (A, E) \subseteq (F, E)\}\).

**Lemma 2.17.** If \((X, \tau, E, \gamma)\) is an operation-soft topological space, then

1. for every \(\gamma\)-soft open set \((G, E)\) and every soft subset \((A, E)\) over \(X\), we have \(\tau_{\gamma}\text{-Cl}(A, E) \cap (G, E) \subseteq \tau_{\gamma}\text{-Cl}(G, E)\).
2. for every \(\gamma\)-soft closed set \((F, E)\) and every soft subset \((A, E)\) over \(X\), we have \(\tau_{\gamma}\text{-Int}(A, E) \cap (F, E) \subseteq \tau_{\gamma}\text{-Int}(F, E)\).

**Definition 2.18.** Let \((X, \tau, E, \gamma)\) be an operation-soft topological space and \((A, E)\) a soft set over \(X\). Then

1. \(e_m\) is called \(\gamma\)-soft cluster point of \((S, E)\) if \((S, E) \cap \tau_{\gamma}\text{-Int}(\tau_{\gamma}\text{-Cl}(U, E)) = \emptyset\) for each \(\gamma\)-soft open set \((U, E)\) containing \(e_m\).
2. the family of all \(\gamma\)-soft cluster point of \((S, E)\) is called the \(\gamma\)-soft closure of \((S, E)\) and is denoted by \(\tau_{\gamma}\text{-Cl}_{\gamma}(S, E)\).
3. A soft subset \((S, E)\) is said to be \(\gamma\)-soft closed if \(\tau_{\gamma}\text{-Cl}_{\gamma}(S, E) = (S, E)\). The complement of a \(\gamma\)-soft closed set is said to be a \(\gamma\)-soft open set.

**Definition 2.19.** [7, 8] A subset \((A, E)\) of an operation-soft topological space \((X, \tau, E, \gamma)\) is said to be

1. \(\gamma\)-soft semiopen of \((A, E)\) if \((A, E) \subseteq \tau_{\gamma}\text{-Cl}(\tau_{\gamma}\text{-Int}(A, E))\).
2. \(\gamma\)-soft preopen of \((A, E)\) if \((A, E) \subseteq \tau_{\gamma}\text{-Int}(\tau_{\gamma}\text{-Cl}(A, E))\).
3. \(\gamma\)-soft \(\beta\)-open of \((A, E)\) if \((A, E) \subseteq \tau_{\gamma}\text{-Cl}(\tau_{\gamma}\text{-Int}(\tau_{\gamma}\text{-Cl}(A, E)))\).
4. soft \(\tau_{\gamma}\)-set of \((A, E)\) if \(\tau_{\gamma}\text{-Int}(A, E) = \tau_{\gamma}\text{-Int}(\tau_{\gamma}\text{-Cl}(A, E))\).
5. \(\gamma\)-soft-semi open.
6. soft \(\beta\)-open set if \((A, E) \subseteq \tau_{\gamma}\text{-Int}(\tau_{\gamma}\text{-Cl}(A, E))\), where \((U, E) \in \tau_{\gamma}\text{ and } (V, E)\) is soft \(\gamma\)-semiregular.
7. \(\gamma\)-soft dense if \(\tau_{\gamma}\text{-Cl}(A, E) = X\).

The family of all \(\gamma\)-soft regular open (resp. \(\gamma\)-soft preopen, \(\gamma\)-soft semiopen, \(\gamma\)-soft \(\beta\)-open) sets of \((X, \tau, E, \gamma)\) is denoted by \(\gamma\)-SRO \((X)\) (resp. \(\gamma\)-SPO \((X)\), \(\gamma\)-SSO \((X)\), \(\gamma\)-SBO \((X)\)).

### 3. On \(\gamma\)-soft submaximal spaces

**Definition 3.1.** An operation-topological space \((X, \tau, E, \gamma)\) is said to be \(\gamma\)-soft submaximal if every \(\gamma\)-soft dense subset over \(X\) is \(\gamma\)-soft open.

**Lemma 3.2.** \((A, E) \in \gamma\)-SPO \((X)\) if, and only if \((A, E) \subseteq (U, E) \cap (D, E)\) for some \((U, E) \in \tau_{\gamma}\) and \(\gamma\)-soft dense set \((D, E) \in S(X)\).

**Proof.** If \((A, E) \in \gamma\)-SPO \((X)\), \((A, E) \subseteq \tau_{\gamma}\text{-Int}(\tau_{\gamma}\text{-Cl}(A, E))\). Let \((U, E) \in \tau_{\gamma}\). Let \((D, E) = X \setminus ((U, E) \setminus (A, E)) \subseteq (A, E)\). So \((D, E)\) is \(\gamma\)-soft dense, \(X = \tau_{\gamma}\text{-Cl}(A, E)\).
If \((X, \tau, E, \gamma)\) is \(\gamma\)-soft submaximal, then \(\tau_\gamma = \gamma\)-SPO(X).

Proof. Clearly \(\tau_\gamma \subset \gamma\)-SPO(X). Now \((A, E) \in \gamma\)-SPO(X), then \((A, E) = (U, E) \cap (D, E)\) for some \((U, E) \in \tau_\gamma\) and \(\tau_\gamma\)-soft dense set \((D, E)\) over \(X\). Hence if \((X, \tau, E, \gamma)\) is \(\gamma\)-soft submaximal, \((D, E) \in \tau_\gamma\), then \((A, E) \in \tau_\gamma\).

Theorem 3.4. The following are equivalent for an operation-soft topological space \((X, \tau, E, \gamma)\):

1. \(\tilde{X}\) is \(\gamma\)-soft submaximal.
2. Every \(\gamma\)-soft preopen set is \(\gamma\)-soft open.
3. Every \(\gamma\)-soft preopen set is \(\gamma\)-soft semiopen and every \(\gamma\)-soft \(\alpha\)-open set is \(\gamma\)-soft open.

Proof. (1) \(\Rightarrow\) (2): It follows from Lemma 3.3.
(2) \(\Rightarrow\) (3): Suppose that every \(\gamma\)-soft preopen set is \(\gamma\)-soft open. Then every \(\gamma\)-soft preopen set is \(\gamma\)-soft semiopen.
(3) \(\Rightarrow\) (1): Let \((A, E)\) be a \(\tau_\gamma\)-soft dense subset of \(X\). Since \(\tau_\gamma\)-Cl((A, E)) = \(\tilde{X}\), then \((A, E)\) is \(\gamma\)-soft preopen. By (3), \((A, E)\) is \(\gamma\)-soft semiopen. Since a soft set is \(\gamma\)-soft \(\alpha\)-open if, and only if it is \(\gamma\)-soft preopen and \(\gamma\)-soft semiopen, then \((A, E)\) is \(\gamma\)-soft \(\alpha\)-open. Thus, by (3), \((A, E)\) is \(\gamma\)-soft open; hence \(\tilde{X}\) is \(\gamma\)-soft submaximal.

Theorem 3.5. The following are equivalent for an operation-soft topological space \((X, \tau, E, \gamma)\):

1. \(\tilde{X}\) is \(\gamma\)-soft submaximal.
2. For all \((A, E) \subset \tilde{X}\), if \((A, E) \notin \tau_\gamma\)-Int((A, E)) \(\neq \emptyset\), then \((A, E) \notin \tau_\gamma\)-Int(\(\tau_\gamma\)-Cl((A, E))) \(\neq \emptyset\).
3. \(\tilde{X}_\gamma^* = \{(A, E) \in \tau_\gamma \mid (A, E) \in \tau_\gamma\} \neq \emptyset\).

Proof. (1) \(\Rightarrow\) (2): Let \((A, E) \subset \tilde{X}\) and \((A, E) \notin \tau_\gamma\)-Int((A, E)) \(\neq \emptyset\). Suppose that \((A, E) \notin \tau_\gamma\)-Int(\(\tau_\gamma\)-Cl((A, E))) \(\neq \emptyset\). Then \((A, E) \subset \tau_\gamma\)-Int(\(\tau_\gamma\)-Cl((A, E))). This implies that \((A, E)\) is \(\gamma\)-soft preopen. Since \(\tilde{X}\) is \(\gamma\)-soft submaximal, by Theorem 3.4, \((A, E)\) is \(\gamma\)-soft open. Thus, \((A, E) \subset \tau_\gamma\)-Int((A, E)) \(\neq \emptyset\). This is a contradiction.
(2) \(\Rightarrow\) (1): Let \((A, E)\) be \(\gamma\)-soft preopen. Then \((A, E) \subset \tau_\gamma\)-Int(\(\tau_\gamma\)-Cl((A, E))). Suppose that \((A, E)\) is not \(\gamma\)-soft open. Then \((A, E)\) is not a subset of \(\tau_\gamma\)-Int((A, E)) and \((A, E) \subset \tau_\gamma\)-Int((A, E)) \(\neq \emptyset\). By (2), \((A, E) \subset \tau_\gamma\)-Int(\(\tau_\gamma\)-Cl((A, E))) \(\neq \emptyset\). Thus, \((A, E)\) is not a subset of \(\tau_\gamma\)-Int(\(\tau_\gamma\)-Cl((A, E))). This is a contradiction.

Definition 3.7. An operation-soft topological space is said to be \(\gamma\)-extremally disconnected if \(\tau_\gamma\)-Cl((A, E)) \(\subset \tau_\gamma\) for every \((A, E) \in \tau_\gamma\).

Theorem 3.8. The following are equivalent for an operation-soft topological space \((X, \tau, E, \gamma)\):

1. \(\tilde{X}\) is \(\gamma\)-soft extremally disconnected.
2. \(\tau_\gamma\)-Int((A, E)) is \(\gamma\)-soft closed for every \(\gamma\)-soft closed subset \((A, E)\) over \(X\).
3. \(\tau_\gamma\text{-Cl}(\tau_\gamma\text{-Int}(A, E))) \subset \tau_\gamma\text{-Int}(\tau_\gamma\text{-Cl}(A, E)))\) for every soft subset \((A, E)\) over \(X\).

4. Every \(\gamma\)-soft semiopen set is \(\gamma\)-soft preopen.

5. The \(\tau_\gamma\)-soft closure of every \(\gamma\)-soft \(\beta\)-open subset over \(X\) is \(\gamma\)-soft open.

6. Every \(\gamma\)-soft \(\beta\)-open set is \(\gamma\)-soft preopen.

7. For every soft subset \((A, E)\) over \(X\), \((A, E)\) is \(\gamma\)-soft \(\alpha\)-open if and only if it is \(\gamma\)-soft semiopen.

**Proof.** (1) \(\Rightarrow\) (2): Let \((A, E) \subset X\) be a \(\gamma\)-soft closed set. Then \(X \setminus (A, E)\) is \(\gamma\)-soft open. By (1), \(\tau_\gamma\text{-Cl}(X \setminus (A, E)) = X \setminus \tau_\gamma\text{-Int}(A, E)\) is \(\gamma\)-soft open. Thus, \(\tau_\gamma\text{-Int}(A, E)\) is \(\gamma\)-soft closed.

(2) \(\Rightarrow\) (3): Let \((A, E)\) be any soft set over \(X\). Then \(X \setminus \tau_\gamma\text{-Int}(A, E)\) is \(\gamma\)-soft closed over \(X\) and by (2), \(\tau_\gamma\text{-Int}(X \setminus \tau_\gamma\text{-Int}(A, E))\) is \(\gamma\)-soft closed over \(X\). Therefore, \(\tau_\gamma\text{-Cl}(\tau_\gamma\text{-Int}(A, E))\) is a \(\gamma\)-soft open set over \(X\) and hence, \(\tau_\gamma\text{-Cl}(\tau_\gamma\text{-Int}(A, E)) \subset \tau_\gamma\text{-Int}(\tau_\gamma\text{-Cl}(A, E))\).

(3) \(\Rightarrow\) (4): Let \((A, E)\) be \(\gamma\)-soft semiopen. By (3), we have \((A, E) \subset \tau_\gamma\text{-Cl}(\tau_\gamma\text{-Int}(\tau_\gamma\text{-Cl}(A, E)))\). Thus, \((A, E)\) is \(\gamma\)-soft preopen.

(4) \(\Rightarrow\) (5): Let \((A, E) \in \gamma\text{-SBO}(X)\). Then \(\tau_\gamma\text{-Cl}(A, E)\) is \(\gamma\)-soft semiopen. By (4), \(\tau_\gamma\text{-Cl}(A, E)\) is \(\gamma\)-soft preopen. Thus, \(\tau_\gamma\text{-Cl}(A, E) \subset \tau_\gamma\text{-Int}(\tau_\gamma\text{-Cl}(A, E))\) and \(\tau_\gamma\text{-Cl}(A, E)\) is \(\gamma\)-soft open.

(5) \(\Rightarrow\) (6): Let \((A, E) \in \gamma\text{-SBO}(X)\). By (5), \(\tau_\gamma\text{-Cl}(A, E) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-Cl}(A, E))\) and hence \((A, E)\) is \(\gamma\)-soft preopen.

(6) \(\Rightarrow\) (7): Let \((A, E)\) be \(\gamma\)-soft semiopen set. Since a \(\gamma\)-soft semiopen set is \(\gamma\)-soft \(\beta\)-open, then by (6), it is \(\gamma\)-soft preopen. Since \((A, E)\) is \(\gamma\)-soft semiopen and \(\gamma\)-soft preopen, \((A, E)\) is \(\gamma\)-soft \(\alpha\)-open.

(7) \(\Rightarrow\) (1): Let \((A, E)\) be a \(\gamma\)-soft open set over \(X\). Then \(\tau_\gamma\text{-Cl}(A, E)\) is \(\gamma\)-soft semiopen by (7). \(\tau_\gamma\text{-Cl}(A, E)\) is \(\gamma\)-soft \(\alpha\)-open. Therefore, \(\tau_\gamma\text{-Cl}(A, E) \subset \tau_\gamma\text{-Int}(\tau_\gamma\text{-Cl}(\tau_\gamma\text{-Int}(A, E))) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-Cl}(A, E))\) and hence, \(\tau_\gamma\text{-Cl}(A, E) = \tau_\gamma\text{-Int}(\tau_\gamma\text{-Cl}(A, E))\). Hence \(\tau_\gamma\text{-Cl}(A, E)\) is \(\gamma\)-soft open and \(\overline{X}\) is \(\gamma\)-soft extremally disconnected.

**Theorem 3.9.** The following are equivalent for an operation-soft topological space \((X, \tau, \gamma, \eta)\):

1. \(\overline{X}\) is \(\gamma\)-soft extremally disconnected.
2. \(\tau_\gamma\text{-Cl}(A, E) \in \tau_\gamma\) for every \((A, E) \in \gamma\text{-SOO}(X)\).
3. \(\tau_\gamma\text{-Cl}(A, E) \in \tau_\gamma\) for every \((A, E) \in \gamma\text{-SOO}(X)\).
4. \(\tau_\gamma\text{-Cl}(A, E) \in \tau_\gamma\) for every \((A, E) \in \gamma\text{-SRO}(X)\).

**Proof.** (1) \(\Rightarrow\) (2): Let \(\overline{X}\) be \(\gamma\)-soft extremally disconnected. Let \((A, E) \subset X\) be a \(\gamma\)-soft semiopen. By Lemma 3.10, we have \(\tau_\gamma\text{-Cl}(A, E) = \tau_\gamma\text{-Cl}(A, E)\). Since \(X\) is \(\gamma\)-soft extremally disconnected, \(\tau_\gamma\text{-Cl}(A, E) = \tau_\gamma\text{-Cl}(A, E)\) is \(\gamma\)-soft open.

(2) \(\Rightarrow\) (1): Suppose that \(\tau_\gamma\text{-Cl}(A, E) \in \tau_\gamma\) for every \((A, E) \in \gamma\text{-SSO}(X)\). Let \((A, E) \subset X\) be \(\gamma\)-soft semiopen set. By Lemma 3.10, \(\tau_\gamma\text{-Cl}(A, E) = \tau_\gamma\text{-Cl}(A, E)\). Thus, \(\tau_\gamma\text{-Cl}(A, E)\) is \(\gamma\)-soft open; hence \(X\) is \(\gamma\)-soft extremally disconnected.

(1) \(\Rightarrow\) (3): Let \((A, E)\) be \(\gamma\)-soft \(\beta\)-open set. By Theorem 3.8, \(\tau_\gamma\text{-Cl}(A, E)\) is \(\gamma\)-soft open and hence by Lemma 3.10, \(\tau_\gamma\text{-Cl}(A, E)\) is \(\gamma\)-soft open.
semiopen (resp. γ-soft preopen) set is γ-soft β-open, by (3) τγ-Clγ(A, E) is γ-soft open.
(2) ⇒ (1): ((4) ⇒ (1)). Let (A, E) be a γ-soft open set of X. Every γ-soft open set is γ-soft semiopen and γ-soft preopen. By (2) (resp. (4)), τγ-Clγ(A, E) is γ-soft open and hence, by Lemma 3.10, τγ-Cl((A, E)) is γ-soft open. Therefore, X is τγ-extremally disconnected. □

Lemma 3.13. A soft subset (A, E) of an operation-topological space (X, τ, E, γ) is γ-soft semiopen if, and only if τγ-Cl((A, E)) ⊆ τγ-Cl(τγ-Int((A, E))).


Theorem 3.14. The following are equivalent for an operation-topological space (X, τ, E, γ):
1. X is γ-soft extremally disconnected.
2. If (A, E) ∈ γSO(X) and (B, E) ∈ γSSO(X), then τγ-Cl((A, E)) ∩ τγ-Cl((B, E)) ⊆ τγ-Cl((A, E) ∩ (B, E)).
3. If (A, E) and (B, E) are γ-soft semioopen sets, then τγ-Cl((A, E)) ∩ τγ-Cl((B, E)) = τγ-Cl((A, E) ∩ (B, E)).
4. τγ-Cl((A, E)) ∩ τγ-Cl((B, E)) = ∅ for every disjoint γ-soft semioopen sets (A, E) and (B, E).
5. If (A, E) is γ-soft preopen and (B, E) is γ-soft semioopen, then τγ-Cl((A, E)) ∩ τγ-Cl((B, E)) ⊆ τγ-Cl((A, E) ∩ (B, E)).

(2) ⇒ (3): It follows from the fact that every γ-soft semioopen set γ-soft β-open.
(3) ⇒ (4): Obvious.
(2) ⇒ (5): It follows from the fact every γ-soft preopen set is γ-soft β-open.
(5) ⇒ (1): Let (A, E) and (B, E) be disjoint γ-soft open sets. Since (A, E) and (B, E) are γ-soft preopen and γ-soft semioopen, respectively, by (5), we have τγ-Cl((A, E)) ∩ τγ-Cl((B, E)) ⊆ τγ-Cl((A, E) ∩ (B, E)) = ∅. So τγ-Cl((A, E)) ∩ τγ-Cl((B, E)) = ∅. By Theorem 3.9, X is γ-soft extremally disconnected. □

Lemma 3.15. If (A, E) is a γ-soft semioopen soft set in an operation-topological space (X, τ, E, γ), then we have τγ-Cl((A, E)) = τγ-Clγ((A, E)).

Corollary 3.16. The following are equivalent for an operation-topological space (X, τ, E, γ):
1. X is γ-soft extremally disconnected.
2. If (A, E) is γ-soft β-open set and (B, E) is a γ-soft semioopen set, then τγ-Cl((A, E)) ∩ τγ-Clγ((B, E)) ⊆ τγ-Cl((A, E)) ∩ (B, E)).
3. If (A, E) and (B, E) are γ-soft semioopen sets over X, τγ-Cl((A, E)) ∩ τγ-Clγ((B, E)) ⊆ τγ-Cl((A, E)) ∩ (B, E)).
4. τγ-Cl((A, E)) ∩ τγ-Clγ((B, E)) = ∅ for every disjoint γ-soft semioopen sets (A, E) and (B, E) over X.
5. If (A, E) and (B, E) are γ-soft preopen sets over X, then τγ-Cl((A, E)) ∩ τγ-Clγ((B, E)) ⊆ τγ-Cl((A, E)) ∩ (B, E)).

Proof. The proof follows from Theorem 3.14 and Lemma 3.15. □

Theorem 3.17. The following are equivalent for an operation-topological space (X, τ, E, γ):
1. X is γ-soft submaximal and γ-soft extremally disconnected.
2. Any soft subset over X is γ-soft β-open if, and only if γ-soft open.

Proof. (1) ⇒ (2): Let X be γ-soft submaximal and γ-soft extremally disconnected. By Theorem 3.8, every γ-soft β-open set is γ-soft preopen. By Theorem 3.4, every γ-soft preopen set is γ-soft open. Thus, every γ-soft β-open set is γ-soft open. The converse follows from the fact that every γ-soft open set is γ-soft β-open.
(2) ⇒ (1): Suppose that any soft subset of X is γ-soft β-open if, and only if it is γ-soft open. Since every γ-soft β-open set is γ-soft open and so γ-soft preopen, by Theorem 3.8, X is γ-soft extremally disconnected. Since every γ-soft preopen set is γ-soft open, by Theorem 3.4 X is τγ-submaximal. □

Corollary 3.18. For a γ-soft submaximal and γ-soft extremally disconnected space (X, τ, E, γ), the following properties are equivalent:
1. (A, E) is γ-soft β-open
2. (A, E) is γ-soft semioopen
3. (A, E) is γ-soft preopen
4. \((A,E)\) is \(\gamma\)-soft \(\alpha\)-open.

5. \((A,E)\) is \(\gamma\)-soft open.

**Proof.** The proof follows from Theorem 3.17.

**Theorem 3.19.** An operation-soft topological space is \(\gamma\)-soft extremally disconnected if, and only if \(\gamma\)-soft regular open sets coincide with \(\gamma\)-soft regular closed sets.

**Proof.** Suppose \((A,E)\) is a \(\gamma\)-soft regular open subset of \(X\). Since \(\gamma\)-soft regular open sets are \(\gamma\)-soft open, \((A,E) = \tau_{\gamma}Cl((A,E)) = \tau_{\gamma}Cl(\tau_{\gamma}Int((A,E)))\) and so \((A,E)\) is \(\gamma\)-soft regular closed. If \((A,E)\) is a \(\gamma\)-soft regular closed set, then \(A = \tau_{\gamma}Cl(\tau_{\gamma}Int((A,E))) = \tau_{\gamma}Int(\tau_{\gamma}Cl(\tau_{\gamma}Int((A,E)))) = \tau_{\gamma}Int((A,E))\). Also, \(A = \tau_{\gamma}Cl(\tau_{\gamma}Int((A,E))) = \tau_{\gamma}Int(\tau_{\gamma}Cl(\tau_{\gamma}Int((A,E))))\). Hence \((A,E)\) is \(\gamma\)-soft regular open. Conversely, let \((A,E)\) be a \(\gamma\)-soft open subset of \(X\). Then \(\tau_{\gamma}Int(\tau_{\gamma}Cl((A,E)))\) is \(\gamma\)-soft regular open and so it is \(\gamma\)-soft regular closed. Hence \(\tau_{\gamma}Int(\tau_{\gamma}Cl(\tau_{\gamma}Int((A,E)))) = \tau_{\gamma}Int(\tau_{\gamma}Cl(\tau_{\gamma}Int((A,E)))))\). Hence \(\tau_{\gamma}Cl((A,E))\) is \(\gamma\)-soft open. This shows that \(X\) is \(\gamma\)-soft extremally disconnected.

**Theorem 3.20.** An operation-soft topological space is \(\gamma\)-soft extremally disconnected if, and only if every \(\gamma\)-soft regular preopen sets is \(\gamma\)-soft preopen set.

**Proof.** If \((A,E) \in \tau_{\gamma}\), then \(\tau_{\gamma}Cl(\tau_{\gamma}Int((A,E)))\) is \(\gamma\)-soft preopen. It is clear that \(\tau_{\gamma}Cl((A,E)) = \tau_{\gamma}Int(\tau_{\gamma}Cl((A,E)))\) which implies that \(\tau_{\gamma}Cl((A,E))\) is \(\gamma\)-soft open. Hence \(X\) is \(\gamma\)-soft extremally disconnected. The converse follows from the fact that every \(\gamma\)-soft regular open set is \(\gamma\)-soft preopen set.

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**References**


