Bloom torus: A potential fixed interconnection architecture

S. Kulanidai Therese¹*, D. Antony Xavier² and Andrew Arokiaraj²

Abstract
The architecture Bloom torus as an extension of bloom graph is efficiently introduced in this paper. We determine its topological properties. Moreover, we compute various degree based indices of Bloom torus. A Hamiltonian decomposition partitions its edge set into disjoint Hamilton cycles. Edge-disjoint Hamiltonian cycles assist to produce efficient and fault tolerant algorithms for ring structures. In this paper, we provide an algorithm for two edge-disjoint Hamiltonian cycles of Bloom torus. Finally, we give construction to find Wiener index of Bloom torus. Several benchmarks such as network diameter, network cost, packing density and network throughput are used for performance evaluation of a network. We prove that the bloom torus has less network diameter than the usual grid torus when the column dimension is less than the row dimension. We determine its topological properties. A new flower structure of bloom torus to find better crossing number of bloom torus is derived. Finally undirected graph product for bloom torus is provided.

Keywords
Bloom torus, Hamiltonian cycle, VLSI layout, interconnection network, topological properties.

AMS Subject Classification
05C62, 05C85, 05C10, 05C51, 68R10.

1. Introduction
The torus portrays how energy flows in its utmost balanced dynamic flow system, not depicting a particular form alone. It has a central axis with a vertex on either side surrounded by a coherent field. The energy flows from one vertex to the other through the central axis, wrapping around itself. Donut is the simplest representation of its overall form despite taking various shapes depending upon the medium of its existence. Plants and trees exhibit same energy flow process in different shapes and sizes. Other examples include hurricanes, tornadoes, galaxies and magnetic fields of planets and stars. Extending this observation to quantum realm, we can conclude that atom structures and systems are also of the same dynamic form. With the torus features, the ecosystem becomes balanced and whole. Without them, system gets dysfunctional thereby making it necessary to restore the torus features. Another aspect of this flow process is the double
torus dynamic consisting of two torus forms stacked together rotating in opposite directions. In this, energy flow is either inward or outward at both the poles of a system. This form is common in cosmos, energy flow of trees, weather patterns of the earth, galaxies and in solar dynamics [6, 10, 15, 22].

An interconnection network consists of hardware and software entities that are inter connected to facilitate efficient computation and communication. These entities may take form of processors, processes, memory modules or computer systems [16]. Interconnection networks are becoming increasingly pervasive in many different applications, with the operational costs and characteristics of these networks depending considerably on the application. For some applications, interconnection networks have been studied in depth for decades. This is the case for telephone networks, computer networks, and backplane buses. In the last 20 years there has been a rapid evolution of the interconnection network technology that is currently being infused into a new generation of multi-processor systems. The technology is mature enough to find its way into commercial products, while constantly presenting new avenues for growth and application. This is the case for the interconnection networks used in multi-computers and distributed shared-memory multiprocessors. The basic design issues governing the new generation of multiprocessor network technology is the main theme of current study in interconnection networks.

An interconnection network can be modeled as an undirected graph $G(V,E)$, where $V$ is the vertex set representing the processors and $E$ is the edge set representing the communication links among the processors. Several topologies have been proposed as interconnection networks for multi computer system [18]. Among these interconnection networks, the hypercube and mesh topologies are two popular networks from a commercial point of view. However, although the hypercube is an efficient network because of its symmetry, regularity, logarithmic diameter, modularity and fault tolerance [23], it suffers from wire-ability and packing problems for VLSI implementation due to a non-constant node degree. Many problems in science and engineering such as matrix problems, multi-grid methods [7, 20], and image processing algorithms have mesh-like communication patterns with constant node-degree. A mesh again has the drawback of a larger diameter and low edge bisection [17]. A good interconnection network must have a relatively small node degree [27]. Due to recent developments in parallel and distributed computing, the design and analysis of various interconnection networks have been a main topic of research for the past few years. Directed networks are networks that allow only direct communication between neighboring processors. Some networks, like meshes with multiple broadcasting, are enhanced by adding buses that allow quick broadcasting of a datum from one processor to a group of other processors. In undirected networks, nodes in underlying graph are distinguished as processors, or switches, or network controllers, etc. Switches are simple elements that can perform a comparison and forward the message in one or other direction. Interconnection networks play an important role in the performance of the massively parallel multi core processing systems. The most widely used interconnection networks are direct interconnection networks. In direct interconnection network mesh interconnections based systems have been used commercially [4, 5]. The various factors associated with the direct interconnection network are diameter, bisection width, node degree, edge length. Various research and development results on how to interconnect multiprocessor components have been reported in literature. Several surveys of parallel computing architectures exist (e.g., [4, 8]). One of the most popular architectures is the mesh-connected computer, in which processors are placed in a square or rectangular grid, with each processor being connected by a communication link to its neighbors in up to four directions. Tori are meshes with wraparound connections to achieve vertex and edge symmetry. IBM and United States Department of Energy designed BlueGene/L supercomputer with processors in the multidimensional torus structure [11]. As the efficiency of an interconnection network lies in communication latency, the Cray research developed T3D network with high scalability [2, 24]. Jellybean Machine multicomputer designed with VLSI components [9]. Tera computer academy made processor using toroidal mesh [3]. Tang et.al. compared diagonal mesh with toroidal mesh [25]. Ding-Ming Kwai and Behrooz Parhami proved honeycomb rectangular torus and diamond networks as the class of Cayley graphs [13]. B.Parhami et.al. demonstrated the additional flexibility of pruned tori [19, 21]. Peng Zhang, Reid Powell, and Yuefan Deng constructed the interlaced bypass torus [11].

The rest of the paper is organised as follows: Section 2 describes the bloom torus. Section 3 proposes the VLSI layout of bloom torus. Topological properties of bloom torus are presented, and Wiener index of bloom torus is computed in section 4. The Hamiltonian property of bloom torus is discussed in Section 5; Comparative analysis of bloom torus and grid torus is performed in section 6. Finally, the undirected graph product of bloom torus is identified.

2. Bloom Torus

Meshes and tori are among the most frequent multiprocessor networks available today in the market. Tori are obtained by adding wraparound edges to the mesh architecture. In line with this definition, bloom torus is defined with the bloom graph as the basic architecture. Xavier et.al. introduced the bloom graph [26]. Bloom graph is naturally a cylinder shaped architecture. While comparing bloom graph $B(m,n)$ with cylinder $C_m \times C_n$, bloom graph yields better average distance and greater packing density than cylinder graph. An architecture named Bloom Torus is introduced as follows:

**Definition 2.1.** The bloom torus denoted by $BT(m,n); m,n > 2$ consists of vertex set $V(BT(m,n)) = \{ (x, y) : 0 \leq x < m - 1, 0 \leq y < n - 1 \}$, two distinct vertices $(x_1, y_1)$ and $(x_2, y_2)$ being adjacent if and
Definition 2.1

name as the diamond representation of the bloom torus. This is an isomorphic representation of the bloom torus which we refer to as the isometric representation of bloom torus. As an illustration we refer to Fig. 1 for the isomorphic representation of bloom torus.

Figure 1. Bloom torus BT(6, 8)

Figure 2. Isomorphic drawing of bloom torus BT(6, 8)

3. VLSI layout of bloom torus

A structure with fewer crossings is applied to a wide range of optimization problems. It is one of the most basic and natural problems. Making better VLSI layout has many practical applications [12].

3.1 Isomorphic representation of bloom torus

Normally, a set of \( mn \) vertices arranged as a rectanguler array in the \((X, Y)\)-plane with \( n \) vertices in each of the \( m \) rows, are represented by the coordinates \((i, j)\), \( 0 \leq i \leq n-1, 0 \leq j \leq m-1 \). This placement of vertices, along with the edges described in Definition 2.1, gives rise to the isometric representation of the bloom torus. See Fig. 2 for the isometric representation of BT(6, 8).

It has always been a challenge to obtain isomorphic representations of architectures which will throw light on topological properties of the architectures. Fortunately, we have obtained an isomorphic representation of the bloom torus which we name as the diamond representation of the bloom torus. This is done by placing the row vertices in a cyclic order. In row \( i \), \( i \) even, \( 0 \leq i \leq m-1 \), the vertices begin with \((i, i/2)\) and are consecutively labeled as \((i, i/2), (i, i/2+1), \ldots (i, n-1), (i, 0), \ldots (i, \frac{n}{2})\). On the other hand, in row \( i \), \( i \) odd, \( 1 \leq i \leq m-1 \), the vertices begin with \((i, i/2)\) and are consecutively labeled as \((i, \frac{i-1}{2}), (i, \frac{i+1}{2}), \ldots (i, n-1), (i, 0), \ldots (i, \frac{n-3}{2})\). If the vertices in each row are placed horizontally in the \((X, Y)\)-plane with equal spacing between the vertices and if the edge joining \((i, \frac{i}{2})\) and \((i+1, \frac{i}{2})\), \( i \) even, \( 0 \leq i \leq m-1 \), makes an acute angle of 45° with the horizontal, then an isomorphic representation of BT\((m, n)\) with the edges defined in Definition 2.1 has a diamond mesh as an underlying structure. To avoid crossings for better understanding, if we duplicate the first row as the last row and the first column as the last column which are to be identified, then we get a representation which we call the diamond representation of BT\((m, n)\), \( m, n \geq 3 \). See Fig. 3 for isometric representation of BT(6, 8).

3.2 Crossing number of bloom torus

In this section we minimize the crossings of bloom torus with flower structure.

Theorem 3.1. Let G be a bloom torus BT\((m, n)\) and Cr(G) be the crossing number of G. Then 2\((m-2)(n-1) + mn \leq Cr(G) \leq 2(m-3)n\).

Proof: We redraw the bloom torus shown in Fig. 1 as in Fig. 3. The bloom torus BT\((m, n)\) has at least 2\((m-2)(n-1) + mn \) crossings. See Fig. 1. But with the isomorphic structure, there are 2\((m-3)n \) crossings in Fig. 3.

4. Topological properties of bloom torus

We enumerate the fundamental network properties. We describe the isomorphic representation of bloom torus. Moreover, we compute some topological indices of bloom torus.

4.1 Properties

The bloom torus BT\((m, n)\) has \( mn \) vertices and \( 2mn \) edges. Vertex connectivity and edge connectivity of bloom torus is 4. The diameter of the bloom torus
Theorem 4.3. Let $G$ be a bloom torus $BT_{2s,t}$, where $s,t \geq 3$. Then Wiener index of $G$ is \[
abla s^2 t (s^2 + 3t^2 - 1) \over 6 \]

5. Hamiltonian property of bloom torus

The hamiltonian property provides the capability of configuring the interconnection network as a linear array, which is the configuration with broadcast practical significance of either $n-1$ or $n$ nodes in the presence of a single faulty node or link. A cycle decomposition is a decomposition such that each subgraph in the decomposition is a cycle. A graph has a cycle decomposition such that every vertex has even degree. A hamiltonian cycle decomposition is a decomposition such that each sub graph $HC_j$ in the decomposition is a hamiltonian cycle. Two cycles $HC_1$ and $HC_2$ in a graph $G$ are said to be link-disjoint hamiltonian if both of them are hamiltonian and there is no edge $u \in E(G)$ such that $u \in HC_1$ and $u \in HC_2$.

Theorem 5.1. Bloom torus $BT(m,n)$ is hamiltonian.

Proof: We label the edges of $BT(m,n)$, $m,n \geq 3$ as follows:

i) If the vertices $(0,y)$ and $(m-1,y+\lfloor n \over 2 \rfloor \text{mod } n)$ are adjacent then label the edge as $(0,y)$.

ii) If the vertices $(0,y)$ and $(m-1,y-1+\lfloor n \over 2 \rfloor \text{mod } n)$ are adjacent then label the edge as $(m,y-1\text{mod } n)$.

iii) If the vertices $(x,y)$ and $(x+1,y)$ are adjacent then label the edge as $(x+1,2y-1\text{mod } 2n)$.

iv) If the vertices $(x,y)$ and $(x+1,y+1\text{mod } n)$ are adjacent then label the edge as $(x+1,2y)$.

We trace a hamiltonian cycle $HC$ using the following three steps.

Step 1: IF $m$ is odd, THEN include the edges $(m,j)$ in $HC$, where $j = 0,1,...,n-1$

ELSE include the edges $(0,j)$ in $HC$, where $j = 0,1,...,n-1$

Step 2: When $1 \leq i \leq m-2$

FOR $l \leftarrow 0$ to $n-1$

IF $i$ is odd, THEN include the edges $(i,j)$ in $HC$, where $j = 0,1,...,n-1$
Step 3: When $i = m - 1$, include edges $(i, j)$ in $HC$, where $j = 2i + 1$, $i = 0, 1, \ldots, n - 1$

END IF

END FOR

Thus $BT(m, n)$ is hamiltonian.

Theorem 5.2. Bloom Torus $BT(2s, t)$, $s \geq 2, t \geq 3$ has two disjoint hamiltonian cycles.

Proof:

We construct a hamiltonian cycle $HC_1$ considering 2 cases.

Case 1: $s$ even

The hamiltonian cycle $HC_1$ includes the following edges connecting the vertices, where $\oplus_t$ denotes addition modulo $t$.

i. $(4i + 1, 2i + j)$ and $(4i, 2i + j)$, $\forall 0 \leq i \leq \frac{s - 2}{2}$, $0 \leq j \leq t - 1$.

ii. $(4i, 2i + j)$ and $(4i + 1, (2i + 1) \oplus_t j)$, $\forall 0 \leq i \leq \frac{s - 2}{2}$, $0 \leq j \leq t - 1$.

iii. $(4i + 3, (2i + 2) \oplus_t j)$ and $(4i + 2, (2i + 1) \oplus_t j)$, $\forall 0 \leq i \leq \frac{s - 2}{2}$, $0 \leq j \leq t - 1$.

iv. $(4i + 2, (2i + 2) \oplus_t j)$ and $(4i + 3, (2i + 1) \oplus_t j)$, $\forall 0 \leq i \leq \frac{s - 2}{2}$, $0 \leq j \leq t - 1$.

v. $(4i, 2j - 1)$ and $(4i - 1, 2j - 1)$, $\forall 1 \leq i, j \leq \frac{s - 2}{2}$.

vi. $(4i + 2, 2j + 1)$ and $(4i + 1, 2j)$, $\forall 1 \leq i, j \leq \frac{s - 2}{2}$.

vii. $(0, t - 1)$ and $(2s - 1, s - 1)$.

Case 2: $s$ odd

The hamiltonian cycle $HC_1$ includes the edges connecting the vertices

i. $(4i + 3, (2i + 2) \oplus_t j)$ and $(4i + 2, (2i + 1) \oplus_t j)$, $\forall 0 \leq i \leq \frac{s - 1}{2}$, $0 \leq j \leq t - 1$.

ii. $(4i + 2, (2i + 2) \oplus_t j)$ and $(4i + 3, (2i + 1) \oplus_t j)$, $\forall 0 \leq i \leq \frac{s - 1}{2}$, $0 \leq j \leq t - 1$.

Further $E(BT(m, n)) \setminus E(HC_1)$ constitutes another hamiltonian cycle.

### Table 1. Diameter of bloom torus and grid torus

<table>
<thead>
<tr>
<th>Number of nodes</th>
<th>Diameter Of bloom torus</th>
<th>Diameter Of grid torus</th>
</tr>
</thead>
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<tr>
<td>5000</td>
<td>100</td>
<td>112</td>
</tr>
<tr>
<td>10000</td>
<td>100</td>
<td>125</td>
</tr>
<tr>
<td>15000</td>
<td>100</td>
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<tr>
<td>20000</td>
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<td>165</td>
</tr>
<tr>
<td>25000</td>
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<td>175</td>
</tr>
<tr>
<td>30000</td>
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<td>185</td>
</tr>
<tr>
<td>35000</td>
<td>140</td>
<td>202</td>
</tr>
<tr>
<td>40000</td>
<td>160</td>
<td>222</td>
</tr>
<tr>
<td>45000</td>
<td>180</td>
<td>242</td>
</tr>
<tr>
<td>50000</td>
<td>200</td>
<td>262</td>
</tr>
</tbody>
</table>

### 6. Comparative analysis of bloom torus with grid torus

In this section, performance of bloom torus and usual grid torus are compared. At a very coarse level, networks can be characterized by certain properties or topological parameters. Most important parameters namely diameter, network cost, packing density and network throughput are reviewed in this section. It is interesting to note that bloom torus $BT(2s, t)$ and grid torus $GT(2s, t)$ are considered throughout this section for the comparison as the networks have equal number $2st$ of nodes and equal number of $4st$ links. In the sequel we observe that bloom torus $BT(2s, t)$ performs much better than grid torus $GT(2s, t)$, when $t < 2s - 1$.

#### 6.1 Network diameter

The diameter of a network is the length of the longest among shortest paths between all pairs of nodes. It is clearly important with packet routing because it dictates the worst case communication latency. This is especially true with very long messages. The diameter should be minimum for an efficient interconnection network. It places a lower bound on the communication delay required to disseminate information throughout the network. The diameter of the bloom torus $BT(2s, t)$ is $max(s, t)$ whereas the diameter of $GT(2s, t)$ is $s + \lfloor t/2 \rfloor$. Table 6.1 is obtained using MATLAB and shows the diameter of networks corresponding to its network size. Fig. 6 shows that the diameter of bloom torus is low whereas the diameter of grid torus is high.

#### 6.2 Network cost

Network cost of Tori is defined as $4 \times$ diameter of the network. Table 6.2 is obtained using MATLAB and shows the cost of networks corresponding to its network size. Fig. 7 exhibits that the bloom torus has the lower network cost compared to that of grid torus. The network cost of the bloom torus $BT(2s, t)$ is $4 \max(s, t)$ whereas the network cost of the grid torus $GT(2s, t)$ is $4(\lfloor s \rfloor + \lfloor t/2 \rfloor)$. 

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**Figure 5.** Hamiltonian cycle of bloom torus $BT(10, 9)$. 

**Table 1. Diameter of bloom torus and grid torus**
6.3 Packing density

Packing density is defined by the ratio of the number of nodes of the given network to its network cost. Packing density of a graph is a measurement of efficient chip area required for VLSI layout and implementation. When the packing density of the network is high, the chip area required for its VLSI layout will be small. Table 6.3 is obtained using MATLAB and shows the packing density of networks corresponding to its network size. Fig. 8, exhibits that the bloom torus has the higher packing density compared to that of grid torus. The packing density of the bloom torus $BT(2s,t)$ is $\frac{st}{2\max(s,t)}$ whereas the packing density of the grid torus $GT(2s,t)$ is $\frac{st}{2\lfloor s \rfloor + \lfloor t/2 \rfloor}$.

6.4 Network throughput

Network throughput is defined as the ratio of the total network bandwidth, which is proportional to the number of links in the network to the diameter. It refers to the maximum amount of messages delivered per unit of time through the network. Table 6.4 is obtained using MATLAB and shows the network throughput of networks corresponding to its network size. Fig. 9 exhibits that the bloom torus has the higher network throughput compared to that of grid torus. The network throughput of the bloom torus $BT(2s,t)$ is $\frac{\text{dist}(s,t)}{\max(s,t)}$ whereas the network throughput of the grid torus $GT(2s,t)$ is $\frac{\text{dist}(s,t)}{\lfloor s \rfloor + \lfloor t/2 \rfloor}$.
Table 4. Network throughput of bloom torus and grid torus

<table>
<thead>
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<th>Number of Nodes</th>
<th>Network Throughput</th>
<th>Network Throughput</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Of Bloom Torus</td>
<td>Of Grid Torus</td>
</tr>
<tr>
<td>5000</td>
<td>100</td>
<td>89.28571</td>
</tr>
<tr>
<td>10000</td>
<td>200</td>
<td>160.0000</td>
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<tr>
<td>15000</td>
<td>300</td>
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</tr>
<tr>
<td>20000</td>
<td>320</td>
<td>242.4242</td>
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<tr>
<td>25000</td>
<td>400</td>
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</tr>
<tr>
<td>30000</td>
<td>480</td>
<td>324.3243</td>
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<tr>
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<td>500</td>
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</tr>
<tr>
<td>50000</td>
<td>500</td>
<td>381.6794</td>
</tr>
</tbody>
</table>

**Figure 9.** Comparison with network throughput

**Figure 10.** Direct product of $C_{2s} \times C_{2t}$. It has $4st$ number of vertices. The graph has two disjoint copy which are isomorphic to each other.

**Figure 11.** One component of $C_{2s} \times C_{2t}$. It has $2st$ number of vertices.

### 7. Graph product in undirected bloom torus $BT(2s, t)$

**Definition 7.1.** The tensor product $G \times H$ of graphs $G$ and $H$ is a graph such that

1. The vertex set of $G \times H$ is the Cartesian product $V(G) \times V(H)$ and
2. Any two vertices $(u, u')$ and $(v, v')$ are adjacent in $G \times H$ if and only if
   - $u'$ is adjacent with $v'$ and
   - $u$ is adjacent with $v$

Let $G = C_{2s} \times C_{2t}$. It consists of the vertex set $V(G) = \{(x, y) : 1 \leq x \leq 2s, 1 \leq y \leq 2t\}, (x_1, y_1)$ is adjacent to $(x_2, y_2)$ if $x_1$ is adjacent to $x_2$ and $y_1$ is adjacent to $y_2$.

**Theorem 7.2.** The direct product of two cycle graphs of even length i.e. $C_{2s} \times C_{2t}$ is the union of two disjoint bloom torus $BT(2s, t)$.

### Conclusion

Focusing on the comparative performance analysis of bloom torus and grid torus, it is concluded that bloom torus is a potential interconnection network. The VLSI layout of the bloom torus as the flower structure has less crossings. Finally depicting bloom torus an undirected product, many applications in cosmology, physics and biological sciences are logically explored.

### References


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