Malaya Journal of Matematik, Vol. S, No. 1, 48-51, 2020
https://doi.org/10.26637/MJM0S20/0010

Connected domination number of cartesian product graphs of Cayley graphs with arithmetic graphs

S. Uma Maheswari[1] M. Siva Parvathi[2]* B.Bhatathi [3] and M. Venkata Anusha [4]

Abstract
The notation $G_1 \square G_2$ is the Cartesian product of two graphs . The vertex set of $G_1 \square G_2$ is $V_1 \times V_2$, ,where $V_1 \times V_2$ is the Cartesian product of the sets V_1 and V_2 and any two distinct vertices (u_1, v_1) and (u_2, v_2) of $G_1 \square G_2$ are adjacent, if $u_1 = u_2$ and $v_1 v_2 \in (G_2)$ or $u_1 u_2 \in E(G_1)$ and $v_1 = v_2$. In this paper we obtain some results related to connected domination number of Cartesian product graphs of Euler Totient Cayley graphs with Arithmetic V_n graphs.

Keywords
Euler totient Cayley graph, Arithmetic V_n graph, Cartesian product graph, Connected domination.

AMS Subject Classification 2010
6905 c

[1,3] *Department of Mathematics, J.M.J. College for Women, Tenali-522202, Andhra Pradesh, India.*
[2,4] *Department of Applied Mathematics,Sri Padmavati MahilaVisvavidyalayam, Tirupati-517502, Andhra Pradesh, India.*
*__Corresponding author__: parvathimani2008@gmail.com
Article History: Received **01** January **2020**; Accepted **12** February **2020**

Contents

1. Introduction

Domination in Graphs is a prospering region of research at present. Domination sets assume a significant job in down to earth applications. for example, co-ordinations and systems structure, portable processing, asset designation and media transmission and so forth. Cayleygraphs are great models for interconnection systems, explored regarding equal handling and circulated calculation.

Nathanson [1] was the pioneer in presenting the ideas of Number Theory, especially, the 'Hypothesis of Congruences' in Graph Theory, hence made ready for the development of another class ofgraphs, in particular "Arithmetic Graphs". Cayley Graphs are another class of graphs related with components of a group. In this group is related with some Arithmetic functions, at that point the Cayley graph turns into an Arithmetic graph. The Cayley graph related with Euler totient function is known as an Euler totient Cayley graph.

The Cartesian product of graphs is a straight forward and common development. As per Imrich and Klavzar [2], Cartesian results of graph were characterized in 1912 by Whitehead and Russell [3]. These products were repeatedly rediscovering later, outstandingly by [4] in 1960.

2. Euler Totient Cayley Graph

The Euler totient Cayley graph $G(Z_n, \phi)$ is a graph whose vertex set V is given by Z_n=$\{0, 1, 2, 3, \ldots, n-1\}$ and the edge set is E=$\{(x, y) \setminus x - y \in or y - x \in S\}$, where S denotes the set of all positive integers less than n and relatively prime to n . That is $S = \{r \setminus 1 \leq r \leq n \quad and \quad GCD(r, n) = 1\}$. Then $|S|$=$\phi(n)$. Some properties of Euler totient Cayley graphs and enumeration of Hamilton cycles and triangles can be found in Madhavi [5]. The Euler totient Cayley graph is a complete graph if n is a prime and it is $\phi(n)$- regular. The domination parameters of these graphs are studied by the authors [6] and we require the following results and we present them without proofs.

Lemma 2.1. *[6] If n is a prime, then the domination number* *of $G(Z_n,\phi)$ is* 1.

Lemma 2.2. *[6] If n is power of a prime, then the domination number of $G(Z_n,\phi)$ is* 2.

Lemma 2.3. *[6] The domination number of $G(Z_n,\phi)$ is 2, if $n=2p$ where p is an odd prime.*

Lemma 2.4. *[6] Suppose n is neither a prime nor 2p. Let $n=p_1^{(\alpha_1)}p_2^{(\alpha_2)}\ldots\ldots.p_k^{(\alpha_k)}$, where $p_1,p_2,\ldots p_k$ are primes and $\alpha_1,\alpha_2,\ldots.\alpha_k$ are integers ≥ 1. Then the domination number of $G(Z_n,\phi)$ is given by $\gamma(G(Z_n,\phi))=\lambda+1$, where λ is the length of the longest stretch of consecutive integers in V, each of which shares a prime factor with n.*

3. Arithmetic V_n graph

Let $n>1$, such that $n=p_1^{(\alpha_1)}p_2^{(\alpha_2)}\ldots\ldots.p_k^{(\alpha_k)}$. Then the Arithmetic V_n graph is characterized as the graph whose vertex set comprises of the divisors of n and two vertices u,v are adjacent in V_n graph if and only if $GCD(u,v)=p_i$, for some prime divisor p_i of n.

In this graph vertex '1' becomes an isolated vertex. Hence we ignore Arithmetic V_n graph with isolated vertex, as the contribution of vertex '1'is nothing, in the study of some domination parameters and properties of these graphs.

For every prime n, V_n contains an isolated vertex so V_n is connected. Otherwise, by the adjacency of V_n graph, there are edges between prime number vertices and their prime power vertices and also to their prime product vertices. So, V_n is connected for every n.
The domination parameters of these graphs are obtained by the author and the proof of the following theorem can be found in [7].

Theorem 3.1. *[7] If $n=p_1^{(\alpha_1)}p_2^{(\alpha_2)}\ldots\ldots.p_k^{(\alpha_k)}$. where $p_1,p_2,\ldots p_k$ are primes and $\alpha_1,\alpha_2,\ldots.\alpha_k$ are integers ≥ 1, then the domination number of $G(V_n)$ is given by*

$$\gamma(G(V_n))=\begin{cases} k-i, & \text{if } \alpha_i=1 \text{ for more than one } i \\ k, & \text{Otherwise.}\end{cases}$$

where k is the core of n.

4. Cartesian Product Graph of $G(Z_n,\phi)$ with $G(V_n)$

In this paper we consider the Cartesian product graph of Euler Totient Cayley graph with Arithmetic V_n graph. The properties and some domination parameters of these graphs are studied by the author in [8].
Let G_1 denote the Euler Totient Cayley graph $G(Z_n,\phi)$ and G_2 denote the Arithmetic graph $G(V_n)$. Then G_1 and G_2 are simple graphs as they have no loops and multiple edges. Hence by the definition of adjacency in Cartesian product, $G_1\square G_2$ is also a simple graph. The Cartesian product graph $G_1\square G_2$is a complete graph, if n is a prime and the degree of a vertex in $G_1\square G_2$ is given by $deg_{(G_1\square G_2)}\ (u_i,v_j)$=$deg_{(G_1)(u_i)}+deg_{(G_2)(v_j)}$.

5. Connected Dominating Sets of Cartesian Product Graph

In this section we find minimum connected dominating sets of Cartesian product graph of $G(Z_n,\phi)$ graph with $G(V_n)$ graph and obtain its connected domination number in various cases.

Let $G(V,E)$ be a graph and $u,v\in V$. A subset D of $V(G)$ is a dominating set of G if every vertex in $V-D$ is adjacent to a vertex in D. The dominating set D of G is said to be a connected dominating set if the induced subgraph $\langle\, D\,\rangle$ is connected. The minimum cardinality of a connected dominating set of G is called the connected domination number of G and is denoted by $\gamma_c(G)$.

Theorem 5.1. *If n is a prime, then the connected domination number of $G_1\square G_2$ is 1 .*

Proof. $n=11$

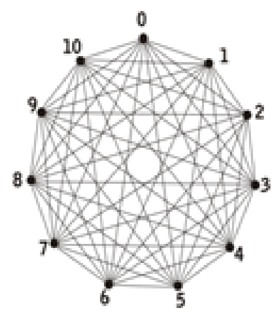

Figure1: $G_1=G(Z_{11},\phi)$

Figure2: $G_2=G(V_{11})$

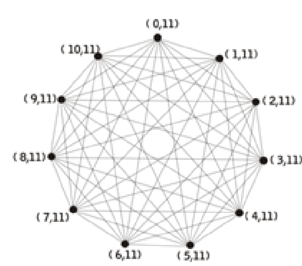

Figure3: $G_1\square G_2$
Connected dominating set: $\{(0,11)\}$

Suppose n is a prime, then the graph $G_1\square G_2$ is a complete graph and every single vertex of $G_1\square G_2$ constitutes a

dominating set. Obviously this dominating set becomes a connected dominating set.
Hence $\gamma_c(G_1 \Box G_2) = 1$. □

Theorem 5.2. *Suppose $n = 2p$ where p is an odd prime, then the connected domination number of $G_1 \Box G_2$ is 8. .*

Proof. n=2 · 3=6

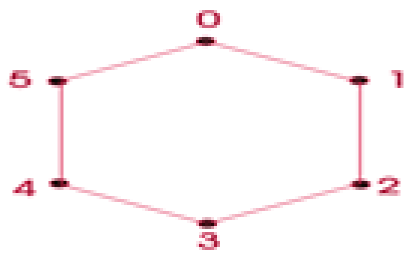

Figure4: $G_1 = G(Z_6, \phi)$

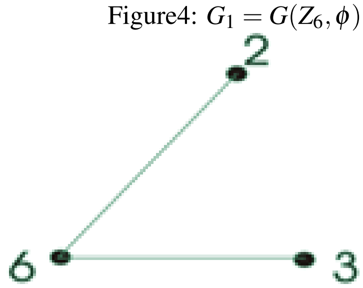

Figure5: $G_2 = G(V_6)$

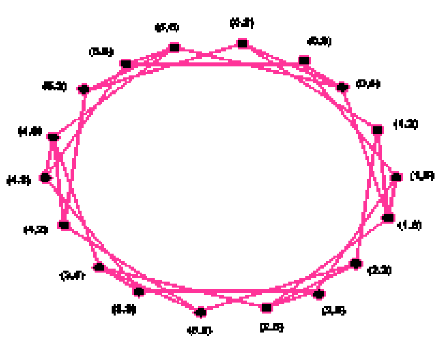

Figure6: $G_1 \Box G_2$
Connected dominating set:
$\{(0,2),(0,3),(0,6),(1,6),(2,6),(3,2),(3,3),(3,6)\}$

Suppose $n = 2p$, where p is an odd prime. Consider the graph $G_1 \Box G_2$. To find a connected dominating set of $G_1 \Box G_2$, we start our investigation with the dominating set D of G_1 G_2 with minimum cardinality, which is defined in Theorem 2.3 as $D = \{(U_{(d_1)},2),(u_{(d_1)},p),(u_{(d_1)},2p),(u_{(d_2)},2),(u_{(d_2)},p),(u_{(d_2)},2p)\}$
Where $|u_{(d_1)} - u_{(d_2)}|$=p.

Since $GCD(2,2p) = 2$ and $GCD(p,2p) = p$, it follows that the vertices $(U_{(d_1)},2),(U_{(d_1)},2p)$ are connected to the vertex $(U_{(d_1)},2p)$. By the same argument, vertices$(u_{(d_2)},2),(u_{(d_2)},p)$, are connected to the vertex $(u_{(d_2)},2p)$.

Further $|u_{(d_1)} - u_{(d_2)}|$=p. implies that $u_{(d_1)} \neq u_{(d_2)}$ and $GCD(u_{(d_1)} - u_{(d_2)},n) \neq 1$. Hence vertex $u_{(d_1)}$ is not adjacent to vertex $u_{(d_2)}$. From this it is clear that no vertex of $\{(U_{(d_1)},2),(u_{(d_1)},p),(u_{(d_1)},2p)\}$ is connected to any vertex of $\{(u_{(d_2)},2),(u_{(d_2)},p),(u_{(d_2)},2p)\}$. Thus D is not a connected dominating set.

In order to connect these two subsets of D, we have to include a vertex $(u_{(d_1)}+1,2p)$ into D. Now the included vertex $(u_{(d_1)}+1,2p)$ is connected to the vertex $(u_{(d_1)},2p)$ as $u_{(d_1)}+1-u_{(d_1)} = 1$ and $GCD(1,n) = 1$.

Further the included vertex $(u_{(d_1)}+1,2p)$ in D is not connected to any one of $(u_{(d_2)},2),(u_{(d_2)},p),(u_{(d_2)},2p)$ because $|u_{(d_1)} - u_{(d_2)}| = p$, p is an odd prime, and hence $|u_{(d_1)}+1-u_{(d_2)}| = p+1$ which is an even number and hence $GCD(u_{(d_1)}+1-u_{(d_2)},n) = GCD(u_{(d_1)}+1-u_{(d_2)},2p) \neq 1$.

Hence we need to include another vertex $(u_{(d_2)}-1,2p)$ into D_c, which is connected to the vertex $(u_{(d_2)},2p)$ as $u_{(d_2)} - (u_{(d_2)}-1) = 1$ and $GCD(1,n) = 1$. (Such a vertex $(u_{(d_2)} - 1,2p)$ must exist because $|u_{(d_1)} - u_{(d_2)}| = p$).). Now the newly included vertex $(u_{(d_2)}-1,2p)$ in D is connected to the vertex $(u_{(d_1)}+1,2p)$ as $(u_{(d_2)}-1)-(u_{(d_1)}+1) = p-2$ and $GCD(p-2,n) = 1$. (Since p is odd we have $p-2$ is odd). Thus the set $D = \{(U_{(d_1)},2),(u_{(d_1)},p),(u_{(d_1)},2p),(u_{(d_1)}+1,2p),(u_{(d_2)},2),(u_{(d_2)},p),(u_{(d_2)},2p)(u_{(d_1)}-1,2p)\}$becomes a dominating set of $(G_1 \Box G_2)$ with minimum cardinality and which is connected.

Hence γ_c $(G_1 \Box G_2)$ = $|D'|$=8.

□

Theorem 5.3. *If n is not a prime, then γ_c $(G_1 \Box G_2)$ =λ+1·$|V_2|$ where λ is the length of the longest stretch of consecutive integers in V_1 of G_1 each of which shares a prime factor with n.*

Proof. $n = 2^2 = 4$
In Lemma 2.4, we proved that
$D_{(c_1)}$=$\{u_{(d_1)},u_{(d_2)},\ldots,u_{(d_(\lambda+1))}\}$ is a dominating set of G_1 with minimum cardinality λ+1.
Let V_2=$\{v_1,v_2,\ldots,v_m\}$ be the vertex set of G_2.
Consider D_c= $D_{(c_1)} \times V_2$
= $\{u_{(d_1)},u_{(d_2)},\ldots,u_{(d_{(\lambda+1)})}\} \times \{v_1,v_2,\ldots,v_m\}$
=$\{(u_{(d_1)},v_1),(u_{(d_1)},v_2),\ldots,(u_{(d_1)},v_m)\}$,
$\{(u_{(d_2)},v_1),(u_{(d_2)},v_2),\ldots,(u_{(d_2)},v_m),\ldots,$
$(u_{(d_{(\lambda+1)})},v_1),(u_{(d_{(\lambda+1)})},v_2),\ldots,(u_{(d_{(\lambda+1)})},v_m)\}$
We now prove that D_c is connected. Since $u_{(d_1)},u_{(d_2)},\ldots,u_{(d_(\lambda+1))}$ are consecutive positive integers, we have $|u_{(d_1)} - u_{(d_2)}|$=1. Hence each $u_{(d_i)}$ is adjacent to $u_{(d_i+1)}$ for 1≤i≤ λ. Since n is not a prime, the graph G_2 it self is

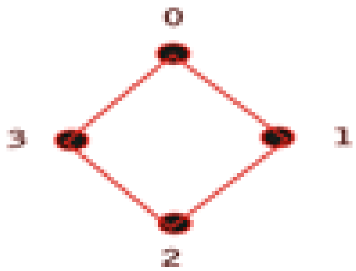

Figure7: $G_1 = G(Z_4, \phi)$

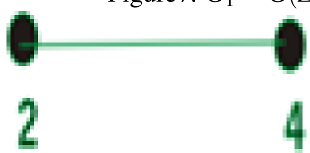

Figure8: $G_2 = G(V_4)$

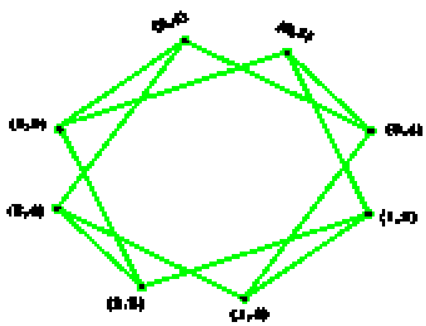

Figure9: $G_1 \Box G_2$
Connected dominating set: $\{(0,2),(0,4),(1,2),(1,4)\}$

a connected graph. It means, if v_j is any vertex in $v_1, v_2, \ldots, v_m$ then v_j is connected to each of the other vertices of $v_1, v_2, \ldots, v_m$. Hence by the definition of the Cartesian product we can see that , each $u_{(d_i)}$,v_j is connected to the vertices $(u_{(d_i+1)}, v_j)$ and $(u_{(d_i)}, v_l)$ where 1≤i≤ λ and v_j , $v_l \in \{v_1, v_2, \ldots, v_m\}$. Thus the dominating set D_c of $G_1 \Box G_2$. becomes a connected dominating set of $G_1 \Box G_2$. Hence γ_c $G_1 \Box G_2 \geq |D_c|$=$(\lambda+1) \cdot |V_2|$.

Suppose we delete a vertex, say $(u_{(d_i)}, v)$ from D for some i, 1≤i≤ λ+1. Since each vertex in G_1 is of degree $\phi(n)$, vertex $u_{(d_i)}$ is adjacent to the vertices, say $u_1, u_2, \ldots, u_{\phi(n)}$ respectively. Then the vertices $(u_1, v), (u_2, v), \ldots, (u_{\phi(n)}, v)$ are all not dominated by other vertices of $D - \{u_{(d_i)}, v)\}$. If so then $u_1, u_2, \ldots, u_{\phi(n)}$ are also dominated by other vertices of $u_1, u_2, \ldots, u_{\phi(n)}$ which implies that D_1 is not a minimum dominating set of G_1, a contradiction.
Therefore D is a minimal connected dominating set of $G_1 \Box G_2$.
Hence $\lambda_c(G_1 \Box G_2)$ =$(\lambda+1)|V_2|$.

□

6. Conclusion

The objective of this work is to familiarize the reader with the Cartesian product graph of Euler Totient Cayley graph with Arithmetic V_n graph. The purpose of studying basic properties and various domination parameters of Cartesian product graphs that arise from the Arithmetic graphs and Cayley graphs is to enhance the behavioural aspects of these graphs and gives new dimensions to the theory of Cartesian product graphs. It is useful other Researchers for further studies of other properties of these product graphs and their relevance in both combinatorial problems and classical algebraic problems.

7. Acknowledgment

We are very much thankful to the reviewers for their suggestions and we incorporated the necessary changes.

References

[1] Nathanson, Melvyn B.,Connected components of arithmetic graphs, Monat. Fur. Math,. 29(1980), 219–220.

[2] Imrich, W. and Klavzar, S. Product Graphs: Structure and Recognition, John, Wiley and Sons, New York, USA., ,(2000).

[3] Whitehead, A.N. and Russel, B. , Principia Mathematica, Volume 2, Cambridge University Press, Cambridge,. (1912).

[4] Sabidussi, G., Graph Multiplication, Math. Z., 72(1960), 446–457.

[5] Madhavi, L., s Studies on domination parameters and enumeration of cycles in some Arithmetic Graphs, Ph. D. Thesis submitted to S.V.University, Tirupati, India, (2002).

[6] Uma Maheswari. S and Maheswari. B,Domination Parameters of Euler Totient Cayley Graphs, in Review Bulletin of Calcutta Mathematical Society, 19(2)(2011), 207–214.

[7] Uma Maheswari. S and Maheswari. B, Some Domination Parameters of Arithmetic Graph Vn, IOSR Journal of Mathematics , 2(6)(24)(2012), 14–18.

[8] Uma Maheswari. S and Maheswari. B, Some Properties of Cartesian Product Graphs of Euler Totient Cayley Graphs with Arithmetic Graphs, *International Journal of Computer Applications*, 138(2013), 26–29.

[9] K. W. Blayneh, Analysis of age structured host-parasitoid model, *Far East J. Dyn. Syst.,* (2016), 125–145.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆