Alpha power transformed Pareto distribution and its properties with application

K.M. Sakthivel\(^1\)* and A. Arul chezhian\(^2\)

Abstract

In this article, a new lifetime distribution sparked by alpha power transformation is studied. This article aims to propose a new generalization of the Pareto distribution known as alpha power transformed Pareto (APTP) distribution. The APTP distribution provides a better fit than the Pareto and transmuted Pareto distributions. Some mathematical properties of APTP distribution such as moments, quantile function, survival reliability function, conditional reliability function, hazard function and order statistics are derived. For parameter estimation, maximum likelihood, least square and weighted least square and maximum product space, percentiles estimators are used. A river flood data sets are occupied to appearance how the APTP distribution works better in real life.

Keywords

Alpha power transformation, Pareto distribution, moments, hazard rate, percentiles, maximum product of spacings and maximum likelihood.

AMS Subject Classification

60E05, 62F10, 62G30, 62H10.

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1. Introduction

Pareto distribution is prominent distribution used in various field such as reliability analysis, actuarial science, survival analysis, life testing, economics, finance, hydrology, telecommunication, physics and engineering. The Pareto distribution of first kind is defined by [11] has the cumulative density function is given by

\[
F(x) = 1 - \left( \frac{\beta}{x} \right)^\theta, \beta > 0, \theta > 0, x \geq \beta
\]  

(1.1)

It has two parameters \(\theta\) and \(\beta\), where has scale parameter and lower bound of the data. In the literature, some extensions of Pareto distribution are available such as the beta-Pareto distribution has introduced by Akinsie and Famoye [1]. Generalized Pareto distribution has introduced by Pickands et al. [20], Exponentiated Pareto distribution has introduced by Nadarajah et al. [17]. Beta Generalized Pareto distribution has introduced by Mahmoudi [15], Weibull Pareto distribution by Alzaatreh et al. [2], Kumaraswamy Pareto distribution by Pereira et al. [19], Kumaraswamy generalized Pareto distribution introduced by Nadarajah et al. [18], Korkmaz et al. [12] introduced the Burr X Pareto distribution, Exponentiated gen-
eralized Pareto distribution introduced by Lee and Kim [13]. The alpha power transformation (APT) is one of method that make models. Mahdavi and Kundu [14] introduced the APT distribution to incorporate skewness to the base line distribution. The cumulative distribution function of APT distribution is specified by

\[
F(x) = \begin{cases} 
\frac{\alpha}{\alpha - 1} & \text{if } \alpha > 0, \alpha \neq 1 \\
F(x) & \text{if } \alpha = 1
\end{cases}
\]

and the probability distribution function of APT distribution is given by

\[
f(x) = \begin{cases} 
\frac{\log \alpha}{\alpha - 1} f(x) \alpha^{F(x)} & \text{if } \alpha > 0, \alpha \neq 1 \\
f(x) & \text{if } \alpha = 1
\end{cases}
\]

Some extension of APT method are treated like APT generalized exponential [6], APT extended exponential [9], APT inverse Lindley [18], APT weibull [5] and APT power Lindley [10]. Alpha power transformed extended exponential distribution by Hassan et al. [9], we proposed a new mathematical model and it named as alpha power transformed Pareto distribution in section 2. We studied the some properties of APTP distribution such as moments, quantile function, reliability function, conditional reliability function, hazard rate function and order statistics are given in section 3. We obtain maximum likelihood, least square and weighted least square, maximum product space and percentiles in section 4. In section 5, the paper finishes with conclusion.

### 2. The APTP Model

The probability function of APTP with parameters \(\alpha, \beta\) and \(\theta\) are obtained by using the cdf and pdf of Pareto distribution in \(\alpha - \) power transformation. The cdf of APTP distribution as follows

\[
F_{\text{APT}}(x) = \begin{cases} 
\frac{1 - (\frac{\theta}{x})^\alpha}{\alpha - 1} & \text{if } \alpha > 0, \alpha \neq 1 \\
1 - (\frac{\theta}{x})^\alpha & \text{if } \alpha = 1
\end{cases}
\]

The probability density function of APTP distribution as follows

\[
f_{\text{APT}}(x) = \begin{cases} 
\left(\frac{\log \alpha}{\alpha - 1}\right) \left(\frac{\theta^\beta}{\alpha - 1}\right) \alpha^{1 - (\frac{\theta}{x})^\alpha} & \text{if } \alpha > 1, \alpha \neq 1 \\
\frac{\theta^\beta}{\alpha - 1} & \text{if } \alpha = 1
\end{cases}
\]

where \(x \geq \beta, \theta > 0\) and \(\beta > 0\). If \(x\) is a random variable comes from APTP distribution, (i.e.,) \(X \sim APTP(\alpha, \beta, \theta)\). Figure 1 and Figure 2 shows the different shapes of pdf and cdf of \(APTP(\alpha, \beta, \theta)\).

### 3. Statistical properties

#### 3.1 Survival, hazard rate, quantile and conditional reliability function

Survival function and quantile functions of APTP distribution as follows

\[
S_{\text{APT}}(x) = \begin{cases} 
\frac{\alpha}{\alpha - 1} \left(1 - \left(\frac{\theta}{x}\right)^\alpha\right) & \text{if } \alpha \neq 0, \alpha > 0 \\
\left(\frac{\theta}{x}\right)^\theta & \text{if } \alpha = 0
\end{cases}
\]

\[
x_p = \theta \left[\frac{\log \alpha}{\log \alpha + \log(p(\alpha - 1) + 1)}\right]^\frac{1}{\beta}
\]
A conditional reliability function is given by
\[ \hat{G}(x+t; \alpha, \beta, \theta/t) = \frac{G(x+t; \alpha, \beta, \theta)}{G(x; \alpha, \beta, \theta)} = 1 - \frac{1 - \left( \frac{x}{\beta} \right)^{\theta}}{1 - \left( \frac{x}{\beta} \right)^{\theta}} \] (3.3)

The failure rate function of APTD distribution as follows
\[ h_{APT}(x) = \begin{cases} \left( \frac{\theta}{\beta} \right)^{\theta} \left( \frac{x}{\beta} \right)^{\theta-1} & \text{if } \alpha \neq 0 \\ \frac{\theta}{\beta} & \text{if } \alpha = 1 \end{cases} \] (3.5)

3.2 Moments
A random variable \( X \) having the APTP distribution of the \( k^{th} \) moment is derived as follows
\[ \mu_k^i = \int_{\beta}^{\infty} x^k \left( \frac{\log \alpha}{\alpha - 1} \right) \left( \frac{\theta \beta^\theta}{\theta^{\theta+1}} \right) \left( \frac{\beta}{x} \right)^{\theta j} \] (3.6)

Since the power series is written as
<table>
<thead>
<tr>
<th>( \alpha^d )</th>
<th>( \sum_{i=0}^{\infty} \frac{(\log \alpha)^i}{i!} d^i )</th>
</tr>
</thead>
</table>

Hence (3.6) can be expressed as follows:
\[ \mu_k^i = \int_{\beta}^{\infty} x^k \left( \frac{\log \alpha}{\alpha - 1} \right) \left( \frac{\theta \beta^\theta}{\theta^{\theta+1}} \right) \left( \frac{\beta}{x} \right)^{\theta j} dx \] (3.7)

by using binomial expansion
\[ \mu_k^i = \frac{\log \alpha}{\alpha - 1} \sum_{i=0}^{\infty} \frac{(\log \alpha)^i}{i!} \sum_{j=0}^{\infty} (-1)^j \binom{i}{j} \int_{\beta}^{\infty} x^k \frac{\theta \beta^\theta}{\theta^{\theta+1}} \left( \frac{\beta}{x} \right)^{\theta j} dx \] (3.8)

The \( k^{th} \) moment of the APTP distribution can be derived as
\[ \mu_k^i = \frac{\log \alpha}{\alpha - 1} \sum_{i=0}^{\infty} \frac{(\log \alpha)^i}{i!} \sum_{j=0}^{\infty} (-1)^j \binom{i}{j} \left( \frac{1}{j+1} \theta (j+1) \right)^{\theta k} \] (3.9)

In particular mean and variance are given by
\[ E(X) = \frac{\log \alpha}{\alpha - 1} \sum_{i=0}^{\infty} \frac{(\log \alpha)^i}{i!} \sum_{j=0}^{\infty} (-1)^j \binom{i}{j} \left( \frac{1}{j+1} \theta (j+1) \right) \] (3.10)
\[ Var(X) = \frac{\log \alpha}{\alpha - 1} \sum_{i=0}^{\infty} \frac{(\log \alpha)^i}{i!} \sum_{j=0}^{\infty} (-1)^j \binom{i}{j} \left( \frac{1}{j+1} \theta (j+1) \right)^2 \] (3.11)

3.3 Order statistics
In statistics, the \( k^{th} \) smallest value is equal to the \( k^{th} \) order statistics of a statistical sample. For a sample of size \( n \), the \( n^{th} \) order statistic is the maximum, that is,
\[ X_{(n)} = \max \{ X_1, X_2, ..., X_n \} \] (3.12)

The difference between maximum and minimum is the range of the sample. It is surely a function of the order statistics
\[ Range \{ X_1, X_2, ..., X_n \} = X_{(n)} - X_{(1)} \] (3.13)

We know if \( X_{(1)}, X_{(2)}, ..., X_{(n)} \) denotes the order statistics of a random sample \( X_1, X_2, ..., X_n \) from a continuous population.
with cdf $F_X(x)$ and $f_X(x)$ then the pdf of $X_{(j)}$ is given by

$$f_X(x) = \frac{n!}{(j-1)!(n-j)!} f_X(x)$$

$$[F_X(x)]^j = \left[1 - F_X(x) \right]^{n-j}, j = 1, 2, 3, ... , n$$

(3.13)

Therefore, the maximum likelihood equation are derived as follows

$$\frac{\partial l}{\partial \alpha} = \frac{n\theta - \log \alpha \sum_{i=1}^{n} \left( \frac{\beta \theta}{x_i} \right)^{\theta - 1}}{x_i}$$

(4.3)

and

$$\frac{\partial l}{\partial \beta} = \frac{n\theta - \frac{n}{\beta} \sum_{i=1}^{n} \log \alpha \sum_{j=1}^{n} \left( \frac{\beta \theta}{x_j} \right)^{\theta - 1}}{x_j}$$

(4.4)

Maximum likelihood estimator of the model parameter are determined by work out numerically by solving the non-linear equation $\frac{\partial l}{\partial \alpha} = 0$, $\frac{\partial l}{\partial \beta} = 0$ and $\frac{\partial l}{\partial \theta} = 0$.  

4.2 Least square estimator and weighted least square estimator

Swin et al. [22] has introduced regression based estimators by using this estimator to estimate the parameters of Pareto distributions. Suppose $X_1, X_2, ..., X_n$ be a random sample from APTP $(\alpha, \beta, \theta)$ and Let $X_{(n)}$ denote the $j^{th}$ order statistics. The pdf of $X_{(j)}$ is given by

$$f_X(x) = \frac{n!}{(j-1)!(n-j)!} \left( \frac{\log \alpha}{\alpha - 1} \right) \left( \frac{\theta \beta \theta}{x^{\alpha+1}} \right)^{\alpha - \left( \frac{x}{\beta} \right)^{\theta}}$$

(3.14)

$$\times \left[ \alpha^{1 - \left( \frac{x}{\beta} \right)^{\theta}} - 1 \right]^{n-j}$$

(3.15)

and the pdf of the smallest order $X_{(1)}$ is given by

$$f_X(x) = n \left( \frac{\log \alpha}{\alpha - 1} \right) \left( \frac{\theta \beta \theta}{x^{\alpha+1}} \right)^{\alpha - \left( \frac{x}{\beta} \right)^{\theta}}$$

$$\times \left[ 1 - \frac{1 - \left( \frac{x}{\beta} \right)^{\theta}}{x - 1} \right]^{n-1}$$

(3.16)

Therefore, the probability density function of the largest order statistics $X(n)$ is given by

$$f_X(x) = \frac{n!}{(j-1)!(n-j)!} \left( \frac{\log \alpha}{\alpha - 1} \right) \left( \frac{\theta \beta \theta}{x^{\alpha+1}} \right)^{\alpha - \left( \frac{x}{\beta} \right)^{\theta}}$$

$$\times \left[ \alpha^{1 - \left( \frac{x}{\beta} \right)^{\theta}} - 1 \right]^{n-j}$$

(3.17)

4. Estimation

4.1 Maximum likelihood estimation

By maximizing the Log likelihood function, the maximum likelihood estimate of $\alpha, \beta$ and $\theta$ are obtained. In case of the APTP distribution, the log-likelihood function of the model

$$l = n log \left( \frac{\log \alpha}{\alpha - 1} + n \log \beta + \theta \log \alpha \right)$$

$$- (\theta + 1) \sum_{i=1}^{n} \log (x_i) + \log \alpha \sum_{i=1}^{n} \left( 1 - \left( \frac{\beta \theta}{x_i} \right)^{\theta} \right)$$

(4.1)

Therefore, the maximum likelihood equation are derived as follows

$$\frac{\partial l}{\partial \alpha} = \frac{n(\alpha - 1 - \alpha \log \alpha)}{\alpha(\alpha - 1) \log \alpha} + \frac{1}{\alpha} \sum_{i=1}^{n} \left( 1 - \left( \frac{\beta \theta}{x_i} \right)^{\theta} \right)$$

(4.2)
with respect to the unknown parameters.

where
\[ W_j = \frac{1}{f(G(Y_{ij}))} \frac{(n+1)^2(n+2)}{j(n-j+1)} \]  
\[ (4.9) \]

we can rewrite APTP distribution, WLSEs of \( \alpha, \beta \) and \( \theta \), say \( \hat{\alpha}_{WLSE}, \hat{\beta}_{WLSE} \) and \( \hat{\theta}_{WLSE} \) respectively, can be obtained by minimizing

\[ \sum_{j=1}^{n} \frac{(n+1)^2(n+2)}{j(n-j+1)} \left( \frac{\alpha^{1-(\frac{j}{n})^\theta} - \frac{j}{n+1}}{\alpha - 1} \right)^2 \]  
\[ (4.10) \]

with respect to \( \alpha, \beta \) and \( \theta \).

### 4.3 Maximum product of spacings estimator

Cheng and Amin [3] have proposed the maximum product of spacings (MPS) method and it is an alternative to the maximum likelihood estimation (MLE) method. The Kullback-Leibler divergence (KLD) is an approximation of MPS method derived by Rennby [21]. The Kullback-Leibler divergence between \( f(x, \theta) \) and \( f(x, \theta_0) \) is given by

\[ KLD(f(x, \theta_0)||f(x, \theta)) = \int f(x, \theta_0) \log \left( \frac{f(x, \theta_0)}{f(x, \theta)} \right) \]  
\[ (4.11) \]

The Kullback-Leibler divergence is zero if and only if \( f(x, \theta) = f(x, \theta_0) \) for all \( x \).

Let \( x_1, x_2, ..., x_n \) be a random sample from a cumulative density function \( F(x, \theta) \). Let \( f(x, \theta) \) denote the analogous probability density function. For estimating \( \theta \) is a excellent method should make the KLD among the model and the true distribution as small as possible. In application, this can be approximated by estimating

\[ \frac{1}{n} \sum_{i=1}^{n} \log \left( \frac{f(x_i, \theta_0)}{f(x_i, \theta)} \right) \]  
\[ (4.12) \]

So, by minimizing (4.12) with respect to \( \theta \), the estimator of \( \theta_0 \) can be found. Rennby [21] suggested another

\[ \frac{1}{n} \sum_{i=1}^{n+1} \log \left( \frac{F(x_i, \theta_0) - F(x(i-1), \theta_0)}{F(x_i, \theta) - F(x(i-1), \theta)} \right) \]  
\[ (4.13) \]

where \( x(i), i = 1, 2, 3, ..., n \) denotes the ordered sample and \( F(x(0)) = 1, F(x(n+1)) = 0 \). The estimator obtained by minimizing (4.13) is called the MPS estimator of \( \theta_0 \). It is clear that minimizing (4.13) is equivalent to minimizing

\[ \sum_{i=1}^{n+1} \log F(x(i), \theta) - F(x(i-1), \theta) \]  
\[ (4.14) \]

In case of the distribution, the MPDs of \( \alpha, \beta \) and \( \theta \), say \( \hat{\alpha}_{MPS}, \hat{\beta}_{MPS} \) and \( \hat{\theta}_{MPS} \), respectively, can be obtained by minimizing

\[ \sum_{i=1}^{n+1} \log F(x(i), \alpha, \beta, \theta) - G(x(i-1), \alpha, \beta, \theta) \]  
\[ (4.15) \]

\[ = \sum_{i=1}^{n+1} \frac{\alpha^{1-(\frac{x_i}{n})^\theta} - 1}{\alpha - 1} - \frac{\alpha^{1-(\frac{x_{i-1}}{n})^\theta} - 1}{\alpha - 1} \]  
\[ (4.16) \]

with respect to \( \alpha, \beta \) and \( \theta \).

### 4.4 Percentile estimator (PE)

Let \( x_1, x_2, ..., x_n \) be a random sample from the APTP distribution. Let \( x_1 < x_2 < ... < x_n \) be the analogous order statistics. The estimators of set of parameters and are obtained by minimizing the following

\[ \sum_{i=1}^{n+1} \left( \frac{\alpha^{1-(\frac{x_i}{n})^\theta}}{\alpha - 1} \right)^2 \]  
\[ (4.17) \]

with respect to \( \alpha, \beta \) and \( \theta \) where \( p_i \) denote some estimator of \( H_{APT}(x(i); \alpha, \beta, \theta) \), and \( p_i = \frac{i}{n+1} \).

### 5. Application

In this section, analysis the river flood data set to display that APTP distribution can be fit superior than Pareto and transmuted Pareto distributions.

#### 5.1 Exceedances of Wheaton River flood data set

In China, the wheaton river near carcross in Youkon Territory the data are the exceedances of flood peaks (in m3/s). In 1958-1984, the 72 data exceedance rounded to one decimal area has given in Choulakian and Stephens [4].

The MLEs of parameters and three information criteria: Akaike information criterion (AIC), Akaike information criterion corrected (AICC) and Bayesian information criteria (BIC) are calculated. These criteria are given by

\[ AIC = -2l(\theta) + 2m \]  
\[ (5.1) \]

\[ AICC = AIC + \frac{2m(m+1)}{n-m-1} \]  
\[ (5.2) \]

\[ BIC = -2l(\theta) + m\log(n) \]  
\[ (5.3) \]

Where \( m \) and \( n \) are the number of parameters and sample size, respectively and \( l \) is the maximized log-likelihood. As a rule of thumb, the model with smaller AIC or AICC or BIC value is considered to provide a better fit. According to Table 1, APTP provide the good model for the exceedances of Wheaton river flood among two models.
Table 1. Information criteria for exceedance of Wheaton river flood data

<table>
<thead>
<tr>
<th>Model</th>
<th>-2$l$</th>
<th>AIC</th>
<th>AICC</th>
<th>BIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pareto</td>
<td>606.128</td>
<td>610.128</td>
<td>610.3</td>
<td>610.405</td>
</tr>
<tr>
<td>T.Pareto</td>
<td>572.402</td>
<td>578.402</td>
<td>578.402</td>
<td>580.955</td>
</tr>
<tr>
<td>APT.Pareto</td>
<td>534.019</td>
<td>540.019</td>
<td>540.372</td>
<td>542.572</td>
</tr>
</tbody>
</table>

6. Conclusion

In this article, we introduced a mathematical model and it is named as alpha power transformed Pareto distribution, based on alpha power transformation. Various properties of APT distribution are obtained such as $r^{th}$ moment, quantile function, hazard function, survival reliability function, conditional reliability function and order statistic are studied. The model parameters are estimated by maximum likelihood estimation, least square and weighted least square estimator. Furthermore, an real time application of the APT Pareto distribution to real river flood data sets are presented to illustrate that this distribution provides a better fit than Pareto and transmuted Pareto distributions.

References