k-Geodetic propagation in graphs

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Abstract
For a graph \( G(V(G), E(G)) \), the set \( Z \subseteq V(G) \) is called a k-geodetic propagation set if \( Z \) is both a k-geodetic set as well as a propagating set. The cardinality of the minimum k-geodetic propagation set is called the k-geodetic propagation number. Few general results and also the computational complexity part of this concept for general, bipartite and chordal graphs are derived.

Keywords
Propagation, k-geodetic number, geodetic set, geodetic number, k-geodetic set.

AMS Subject Classification
05C12, 68Q17, 94C15.

1. Introduction

Communication is a generally used term in the field of Engineering and Technology with various methods of communication, to be specific satellite, radio, web, broadband and remote telephone utilities. In each one of these modes there are three fundamental components - a data source, communication channel and the recipients which caters the passing of message from the sender to the beneficiary and is invariant to any method of correspondence. A communication framework may have security breaks and can be demonstrated as a graphs or networks. But there can be some outside sources or bugs attempting to get to the system and impedes the correspondence of the message. Likewise there can be some impedance like cross-talk or nearby channel obstruction that may make the entire communication framework broken. To defeat these situations, we need to assign security powers at stations in a specific distance to administer the whole communication network. The idea of assigning security powers within a specific distance for a communication network inspires us to define k-geodetic propagation in graphs. Thus propagation set caters communication of the message while the geodetic set secures the communication system.

Consider a graph \( G(V(G), E(G)) \), with order \( |V(G)| \) denoted as \( o(G) \). The shortest path between the vertices \( x \) and \( y \) is called an \( (x-y) \) geodesic. For a graph \( G \) the diameter, denoted as \( diam(G) \) is the length of the maximum geodesic [5]. Harary et al introduced a graph theoretical parameter called the geodetic number of a graph in [3] and later different variants of the geodetic problem were introduced. In [3] the geodetic number of a graph is as follows, let \( I[u,v] \) be the set of all vertices lying on some \( u-v \) geodesic of \( G \), and for some non empty subset \( S \) of \( V(G) \), \( I[S] = \cup_{u,v \in S} I[u,v] \). The set \( S \) of vertices of \( G \) is called a geodetic set of \( G \), if \( I[S] = V \) and a geodetic set of minimum cardinality is called minimum geodetic set of \( G \). The cardinality of the minimum geodetic set of \( G \) is the geodetic number \( g(G) \) of \( G \). A geodesic in \( G(V(G), E(G)) \) with length exactly \( k \) is called a k-geodesic, where \( k \geq 1 \). The problem to find a \( S \subseteq V(G) \) where all vertices of the graph \( G \) are covered by k-geodesics between some pair of vertices of \( S \) is called the k-geodetic problem.[2] The cardinality of the minimum k-geodetic set of \( G \) is the k-geodetic number \( g_k(G) \) of \( G \).

A vertex \( \alpha \) of the set \( Z \subseteq V(G) \) propagates to \( \beta \) of \( V(G) \setminus Z \) only when \( \beta \) is the only one member of \( V(G) \setminus Z \) which is adjacent to \( \alpha \). For a vertex set \( Z \subseteq V(G) \), a set \( Z \cup \beta \) is a propagating extension of \( Z \) if there exists a vertex \( \alpha \in Z \) with the property that \( \beta \) is the only one member of \( V(G) \setminus Z \) which is adjacent to \( \alpha \). We call a set \( Z \subseteq V(G) \) a propagating set if there exists a sequence \( Z = Z_0 \subseteq Z_1 \subseteq \ldots \subseteq Z_t = V(G) \), where \( Z_k \) is a propagating extension of \( Z_{k-1} \) for \( k = 1,2,\ldots,t \). Here the sequence \( Z = Z_0 \subseteq Z_1 \subseteq \ldots \subseteq Z_t = V(G) \) is called a
propagating sequence. The cardinality of minimum propagat-
ing set of \( G \) is called propagation number of \( G \) and is denoted by \( p(G) \) \[1\]. The propagation problem of a graph \( G \) is to find a minimum propagating set of \( G \).

### 2. k-Geodetic propagation

For a connected graph \( G(V(G), E(G)) \), a set \( Z \subseteq V(G) \) is called a k-geodetic propagation set if \( Z \) is both a k-geodetic set as well as a propagating set. The cardinality of the minimum k-geodetic propagation set is called the k-geodetic propagation number denoted by \( g_k p(G) \). In Fig 1: for \( k = 2 \) the set \( S = \{v_1, v_3, v_5, v_7\} \) forms a 2-geodetic set and set \( Z = \{v_1, v_3, v_5, v_7, v_8\} \) forms a 2-geodetic propagating set. A geodesic in \( G(V(G), E(G)) \) with length exactly 2 is called a 2-geodesic. A set \( Z \subseteq V(G) \) is called a 2-geodetic propagation set if \( Z \) is both a 2-geodetic set as well as a propagating set. The cardinality of the minimum 2-geodetic propagation set is called the 2-geodetic propagation number denoted by \( g_2 p(G) \).

#### Figure 1. \( G = C_8 \)

![Figure 1.](image)

### 3. Computational complexity for k-geodetic propagation

The proof for the NP-completeness of the geodetic prop-
gagation problem for general graphs can be reduced from the
total dominating set problem which is already proved to NP-
complete \[4\].

**Theorem 3.1.** k-geodetic propagation problem is NP- com-
plete for General Graphs.

**Proof.** The graph \( \bar{G}(\bar{V}, \bar{E}) \) can be constructed from \( G(V, E) \) as follows. The vertex set \( \bar{V} \) is \( \bar{V} = \bar{V} \cup V' \) where the vertex set \( V' \) are the vertices of a path of length \((k-1)\). The edge set of \( \bar{G} \) is \( \bar{E} = E \cup E' \) where \( E' \) is the edge set of a path of length \((k-1)\). The \( \bar{G} \) is composed of two layers, the top layer consist of \( G \) itself, while the second layer consists of \( |V| \) paths, each of length \((k-1)\). In other words, \( \bar{G}(\bar{V}, \bar{E}) \) is constructed by adding a degree 1 vertex from the path \( P_{k-1} \) to each \( v \in V(G) \). Thus there exists \(|V|\) simplicial vertices in \( \bar{G}(\bar{V}, \bar{E}) \). Denote the set of set of simplicial vertices in \( \bar{G}(\bar{V}, \bar{E}) \) by \( V'' \). Let \( T \) be a total dominating set for \( G \). Since \( V'' \) is a set of simplicial

![Figure 2. Computational complexity for k-geodetic propagation](image)
V" initiates the propagation process and all the neighbouring vertices of the path P_{k-1} will be propagated. In this process all vertices of \( \bar{G} \) are propagated. Conversely, assume that \( T \cup V' \) is a k-geodetic propagation set of \( \bar{G} \). It is straightforward to see that the every k-geodesics from each vertex in \( V' \) will have one end point in \( v \in V(G) \). Clearly, \( T' \) forms a total domination set for \( G \).

**Theorem 3.2.** k-geodetic propagation problem is NP-complete for Bipartite graphs.

**Proof.** The proof for the NP-completeness of the geodetic propagation problem for bipartite graphs can be reduced from the total dominating set problem which is already proved to NP-complete for bipartite graphs [4]. We follow the same construction as given in Theorem 1 for the construction of \( G(\bar{X}, \bar{Y}) \) from the given bipartite graph \( G(X, Y) \). Here \( \bar{G} \) is also a bipartite graph with bipartition \( \bar{X} \) and \( \bar{Y} \). The remaining proof is similar to proof of Theorem 1. \( \square \)

**Theorem 3.3.** k-geodetic propagation problem is NP-complete for chordal graphs.

**Proof.** The proof for the NP-completeness of the geodetic propagation problem for chordal graphs can be reduced from the total dominating set problem which is already proved to NP-complete for chordal graphs [4]. We follow the same construction as given in Theorem 1 for the construction of \( G(V, E) \) from the given chordal graph \( G(V, E) \). Here \( G(V, E) \) is also a chordal graph. The remaining proof is similar to proof of Theorem 1. \( \square \)

**Proposition 3.4.** For a graph \( G \), every k-geodetic propagation set contains all its simplicial vertices.

**Proposition 3.5.** If \( G \) is a connected graph of order \( n \geq 2 \), then \( 2 \leq \max(\gamma_k(G), \rho(G)) \leq \gamma_k(G) \leq n \)

**Definition 3.6.** For a path \( P_n \) with \( n \geq 3 \) and \( 3 \leq k \leq n - 2 \),
\[
g_k(P_n) = \left\lfloor \frac{n}{k} \right\rfloor + i \quad \text{where} \quad
i = 1 \quad \text{if} \quad n \equiv 1 \mod k \\
2 \quad \text{if} \quad n \equiv 0, 2 \mod k \\
3 \quad \text{otherwise}.
\]
Also, \( g_{n-1}(P_n) = 2 \) and \( g_n(P_n) = n \).

**Definition 3.7.** For a cycle \( C_n \) with \( n \geq 3 \) and \( 1 < k < n - 2 \),
\[
g_k(C_n) = \left\lfloor \frac{n}{k} \right\rfloor + l \quad \text{where} \quad
l = 1 \quad \text{if} \quad n \equiv 0, 1 \mod k \\
2 \quad \text{if} \quad n \equiv 2 \mod k \\
3 \quad \text{otherwise}.
\]
Also, \( g_n(C_n) = n \).

**Definition 3.8.** Let \( G \) be a graph with a vertex \( v \) such that the graph \( (G - v) \) is the union of at least two complete graphs and the vertex \( v \) is adjacent to all other vertices of \( G \). Then \( g_k(G) = n - 1 \) for \( k > 1 \).

The converse need not be true. For example, \( g_k(K_n - e_1) = n - 1 \) and \( g_k(K_n - \{e_1, e_2\}) = n - 1 \) where \( e_1, e_2 \in E(K_n) \).

**Proposition 3.9.** For a connected graph \( G \) of diameter 2, \( g_2(G) = g_4(G) \).

**Proof.** Clearly for any connected graph \( G \), \( g_2(G) \geq g_4(G) \). Now, we have to prove that for a connected graph \( G \) of diameter 2, \( g_2(G) \leq g_4(G) \). Let \( Z \subset V(G) \) be a geodetic propagation set for \( G \). Consider \( u \in V' \cup Z \) and \( x, y \in Z \). Let the vertex \( u \) belongs to the \( x-y \) geodesic. Since the diameter of \( G \) is assumed to be 2, the length of \( x-y \) geodesic will be 2. Thus \( Z \) is also a 2-geodetic propagation set for \( G \) and \( g_2(G) \leq g_4(G) \).

**Proposition 3.10.** Let \( G = (\bigcup_{i=1}^{n} G_i + \{v\}) \) such that each \( G_i \) is a graph with diameter 2. Then \( g_2(G) = \sum_{i=1}^{n} g_2(G_i) + 1 \)

**Proof.** Let \( Z_i \) be the minimum 2-geodetic propagation set of \( G_i \) such that \( |Z_i| = g_2(G_i) \). But the diameter of each \( G_i \) is assumed to be 2, thus \( I(Z_i) = V(G_i) \cup \{v\} \). Therefore \( Z = \bigcup_{i=1}^{n} Z_i \) is a 2-geodetic set. But \( Z \) is not enough to propagate to the remaining vertices of \( G \). Clearly \( Z' = Z \cup \{v\} \) is a 2-geodetic propagation set for \( G \). Therefore \( g_2(G) = \sum_{i=1}^{n} g_2(G_i) + 1 \).

**Proposition 3.11.** For a connected graph \( G \) with geodetic domination number \( \gamma_k(G) \) and 2-geodetic domination number \( g_2(G) \), \( \gamma_k(G) \leq g_2(G) \)

**Proof.** Consider a set \( S \subset V(G) \). Let \( S \) be a 2-geodetic propagation set. Consider a vertex \( x \in V(G) \) \( \setminus S \). Then there exists a \( u-v \) path of length 2 such that \( u, v \in S \) and \( x \) lies on \( P \). This implies that the vertex \( x \) is dominated by the vertices \( u \) and \( v \). Therefore the set \( S \) forms a geodetic domination set. Thus \( g_2(G) \), \( \gamma_k(G) \leq g_2(G) \).

**Theorem 3.12.** For a graph \( G \) with minimum degree \( \delta \), \( girth > 2k \) and order \( n \), the k-geodetic propagation number is \( g_k(G) \geq \frac{k-3+\sqrt{(k-3)^2-4(k-1)(8k-\delta^2k+\delta^2-\delta-2n)}}{2(k-1)} \)

**Proof.** Consider a graph \( G \) with minimum degree \( \delta \), \( girth > 2k \) and order \( n \) and k-geodetic propagation set \( Z \). Consider \( u, v \in Z \) and let \( P_1 \) and \( P_2 \) be the k-geodesics between the vertices \( u \) and \( v \). Then \( P_1 + P_2 \) is a cycle of length \( 2k \) which is a contradiction to the assumption that \( girth(G) > 2k \). Therefore \( I_k(u,v) \) contains a unique path of length \( k \). Therefore any two pair of vertices in \( Z \) can be the end points of some \( k \)-geodesic. Also, a vertex \( x \in Z \) can propagate to its neighbour \( y \in V(G) \setminus Z \) only if all the remaining neighbours are already propagated or are members of \( Z \). Thus the number of vertices in the set \( V(G) \setminus Z \) is bounded above by \( \left( \left\lfloor \frac{Z}{2} \right\rfloor - \left( \frac{\delta}{2} \right) \right)(k-1) \). Therefore \( n - |Z| \leq \left( \left\lfloor \frac{Z}{2} \right\rfloor - \left( \frac{\delta}{2} \right) \right)(k-1) \). From this we obtain \( |Z| \geq \frac{k-3+\sqrt{(k-3)^2-4(k-1)(8k-\delta^2k+\delta^2-\delta-2n)}}{2(k-1)}, \) i.e. \( g_i(G) \geq \frac{k-3+\sqrt{(k-3)^2-4(k-1)(8k-\delta^2k+\delta^2-\delta-2n)}}{2(k-1)} \). Here only the positive solutions are considered. \( \square \)
Theorem 3.13. For a graph $G$ with order $n$, $\delta \geq 2$, $k \geq 3$ and girth $> 4$, the $k$-geodetic propagation number is $g_k p(G) \geq \frac{(2n + k\delta \Delta (k-1))}{(k\Delta (k-1) + 2)}$

Proof. Consider a graph $G$ with order $n$, $\delta \geq 2$, girth $> 4$, the minimum $k$-geodetic propagation set $Z$. The number of $k$-geodesics from a vertex $x \in V$ is bounded above by $\Delta (k-1)$. Thus the number of $k$-geodesics with end points in $S$ is bounded above by $\frac{(S-\delta)}{2} \Delta (k-1)$ and the number of vertices in $V(G) \setminus Z$ lying on these $k$-geodesics is at most $\frac{(S-\delta)}{2} \Delta (k-1) k$. This implies that $n - |Z| \leq \frac{(S-\delta)}{2} \Delta (k-1) k$. From this we obtain $g_k p(G) \geq \frac{(2n + k\delta \Delta (k-1))}{(k\Delta (k-1) + 2)}$.

Theorem 3.14. For $G$ with order $n \geq 2$ and $1 \leq k \leq d$, $g_p(G) \leq n(G) - k + 1$

Proof. Let $P = v_0, v_1 \ldots v_n$ be a $k$-geodesic path of $G$ and $Z = V(P) \cup \{v_0, v_n\}$. Clearly $Z$ is a $k$-geodesic set for $G$. Since all the neighbouring vertices except $v_1$ of $v_0$ are in $Z$, $v_0$ initiates the propagation to $v_1$ and similarly $v_1$ propagates to $v_2$ and so on. Hence $g_p(G) \leq |Z| = n(G) - k + 1$

Theorem 3.15. For a complete bipartite graph $K_{m,n}$, $g_2 p(K_{m,n}) = m + n - 2$

Proof. For a bipartite graph $K_{m,n}$ it is proved that the propagation number, $p(G) = m + n - 2$.[9] Clearly, the set these $(m+n-2)$ vertices forms a minimum 2-geodesic set for $K_{m,n}$. Thus $g_2 p(K_{m,n}) = m + n - 2$.

The corona product of two graphs $G$ and $H$ denoted as $G\circ H$ is obtained by taking one copy of $G$ and $|V(G)|$ copies of $H$, and by joining each vertex of the $i$-th copy of $H$ with the $i$-th vertex of $G$, where $i = 1,2,\ldots,|V(G)|$.

Theorem 3.16. Let $G$ and $H$ be connected graphs with order $m$ and $n$ respectively, then $g_2 p(G \circ H) \leq ng_2 p(H) + n$

Proof. Consider connected graphs $G$ and $H$ with $o(G) = m$ and $o(H) = n$. The corona product of graphs $G$ and $H$, i.e $(G \circ H)$ contains $m$ copies of $H$. Denote the vertices in each copy of $H_i$ as $(u_i, v_i)$ where $1 \leq i \leq m$ and $1 \leq j \leq n$. Choose a set $S \subseteq V(G \circ H)$ such that it contains all the vertices of the 2-geodesic propagating set in each copies of $H_i$ and $|V(G)| - 1$ where $i = 1,2,\ldots,|V(G)|$. Clearly, all the vertices in each copy of $H_i$ will belong to some 2-geodesic between the vertices of $S$. But a vertex in $S$ can propagate to a vertex in $V(G) \setminus S$ only if all its other neighbours are already propagated. But from the construction of $S$, there exists at least one vertex in $G$ and in any one $H_i$ left non propagated. Thus $S$ does not form a 2-geodesic propagation set for $(G \circ H)$. Thus $g_2 p(G \circ H) \leq ng_2 p(H) + n$.

For the graph given in Fig 3: $g_2 p(G \circ H) = ng_2 p(H) + n$

Theorem 3.17. For $2 \leq m \leq n$ the $g_2 p(P_m \square P_n) \leq \frac{m(n-1)}{2}$

Proof. Let $G = P_m \square P_n$ with $V(G) = V(P_m) \times V(P_n) = \{(u_i, v_j) \mid u_i \in V(P_m), v_j \in V(P_n)\}$ where $i = 1,2,\ldots,m$ and $j = 1,2,\ldots,n$. Define $S \subseteq V(P_m \square P_n)$ such that $S = \{(u_a, v_b) \cup (u_a, v_b')\}$ where $|S| = \frac{mn}{2}$. Here both $a$ and $d$ are odd with $|a - d| = 0$ or $2$ and both $b$ and $b'$ are even and $|b - b'| = 0$ or $2$. Clearly all the vertices of $V(P_m \square P_n) \setminus S$ belongs to some 2-geodesic between the vertices in $S$. But all the vertices in $S$ is adjacent to at least two non propagated vertices of $V(P_m \square P_n) \setminus S$, hence cannot initiate the propagation process. From [10] the propagation number for grids, i.e $p(P_m \square P_n) = m$. Thus the $\frac{m(n-1)}{2}$ from $S$ together with the $m$ boundary vertices can initiate the propagation process and propagate the remaining non propagated vertices of $V(P_m \square P_n) \setminus S$ simultaneously. Thus $g_2 p(P_m \square P_n) \leq \frac{m(n-1)}{2} + m = \frac{m(n+1)}{2}$.

References


