Continuous mappings on cubic topological spaces

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Abstract
In this paper we have introduced and investigated some continuous mappings on P-cubic topological spaces and R-cubic topological spaces and obtained interrelations between them. We have proved and analysed some basic properties and characterization of the newly defined continuous mappings.

Keywords
Cubic sets, P-cubic topology, R-cubic topology, P-cubic continuous mappings, R-cubic continuous mappings.

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1 Introduction


2. Preliminaries

In this section some preliminary definitions with references are given.

Definition 2.1. [2] Let X be a non-empty set. Then A = \{⟨x, µ(x), λ(x)⟩ | x ∈ X\} structure is a cubic set in X in which µ is an IVFS in X and λ is a fuzzy set in X. Simply a cubic set is denoted by A = ⟨µ, λ⟩ and C^c denotes the collection of all cubic sets in X.

1. Cubic set A = ⟨µ, λ⟩ in which µ(x) = 0 and λ(x) = 1 (resp. µ(x) = 1 and λ(x) = 0) ∀x ∈ X is denoted by 0 (resp. 1).

2. Cubic set A = ⟨µ, λ⟩ in which µ(x) = 0 and λ(x) = 1 (resp. µ(x) = 1 and λ(x) = 0) ∀x ∈ X is denoted by 0 (resp. 1).

Definition 2.2. [2] Let A = ⟨µ, λ⟩ and B = ⟨β, η⟩ be two cubic sets in X. Then we define:

(a) Equal: A = B ⇔ µ = β and λ = η

(b) P-order: A = B ⊆ P µ ⊆ β and λ ≤ η

(c) R-order: A = B ⊆ R µ ⊆ β and λ ≥ η

Definition 2.3. [2] The complement of a cubic set A = ⟨µ, λ⟩ = \{⟨x, [µ^−(x), µ^+(x)], λ(x)⟩ | x ∈ X\} in X is defined to be A^c = ⟨µ^c, 1 − λ⟩ = \{⟨x, [1 − µ^+(x), 1 − µ^−(x)], 1 − λ(x)⟩ | x ∈ X\}. Obviously, (A^c)^c = A, 0^c = 1, 1^c = 0, 0 = 1 and 1 = 0

Definition 2.4. [2] For any cubic set A_i = \{⟨x, µ_i(x), λ_i(x)⟩ | x ∈ X\} where i ∈ N, we define...
(a) P-Union
\[ \bigcup_{i \in N} p A_i = \{ (x, \bigcup_{i \in N} \mu_i(x), \bigvee_{i \in N} \lambda_i(x)) | x \in X \} \]

(b) R-Union
\[ \bigcup_{i \in N} R A_i = \{ (x, \bigcup_{i \in N} \mu_i(x), \bigwedge_{i \in N} \lambda_i(x)) | x \in X \} \]

(c) P-Intersection
\[ \bigcap_{i \in N} p A_i = \{ (x, \bigcap_{i \in N} \mu_i(x), \bigwedge_{i \in N} \lambda_i(x)) | x \in X \} \]

(d) R-Intersection
\[ \bigcap_{i \in N} R A_i = \{ (x, \bigcap_{i \in N} \mu_i(x), \bigvee_{i \in N} \lambda_i(x)) | x \in X \} \]

Definition 2.5. [1] A P-cubic topology \( \mathcal{F}_P \) is the family of cubic sets in \( X \) which satisfies the following conditions:

(i) \( \hat{0}, \hat{1} \in \mathcal{F}_P \)
(ii) If \( A_i \in \mathcal{F}_P \) then
\[ \bigcup_{i \in N} p A_i \in \mathcal{F}_P \]
(iii) If \( A, B \in \mathcal{F}_P \) then
\[ A \cap_R B \in \mathcal{F}_P \]

The pair \((X, \mathcal{F}_P)\) is called the P-cubic topological space and any cubic set in \( \mathcal{F}_P \) is known as R-cubic open set in \( X \). The complement \( A' \) of a P-cubic open set \( A \) in P-cubic topological space \((X, \mathcal{F}_P)\) is called a P-cubic closed set in \( X \).

Definition 2.6. [1] A R-cubic topology \( \mathcal{F}_R \) is the family of cubic sets in \( X \) which satisfies the following conditions:

(i) \( \hat{0}, \hat{1}, \hat{0}, \hat{1} \in \mathcal{F}_R \)
(ii) If \( A_i \in \mathcal{F}_R \) then
\[ \bigcup_{i \in N} R A_i \in \mathcal{F}_R \]
(iii) If \( A, B \in \mathcal{F}_R \) then
\[ A \cap_R B \in \mathcal{F}_R \]

The pair \((X, \mathcal{F}_R)\) is called the R-cubic topological space and any cubic set in \( \mathcal{F}_R \) is known as R-cubic open set in \( X \). The complement \( A' \) of a R-cubic open set \( A \) in R-cubic topological space \((X, \mathcal{F}_R)\) is called a R-cubic closed set in \( X \). Throughout this paper \((X, \mathcal{F}_P)\) or \( X_P \) denotes the P-cubic topological space and \((X, \mathcal{F}_R)\) or \( X_R \) denotes the R-cubic topological space.

### 3. Continuous Mappings on P-Cubic Topological Spaces

In this section we have defined and analysed the basic properties and characterization of some continuous mappings like \( \alpha \) continuous mappings, semi continuous mappings, pre-continuous mappings and \( \beta \)-continuous mappings in P-cubic topological spaces.

**Definition 3.1.** Let \( f_P : X \rightarrow Y \) be a mapping and let \( A = (\mu, \lambda) \) be a cubic set in \( X \). Then the image of \( A \) under \( f_P \), denoted by \( f_P(A) = (f_P(\mu), f_P(\lambda)) \), is defined by

\[
[f_P(\mu)](y) = \begin{cases} 
\sup_{x \in X} \mu(x) & \text{if } f_P(\mu)(y) \neq \phi, \\
0 & \text{otherwise}
\end{cases}
\]

\[
[f_P(\lambda)](y) = \begin{cases} 
\sup_{x \in X} \lambda(x) & \text{if } f_P(\lambda)(y) \neq \phi, \\
0 & \text{otherwise}
\end{cases}
\]

for all \( y \in Y \), where \( f_P^{-1}(y) = \{ x \mid f_P(x) = y \} \) Let \( B = (\beta, \eta) \) be an cubic set in \( Y \). Then the inverse image of \( B \) under \( f_P \), denoted by \( f_P^{-1}(B) = (f_P^{-1}(\beta), f_P^{-1}(\eta)) \), is defined by \( f_P^{-1}(\beta)(x) = \{ y \mid (f_P(x))(y) = \beta(y) \} \) and \( f_P^{-1}(\eta)(x) = \{ y \mid (f_P(x))(y) = \eta(y) \} \).

**Definition 3.2.** Let \( X_P \) and \( Y_P \) be any two P-cubic topological spaces. A mapping \( f_P : X_P \rightarrow Y_P \) said to be a mapping.

(i) P-cubic continuous mapping if \( f_P^{-1}(A) \) is a P-cubic open set in \( X_P \) for each P-cubic open set \( A \) in \( Y_P \).

(ii) P-cubic semi continuous (resp. \( \alpha \)-continuous, \( \beta \)-continuous) mapping if \( f_P^{-1}(A) \) is a P-cubic open set (resp. \( \alpha \)-open, \( \beta \)-open) set \([4]\) in \( X_P \) for each P-cubic open set \( A \) in \( Y_P \).

Proposition 3.3. The identity mapping \( f_P : X_P \rightarrow X_P \) is a P-cubic continuous mapping.

**Proof.** Straightforward.

Proposition 3.4. The composition of two P-cubic continuous mappings is again a P-cubic continuous mapping in general.

**Proof.** Straightforward.

Proposition 3.5. Every P-cubic continuous mapping is a P-cubic semi continuous (resp. \( \alpha \)-continuous, \( \beta \)-continuous) mapping but the converses are not true in general.

**Proof.** Since every P-cubic open set is a P-cubic semi open (resp. \( \alpha \)-open, pre-open, \( \beta \)-pen) set, the result follows obviously.

The following examples show that the converses of proposition 3.5 is not true.
Example 3.6. Let $X \neq \emptyset$, $\mathcal{F}_P = \{\emptyset, A_1, A_2, A_3, 1\}$ and $\mathcal{F}_P' = \{\emptyset, B, 1\}$ be two $P$-cubic topologies on $X$ where $A_1 = \langle[0.2, 0.4], 0.2, \rangle$, $A_2 = \langle[0.3, 0.4], 0.3, \rangle$, $A_3 = \langle[0.4, 0.6], 0.5, \rangle$ and $B = \langle[0.2, 0.5], 0.25, \rangle$. Define a mapping $f_p : (X, \mathcal{F}_P) \to (X, \mathcal{F}_P')$ by $f_p(x) = x$, then $f_p$ is a $P$-cubic $\alpha$-continuous mapping but not a $P$-cubic continuous mapping.

Example 3.7. Let $X \neq \emptyset$, $\mathcal{F}_P = \{\emptyset, A_1, A_2, A_3, 1\}$ and $\mathcal{F}_P' = \{\emptyset, C, 1\}$ be two $P$-cubic topologies on $X$ where $A_1 = \langle[0.2, 0.4], 0.2, \rangle$, $A_2 = \langle[0.3, 0.4], 0.3, \rangle$, $A_3 = \langle[0.4, 0.6], 0.5, \rangle$, $A_4 = \langle[0.5, 0.7], 0.6, \rangle$ and $C = \langle[0.3, 0.5], 0.4, \rangle$. Define a mapping $g_p : (X, \mathcal{F}_P) \to (X, \mathcal{F}_P')$ by $g_p(x) = x$, then $g_p$ is a $P$-cubic semi continuous mapping but not a $P$-cubic continuous mapping.

Example 3.8. Let $X \neq \emptyset$, $\mathcal{F}_P = \{\emptyset, A_1, A_2, A_3, 1\}$ and $\mathcal{F}_P' = \{\emptyset, D, 1\}$ be two $P$-cubic topologies on $X$ where $A_1 = \langle[0.2, 0.4], 0.2, \rangle$, $A_2 = \langle[0.3, 0.4], 0.3, \rangle$, $A_3 = \langle[0.4, 0.6], 0.5, \rangle$ and $D = \langle[0.1, 0.5], 0.3, \rangle$. Define a mapping $h_p : (X, \mathcal{F}_P) \to (X, \mathcal{F}_P')$ by $h_p(x) = x$, then $h_p$ is a $P$-cubic pre-continuous mapping and $P$-cubic $\beta$-continuous mapping, but not a $P$-cubic continuous mapping.

Theorem 3.9. A mapping $f_p : X_P \to Y_P$ is a $P$-cubic $\alpha$-continuous mapping if and only if it is both a $P$-cubic semi continuous mapping and $P$-cubic pre-continuous mapping.

Proof. Let $f_p$ be both a $P$-cubic semi continuous mapping and a $P$-cubic pre-continuous mapping. Let $A$ be a $P$-cubic open set in $Y_P$, then by hypothesis $f_p^{-1}$ is a $P$-cubic semi open set and $P$-cubic pre-open set. Hence by proposition 3.25[6], $f_p^{-1}$ is a $P$-cubic $\alpha$ open set and hence it is a $P$-cubic $\alpha$-continuous mapping. The converse is immediate.

Theorem 3.10. Let $f_p : X_P \to Y_P$ be a mapping. Then

(i) $f_p^{-1}(B^c) = [f_p^{-1}(B)]^c$ for all cubic sets $B$ in $Y$.

(ii) $[f_p(A)]^c \subseteq P f_p(A^c)$ for all cubic sets $A$ in $X$.

(iii) $B_1 \subseteq P B_2$ implies $f_p^{-1}(B_1) \subseteq P f_p^{-1}(B_2)$, where $B_1$ and $B_2$ are cubic sets in $Y$.

(iv) $A_1 \subseteq P A_2$ implies $f_p(A_1) \subseteq P f_p(A_2)$, where $A_1$ and $A_2$ are cubic sets in $Y$.

(v) $f_p(f_p^{-1}(B)) \subseteq P B$ the equality holds if $f_p$ is surjective, for all cubic sets $B$ in $Y$.

(vi) $A \subseteq P f_p^{-1}(f_p(A))$ the equality holds if $f_p$ is injective, for all cubic sets $A$ in $X$.

(vii) $f_p^{-1}(\bigcup_{i \in \Lambda} pB_i) = \bigcup_{i \in \Lambda} f_p^{-1}(B_i)$

for all cubic sets $B_i$ in $Y$.

$viii$ $f_p^{-1}(\bigcap_{i \in \Lambda} pB_i) = \bigcap_{i \in \Lambda} f_p^{-1}(B_i)$

for all cubic sets $B_i$ in $Y$.
Therefore, by cases I and II we get $f_P(f_P^{-1}(B)) \subseteq P B$ when $f_P$ is surjective, for all $y \in Y$. So by case I the equality holds. (i)

Let $A = \langle \mu, \lambda \rangle$ be a cubic set in $X$. Then

$$f_P^{-1}(f_P(A))(x) = \langle \phi, [\mu(f_P(x))]^-, \phi, \mu(f_P(x))^+ \rangle, \phi, \lambda(f_P(x)) \rangle$$

$$= \langle z, \sup_{z = f_P^{-1}(f_P(x))} \mu^-(z), \sup_{z = f_P^{-1}(f_P(x))} \mu^+(z), \sup_{z = f_P^{-1}(f_P(x))} \lambda(z) \rangle$$

$$= \langle z, [\mu^-(x), \mu^+(x), \lambda(x)] \rangle \forall x \in X$$

Therefore $A \subseteq f_P^{-1}(f_P(A))$

(ii) Let $B_1 = \langle \beta, \eta \rangle$ be a cubic set in $Y$ and $y \in Y$. Then

$$f_P^{-1}(\bigcup_{i \in A} pB_i)(y) = \left( \bigcup_{i \in A} pB_i \right)(y)$$

$$= \bigcup_{i \in A} pB_i(f_P(y))$$

$$= \bigcup_{i \in A} p f_P^{-1}(B_i(y))$$

(iii) Let $B_1 = \langle \beta, \eta \rangle$ be a cubic set in $Y$ and $y \in Y$. Then

$$f_P^{-1}(\bigcap_{i \in A} pB_i)(y) = \left( \bigcap_{i \in A} pB_i \right)(y)$$

$$= \bigcap_{i \in A} pB_i(f_P(y))$$

$$= \bigcap_{i \in A} p f_P^{-1}(B_i(y))$$

$$\Box$$

**Theorem 3.11.** If $X_P$ and $Y_P$ are any two $P$-cubic topological spaces and $f_P$ is a mapping from $X_P$ to $Y_P$, then the following statements are equivalent:

(i) The mapping $f_P$ is continuous

(ii) The inverse image of every $P$-cubic closed set is $P$-cubic closed

(iii) For each cubic point $P_i$ [3] in $X$ the inverse image of every neighbourhood of $f_P(P_i)$ under $f_P$ is a neighbourhood of $P_i$.

(iv) For each cubic point $P_i$ in $X$ and each neighbourhood $V$ of $f_P(P_i)$, there is a neighbourhood $W$ of $P_i$ such that $f_P(W) \subseteq P V$.

**Proof.** (i) $\Leftrightarrow$ (ii): The result is obvious as $f_P^{-1}(B^c) = [f_P^{-1}(B)]^c$ for any cubic set $B$.

(i) $\Leftrightarrow$ (iii) Assume that the mapping $f_P$ is continuous and let $B$ be a neighbourhood of $f_P(P_i)$. Then there exists a $P$-cubic open set $U$ such that $f_P(P_i) \subseteq U \subseteq P B$. Now $P_i \in f_P^{-1}(f_P(P_i)) \subseteq \subseteq P$, where $f_P^{-1}(B)$ is a $P$-cubic open set in $X$ implying that the inverse of every neighbourhood of $f_P(P_i)$ under $f_P$ is a neighbourhood of $P_i$.

(iii) $\Leftrightarrow$ (i): Let $f_P(P_i)$ be an arbitrary $P$-cubic point of a $P$-cubic open set of $f_P$. Then $B$ is a neighbourhood of $f_P(P_i)$. By hypothesis, $f_P^{-1}(B)$ is a neighbourhood of $P_i$, then there is a $P$-cubic open set $U_i$ such that $P_i \in U_i \subseteq P f_P^{-1}(B)$. Then $B = \bigcup_{P_i \in B} U_i$ is a $P$-union of $P$-cubic open set of $X_P$ which implies $f_P^{-1}(B)$ is a $P$-cubic open set of $X_P$. (iv) $\Leftrightarrow$ (iii): Let $P_i$ be a cubic point in $X$ and $V$ be a neighbourhood of $f_P(P_i)$, then by (iii) $f_P^{-1}(V)$ is a neighbourhood of $P_i$, we have $f_P(W) = f_P[f_P^{-1}(V)] \subseteq P V$ where $W = f_P^{-1}(V)$

(iv) $\Leftrightarrow$ (iv): Then there is a neighbourhood $W$ of $P_i$ such that $f_P(W) \subseteq P V$ . Hence $f_P[f_P^{-1}(V)] \subseteq P f_P^{-1}(V)$. Futhermore, since $W \subseteq P f_P^{-1}(W)$, $f_P^{-1}(V)$ is a neighbourhood of $P_i$.

\[ \Box \]

**4. Continuous mappings on $R$-cubic topological spaces**

In this section we have defined and analysed the basic properties and characterization of some continuous mappings like $\alpha$-continuous mappings, semi continuous mappings, pre-continuous mappings and $\bar{\beta}$-continuous mappings in $R$-cubic topological spaces.

**Definition 4.1.** Let $f_R : X \to Y$ be a mapping and let $A = \langle \mu, \lambda \rangle$ be a cubic set in $X$. Then the image of $A$ under $f_R$, denoted by $f_R(A) = (f_R(\mu), f_R(\lambda))$, is defined by

$$[f_R(\mu)(y)]^c = \begin{cases} \sup_{A(x) = \eta} [\mu(x)]^c, & \text{if } f_R^{-1}(y) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

$$[f_R(\mu)(y)]^c = \begin{cases} \sup_{A(x) = \eta} [\mu(x)]^c, & \text{if } f_R^{-1}(y) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

$$[f_R(\lambda)(y)] = \begin{cases} \inf_{x \in f_R^{-1}(y)} [\lambda(x)], & \text{if } f_R^{-1}(y) \neq \phi \\ 0, & \text{otherwise} \end{cases}$$

for all $y \in Y$, where $f_R^{-1}(y) = \{ x \in f_R(x) = y \}$. Let $B = \langle \beta, \eta \rangle$ be an cubic set in $Y$. Then the inverse image of $B$ under $f_R$, denoted by $f_R^{-1}(B) = (f_R^{-1}(\beta), f_R^{-1}(\eta))$, is defined by

$$f_R^{-1}(\beta)(x) = \begin{cases} \beta(f_R(x)), & \text{if } f_R^{-1}(\beta)(x) \neq \phi \\ \eta(f_R(x)), & \text{otherwise} \end{cases}$$

$$f_R^{-1}(\eta)(x) = \eta(f_R(x)), \forall x \in X.$$

**Definition 4.2.** Let $X_R$ and $Y_R$ be any two $R$-cubic topological spaces. A mapping $f_R : X_R \to Y_R$ said to be a

(i) $R$-cubic continuous mapping if $f_R^{-1}(A)$ is a $R$-cubic open set in $X_R$ for each $R$-cubic open set $A$ in $Y_R$.

(ii) $R$-cubic semi continuous (resp. $\alpha$-continuous, pre-continuous, $\beta$-continuous) mapping if $f_R^{-1}(A)$ is a $R$-cubic semi open (resp. open, pre-open, $\beta$-open) set in $X_R$ for each $R$-cubic open set $A$ in $Y_R$.

**Proposition 4.3.** The identity mapping $f_R : X_R \to X_R$ is a $R$-cubic continuous mapping.
Theorem 4.10. Let $f : A \to X$ be a mapping (resp. $Y \to R$).

Proof. Straightforward.

Proposition 4.4. The composition of two $R$-cubic continuous mappings is again a $R$-cubic continuous mapping in general.

Proof. Straightforward.

Proposition 4.5. Every $R$-cubic continuous mapping is a $R$-cubic semi continuous (resp. $\alpha$-continuous, pre-continuous, $\beta$-continuous) mapping but the converses are not true in general.

Proof. Since every $R$-cubic open set is a $R$-cubic semi open (resp. $\alpha$-open, pre-open, $\beta$-open) set, the result follows obviously.

The following examples show that the converses of proposition 4.5 is not true.

Example 4.6. Let $X \neq \emptyset, F_R = \{X, B, 1\}$ and $\overline{F}_R = \{X, B, \hat{1}\}$ be two $R$-cubic topologies on $X$ where $A_1 = (X, B, 0.4), A_2 = (X, B, 0.6), A_3 = (X, B, 0.8), A_4 = (X, B, 0.4), A_5 = (X, B, 0.4), A_6 = (X, B, 0.3), A_7 = (X, B, 0.4)$. Define a mapping $f_R : (X, F_R) \to (X, \overline{F}_R)$ by $f_R(x) = x$, then $f_R$ is a $R$-cubic $\alpha$-continuous mapping but not a $R$-cubic continuous mapping.

Example 4.7. Let $X \neq \emptyset, F_R = \{X, B, 1\}$ and $\overline{F}_R = \{X, B, \hat{1}\}$ be two $R$-cubic topologies on $X$ where $A_1 = (X, B, 0.4), A_2 = (X, B, 0.6), A_3 = (X, B, 0.8), A_4 = (X, B, 0.4), A_5 = (X, B, 0.4), A_6 = (X, B, 0.3), A_7 = (X, B, 0.4)$. Define a mapping $g_R : (X, F_R) \to (X, \overline{F}_R)$ by $g_R(x) = x$, then $g_R$ is a $R$-cubic semi continuous mapping but not a $R$-cubic continuous mapping.

Example 4.8. Let $X \neq \emptyset, F_R = \{X, B, 1\}$ and $\overline{F}_R = \{X, B, \hat{1}\}$ be two $R$-cubic topologies on $X$ where $A_1 = (X, B, 0.4), A_2 = (X, B, 0.6), A_3 = (X, B, 0.8), A_4 = (X, B, 0.4), A_5 = (X, B, 0.4), A_6 = (X, B, 0.3), A_7 = (X, B, 0.4)$. Define a mapping $h_R : (X, F_R) \to (X, \overline{F}_R)$ by $h_R(x) = x$, then $h_R$ is a $R$-cubic pre-continuous mapping and a $R$-cubic $\beta$-continuous mapping, but not a $R$-cubic continuous mapping.

Theorem 4.9. A mapping $f_R : X_R \to Y_R$ is a $R$-cubic $\alpha$-continuous mapping if and only if it is both a $R$-cubic semi continuous mapping and $R$-cubic pre-continuous mapping.

Proof. Let $f_R$ be both a $R$-cubic semi continuous mapping and a $R$-cubic pre-continuous mapping. Let $A$ be a $R$-cubic open set in $Y_R$, then by hypothesis $f_R^{-1}(A)$ is a $R$-cubic semi open set and $R$-cubic pre-open set. Hence by proposition 4.25[6], $f_R^{-1}$ is a $P$-cubic $\alpha$-open set and hence it is a $R$-cubic $\alpha$-continuous mapping. The converse is immediate.

Theorem 4.10. Let $f_R : X_R \to Y_R$ be a mapping. Then

(i) $f_R^{-1}(B^c) = f_R^{-1}(B)^c$ for all cubic sets $B$ in $Y$.

(ii) $[f_R(A)]^c \subseteq f_R(A^c)$ for all cubic sets $A$ in $X$.

(iii) $B_1 \subseteq B_2 \implies f_R^{-1}(B_1) \subseteq f_R^{-1}(B_2)$, where $B_1$ and $B_2$ are cubic sets in $Y$.

(iv) $A_1 \subseteq A_2 \implies f_R(A_1) \subseteq f_R(A_2)$, where $A_1$ and $A_2$ are cubic sets in $Y$.

(v) $f_R^{-1}(B) \subseteq B$ the equality holds if $f_R$ is surjective, for all cubic sets $B$ in $Y$.

(vi) $A \subseteq f_R^{-1}(f_R(A))$ the equality holds if $f_R$ is injective, for all cubic sets $A$ in $X$.

(vii) $f_R^{-1}(\bigcup_{i \in A} rB_i) = \bigcup_{i \in A} r f_R^{-1}(B_i)$ for all cubic sets $B_i$ in $Y$.

(viii) $f_R^{-1}(\bigcap_{i \in A} rB_i) = \bigcap_{i \in A} r f_R^{-1}(B_i)$ for all cubic sets $B_i$ in $Y$.

Proof. (i) Let $B = (\beta, \eta)$ be a cubic set in $Y$. Then

$$f_R^{-1}(B^c)(x) = (x, f_R^{-1}(B^c)(y), f_R^{-1}(\eta^c)(y))$$

(ii) $A = (\mu, \lambda)$ be a cubic set in $X$ and $f_R^{-1}(A) \neq \emptyset$. Then $A^c = \{x, [1 - \mu^{-}(x), 1 - \mu^{-}(x)], 1 - \lambda(x), \}$, we have

$$f_R(A)^c(y) = 1 - f_R(A)(y)$$

$$= 1 - \gamma, [\sup(\mu^-(x)), \sup(\mu^+(x)), \sup(\lambda(x))]$$

$$= (1 - y, [1 - \sup(\mu^-(x)), 1 - \sup(\mu^+(x)), 1 - \sup(\lambda(x))]$$

$$f_R(A)^c(y) = \gamma, [\sup(1 - \mu^-(x)), \sup(1 - \mu^+(x)), \sup(1 - \lambda(x))]$$

Therefore, $[f_R(A)]^c \subseteq f_R(A^c)$

Therefore, $[f_R(A)]^c \subseteq f_R(A^c)$
(v) Let \( B = \langle \beta, \eta \rangle \) be a cubic set in \( Y \) and \( y \in Y \). 

Case I: \( f_R^{-1}(y) \neq \phi \)

\[
[f_R(f_R^{-1}(\beta))(y)]^- = \sup_{y = f(x)} [f_R^{-1}(\beta)(x)]^- \\
= \sup_{y = f(x)} [\beta f(x)]^- = [\beta(y)]^-
\]

Similarly,

\[
[f_R(f_R^{-1}(\eta))(y)]^- = [\eta(y)]
\]

(iii): Let \( B_i = \langle \beta_i, \eta_i \rangle \) be a cubic set in \( X \) and \( y \in Y \). Then

\[
f_R^{-1}(\bigcup_{i \in \Lambda} B_i)(y) = \left( \bigcup_{i \in \Lambda} B_i \right)(y) \\
= \bigcup_{i \in \Lambda} R(B_i(y)) \\
= \bigcup_{i \in \Lambda} R f_R^{-1}(B_i(y))
\]

Theorem 4.11. If \( X_R \) and \( Y_R \) are any two R-cubic topological spaces and \( f_R \) is a mapping from \( X_R \) to \( Y_R \), then the following statements are equivalent:

(i) The mapping \( f_R \) is continuous

(ii) The inverse image of every R-cubic closed set is R-cubic closed

(iii) For each cubic point \( R_x \) in \( X \) the inverse image of every neighbourhood of \( f(R_x) \) under \( f_R \) is a neighbourhood of \( R_x \).

(iv) For each cubic point \( R_x \) in \( X \) and each neighbourhood \( V \) of \( f(R_x) \), there is a neighbourhood \( W \) of \( R_x \) such that \( f_R(W) \subset R \).

Proof. (i) \( \Leftrightarrow \) (ii): The result is obvious as \( f_R^{-1}(B^c) = [f_R^{-1}(B)]^c \) for any cubic set \( B \).

(i) \( \Leftrightarrow \) (iii): Assume that the mapping \( f_R \) is continuous and let \( B \) be a neighbourhood of \( f_R(R_x) \). Then there exists a R-cubic open set \( U \) such that \( f_R(R_x) \subset U \subset B \). Now \( R_x \subset f_R^{-1}(f(R_x)) \subset R \), where \( f_R^{-1}(B) \) is a R-cubic open set in \( X \) implying that the inverse of every neighbourhood of \( f_R(R_x) \) under \( f_R \) is a neighbourhood of \( R_x \).

(iii) \( \Leftrightarrow \) (i): Let \( f_R(R_x) \) be an arbitrary R-cubic point of a R-cubic open set \( B \) of \( Y_R \). Then \( B \) is a neighbourhood of \( f_R(R_x) \). By hypothesis, \( f_R^{-1}(B) \) is a neighbourhood of \( R_x \), then there is a R-cubic open set \( U_x \) such that \( R_x \subset U_x \subset f_R^{-1}(B) \). Then \( B = \bigcup_{R_x \subset U_x} \) is a R-bounded union of R-cubic open sets of \( X_R \) which implies \( f_R^{-1}(B) \) is a R-cubic open set of \( X_R \). (iii) \( \Leftrightarrow \) (iv): Let \( R_x \) be a cubic point in \( X \) and \( V \) be a neighbourhood of \( f(R_x) \), then by (iii) \( f_R^{-1}(V) \) is a neighbourhood of \( R_x \). (iv) \( \Leftrightarrow \) (iii): Let \( V \) be a neighbourhood of \( f(R_x) \). Then there is a neighbourhood \( W \) of \( R_x \) such that \( f_R(W) \subset R \) . Hence \( f_R^{-1}[f_R(W)] \subset R \). Futhermore, since \( W \subset R \), \( f_R^{-1}(W) \) is a neighbourhood of \( R_x \).

References


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