Degree based partition of the power graph of the finite Abelian group

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Abstract
The power graph of the group $G$ is the graph, whose vertex set is the group $G$ itself, and there is an edge between any two distinct vertices if one is a power of the other. It is denoted as $\zeta_G$. In this paper, we mainly focused only on the undirected power graph of a finite abelian group of an order $pq$, where $p$ and $q$ are two distinct primes, $p < q$ and $p \neq q$. First, we shall give the generalization result for the vertex partition set and edge partition set based on the degrees of the power graph $\zeta_G$. The generators of the group $\mathbb{Z}_{pq}$ can be found from the partition. Then we can derive some types of topological indices of $\zeta_G$ of the group $\mathbb{Z}_{pq}$.

Keywords
Vertex partition, edge partition, generators of the group, zagreb indices.

AMS Subject Classification
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1. Introduction

Groups as graphs contains the most merging combination which is used regularly in the algebraic graph theory. The undirected power graph $G \ [1]$ in which two distinct vertices $x$ and $y$ are adjacent if one is a power of the other. The concept of the power graphs in group is the recent development area of research. It is easy to see that the power graph of a given group is connected. The power graph $G$ of the group $G$ is complete iff $G$ is cyclic group of prime power order. The two abelian group with two isomorphic power graphs are isomorphic $[2, 3]$.

Next, we proceed to generalize the power graph of the finite abelian group of an order $pq$, $|G| = pq$ where $p$ and $q$ are two distinct prime. A cyclic group is the group that is generated by a single element $g$ i.e, there exist an element $g$, such that every element of the group can be written as a power of $g$. This element $g$ is the generator of the group.

In general case, finding generator of cyclic group is the difficult task. we believe that there is no fast algorithm to find the generator for the larger number of the group. For example, Every element of a cyclic group of the prime order except the identity does generate the whole group. The generators of the cyclic group depends upon the order of the group. By using that, we may consider the power graph of an order may be $pq$. It is followed that we may find the generators of the large group orders, This method may be suitable to explicitly evaluate all groups under the product of two distinct primes, $p < q$ and $p \neq q$.

The main intuition of this paper is to find the vertex partition and edge partition $[8]$ based on the degree of the vertices of the power graph $\zeta_G$ of the group $G$. we may simplify the methodology for finding the generators of the group $G$ using the vertex partition of the power graph $\zeta_G$.

The power graph may represent the molecular structure of a certain chemical compound and it is mainly associated with the different molecular-biology, specially in the graph neural networks, network navigation, designing network tools etc.,
It is also used to maintain state, the information to capture the adjacency properties of nodes. The molecular descriptors are based on the sense of ultimate outcome of logic and the numerical procedure which converts the preset chemical information into a symbolic representation of molecule will reach a practical number. Therefore, the topological indices [7] are very essential tool in the field of nanotechnology, drug discovery, computer networks, designing tool, information technology etc., It is mainly motivated to explore classification purpose, such as Quantitative Structure Activity Relationship (QSAR) and Quantitative Structure Property Relation (QSPR) and Fuzzy Lattice Neural Networks (FLNN) [9] etc. since we can derive some expression about the first zagreb indices, second zagreb indices, the first zagreb co indices, second zagreb co indices and also complements of the power graph \( \zeta_G \).

## 2. Preliminaries

Some of the well-known basic definitions which are defined earlier for any simple graph as follows

**Definition 2.1.** A partition of a set is a grouping of its elements into non-empty, in such a way that element is included in exactly one subset.

**Definition 2.2.** The First Zagreb Index [6] of a graph \( \zeta_G \), denoted by \( M_1(\zeta_G) \) is defined as

\[
M_1(\zeta_G) = \sum_{u,v \in E(\zeta_G)} [d(u) + d(v)]
\]

**Definition 2.3.** The Second Zagreb Index [6] of a graph \( \zeta_G \), denoted by \( M_2(\zeta_G) \) is defined as

\[
M_2(\zeta_G) = \sum_{u,v \in E(\zeta_G)} |d(u) - d(v)|
\]

**Definition 2.4.** The Third Zagreb Index [4] of a graph \( \zeta_G \), denoted by \( M_3(\zeta_G) \) is defined as

\[
M_3(\zeta_G) = \sum_{u,v \in E(\zeta_G)} |d(u) - d(v)|
\]

**Definition 2.5.** The First Zagreb Co-Index [5] of a graph \( \zeta_G \), denoted by \( \overline{M}_1(\zeta_G) \) is defined as

\[
\overline{M}_1(\zeta_G) = \sum_{u,v \in E(\zeta_G)} [d(u) + d(v)]
\]

**Definition 2.6.** The Second Zagreb Co-Index [5] of a graph \( \zeta_G \), denoted by \( \overline{M}_2(\zeta_G) \) is defined as

\[
\overline{M}_2(\zeta_G) = \sum_{u,v \in E(\zeta_G)} |d(u)d(v)|
\]

The complement of the graph \( \zeta_G \) has the same vertex set as that of \( \zeta_G \); if and if they are not adjacent in \( \zeta_G \). The relationship between the zagreb indices and co-indices of the graph can be shown in [5]. The set of vertices and set of edges are always denoted by \( V \) and \( E \) respectively.

**Theorem 2.7.** Let \( \zeta \) be any graph with \( n \) vertices and \( m \) edges. Then

1. \( M_1(\zeta) = M_1(\zeta) + n(n-1)^2 - 4m(n-1) \)
2. \( \overline{M}_1(\zeta) = 2m(n-1) - M_1(\zeta) \)
3. \( \overline{M}_1(\zeta) = 2m(n-1) - M_1(\zeta) \)
4. \( M_2(\zeta) = n(n-1)^3 - 3m(n-1)^2 + 2m^2 + \frac{3n-3}{2}M_1(\zeta) - M_2(\zeta) \)
5. \( \overline{M}_2(\zeta) = 2m^2 - \frac{1}{2}M_1(\zeta) + M_2(\zeta) \)
6. \( \overline{M}_2(\zeta) = m(n-1)^2 - (n-1)M_1(\zeta) + M_2(\zeta) \)

The above definitions and theorems are the basic informations which are useful in subsequent sections. The computation of different types of zagreb indices of various graph families may be referred in [10, 11].

## 3. Main Results

Let \( G \) be a finite abelian group. The graph \( \zeta_G(V,E) \) is the power graph of \( G \) where \( V \) is the vertex set and \( E \) is the edge set of the graph \( \zeta_G \). We denote the minimum degree as \( \delta = \text{Min} \{d(u)|u \in V(\zeta_G)\} \) and the maximum degree as \( \Delta = \text{Max} \{d(u)|u \in V(\zeta_G)\} \). The vertex set and edge set can be partitioned based on the degree as follows [8]. There are three different degrees of the vertices of the power graph \( \zeta_G \). According to the degrees, the vertex set \( V \) can be partitioned into \( V_x, V_y, V_z \). Assume that \( x < y < z \).

\[
V_x = \{v \in V(\zeta_G)|x = d(v) = q(p-1)\}
V_y = \{v \in V(\zeta_G)|y = d(v) = p(q-1)\}
V_z = \{v \in V(\zeta_G)|z = d(v) = pq - 1\}
\]

Now we have

\[
V = V_x \cup V_y \cup V_z
\]

The cardinality of the vertex partition can be calculated and generalized as follows,

\[
|V_x| = p - 1
|V_y| = q - 1
|V_z| = pq - p - q + 2
\]
The degrees of the elements of $Z$ are the elements of the power graph $\zeta_G$, denoted by $N$, is given by

$$N = \{1, 2, 4, 7, 8, 11, 13, 14\}$$

By the definition of the power graph, the vertices of the power graph $\zeta_G$ are the elements of the power graph

$$Z_{15} = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14\}$$

The degrees of the elements of $Z_{15}$ are

$$\{14, 14, 14, 12, 14, 10, 12, 14, 14, 12, 10, 14, 12, 14, 14\}$$

Based on the degrees 10, 12, 14, the vertex set can be partitioned into $V_x, V_y, V_z$, where $x = 10, y = 12, z = 14$. Now we have

$$V_x = \{5, 10\}$$
$$V_y = \{3, 6, 9, 12\}$$
$$V_z = \{0, 1, 2, 4, 7, 8, 11, 13, 14\}$$

which is shown in Figure 2.

The cardinality $V_z$ is maximum and hence the nonzero elements of $V_z$ will give us generators of $Z_{15}$ i.e.,

$$N = V_z \setminus \{0\} = \{1, 2, 4, 7, 8, 11, 13, 14\}$$

### 3.2 The zagreb indices of the power graph of the finite abelian group

In this section, we determine some types of Zagreb Indices of the power graph of the finite abelian group $G$ as follows,

**Theorem 3.2.** Let $\zeta_G$ be the power graph of the finite abelian group of an order $pq$. Then its Zagreb indices are given by

1. $M_1(\zeta_G) = |G|^3 - 6|G|^2 - |G| - (p + q)(|p + q| - 4|G|) + 2$

2. $M_2(\zeta_G) = \frac{1}{2}(|G|^4 - 9|G|^3 + 11|G|^2(p + q) - |G| - 13(p + q)(p + q) + 10 + (p + q)|5(p + q) - 7| + 2$
3. $M_3(\xi_G)$

$$M_3(\xi_G) = |G|^2 - 2|G|((p + q + 1) + 5|G|) + 3(p + q - 1) + 2$$

**Proof.** Consider the power graph $\xi_G$ of the finite abelian group $G$. Using the number of vertex partition set and edge partition set of the power graph $\xi_G$, we shall obtain results as follows.

1. $M_1(\xi_G)$

$$M_1(\xi_G) = \sum_{uv \in E(\xi_G)} [d(u) + d(v)]$$

$$= p(q - 1) + p(q - 1) \frac{(q - 1)(q - 2)}{2} + (q(p - 1) + (pq - 1) \frac{(pq - 1)(pq - 2)}{2} + ((pq - 1) + (pq - 1)$$

$$+ (pq - q + 2(pq - q + 1)) + ((pq - q - 1) + (pq - q - 1))$$

$$= (p^3 - 4p^2q + 10pq + pq^3 - 4pq^2 - 2p - 2q) +$$

$$+ (p^2q^2 - pq^2 - 3q - 5pq + 2)(pq - p - q + 2)$$

$$+ p^3q^3 - 6p^2q^2 - 3pq - p^2 - q - 5p^2q + 5pq^2 + 2$$

$$+ 5pq^2 + 2$$

$$= |V|^3 - 6|V| - |V| - (p + q)((p + q) - 4|V|) + 2$$

$$= |G|^3 - 6|G| - |G| - (p + q)((p + q) - 4|G|) + 2$$

2. $M_2(\xi_G)$

$$= \sum_{uv \in E(\xi_G)} d(u)d(v)$$

$$= (p(q - 1) + p(q - 1)) \frac{(q - 1)(q - 2)}{2} + (q(p - 1) +$$

$$+ q(p - 1) \frac{(p - 1)(p - 2)}{2} + ((pq - 1) - (pq - 1))$$

$$+ (pq - q - 2(pq - q + 1)) + ((pq - q - 1) - (pq - 1))$$

$$= (p - 1)(pq - p - q + 2) + ((pq - p - q))$$

$$= p^2q^2 - 2p^2q - 2pq^2 + p^2 + q^2 + 5pq - 3p - 3q + 2$$

$$= |V|^2 - 2|V|(p + q + 1) + 5|V| + 3(p + q - 1) + 2$$

$$= |G|^2 - 2|G|(p + q + 1) + 5|G| + 3(p + q - 1) + 2$$

3.3 The Zagreb Co-Indices of the power graph of the finite abelian Group

In this section, we determine some types of Zagreb Co-indices and of its complement of power graph $\xi_G$ of the finite abelian group $G$ as follows,

**Theorem 3.3.** Let $\xi_G$ be the power graph of the finite abelian group. Then its Zagreb Co-indices $\xi_G$ of and of its complements are given by
1. $M_1(\zeta_G) = (p + q)(3 + |G|) - 4|G| - p^2 - q^2 - 2$

2. $M_1(\zeta_G) = 2|G|^2 + |G|[4 - 3p - 3q] - p - q + p^2 + q^2$

3. $M_1(\zeta_G) = 2|G|^2 + |G|[4 - 3p - 3q] - p - q + p^2 + q^2$

4. $M_2(\zeta_G) = |G|^4 - 7|G|^3 + 10|G|^2 + 2|G|^2 +$
\[+ 2(p + q)[2|G|^2 - 6|G|] + 1 + 2(p^2 + q^2) - 6\]

5. $M_2(\zeta_G) = |G|^3 - 2|G|^2(p + q - 2) - |G|[2(p + q + 1]$
\[+ p^2 + q^2]\]

6. $M_2(\zeta_G) = \frac{1}{2}[10|G|(1 - |G|) + (p + q)[7n + 5]$
\[+ 3(p^2 + q^2)]

**Proof.** Consider the power graph $\zeta_G$ of the finite abelian group. Using the number of vertex partition set and edge partition set of the power graph $\zeta_G$, we shall obtain results as follows.

1. $M_1(\zeta_G)$
\[= M_1(\zeta_G) + |V|(|V| - 1) - 4|E|(|V| - 1)\]
\[= M_1(\zeta_G) + pq(pq - 1)^2 - 2(p^2 q^2 - 3pq + 2p + 2q - 2)(pq - 1)\]
\[= M_1(\zeta_G) + p^3 q^3 - 2p^2 q^2 - pq - 2p^2 - q^2 + 4p^2 q^2\]
\[= -4p^2 q^2 - 4pq - 4p + 4q - 4\]
\[= -4p^2 q^2 - 4pq - 3pq - p^2 - q^2 + 3p - 3|G| - 2\]
\[= -4|G|^2 + |G|(p + q) - 3|V| - p^2 - q^2 + 3(p + q) - 2\]
\[= -4|G|^2 + |G|(p + q) - 3|G| - p^2 - q^2 + 3(p + q) - 2\]
\[= (p + q)(3 + |G|) - 4|G| - p^2 - q^2 - 2\]

2. $M_1(\zeta_G)$
\[= 2|E|(|V| - 1) - M_1(\zeta_G)\]
\[= p^2 q^2 + 4pq - 3pq - 3pq - p - q + p^2 + q^2\]
\[= 2|V|^2 + |V|(4 - 3p - 3q) - p - q + p^2 + q^2\]
\[= 2|G|^2 + |G|(4 - 3p - 3q) - p - q + p^2 + q^2\]

3. $M_2(\zeta_G)$
\[= \frac{1}{2} \times |V|(|V| - 1)^3 - 3|E|(|V| - 1)^2\]
\[+ 2|E|^2 + 2\frac{n - 3}{2} M_1(\zeta_G) - M_2(\zeta_G)\]
\[= |V|^4 - 7|V|^3 + 10|V|^2 + 2|V|^4\]
\[+ 2(p + q)[2|V|^2 - 6|V| + 1] + 2(p^2 + q^2) - 6\]
\[= |G|^4 - 7|G|^3 + 10|G|^2 + 2|G|^2\]
\[+ 2(p + q)[2|G|^2 - 6|G|] + 1 + 2(p^2 + q^2) - 6\]

4. $M_2(\zeta_G)$
\[= 2(|E|^2 + \frac{1}{2} M_1(\zeta_G) - M_2(\zeta_G)\]
\[= p^3 q^3 + 4p^2 q^2 - 2p^2 q^2 - 4pq(p + q)\]
\[+ 2pq(p^2 + q^2) - 4pq\]
\[= |V|^3 - 2|V|^2(p + q - 2) - |V|[2(p + q + 1) + p^2 + q^2]\]
\[= |G|^3 - 2|G|^2(p + q - 2) - |G|[2(p + q + 1) + p^2 + q^2]\]

5. $M_2(\zeta_G)$
\[= 2|E|^2 + 2|V|(|V| - 1) M_1(\zeta_G) + M_2(\zeta_G)\]
\[= \frac{1}{10} |V|(1 - |V|) + (p + q)[7n + 5] + 3(p^2 + q^2)\]
\[= \frac{1}{10} |G|(1 - |G|) + (p + q)[7n + 5] + 3(p^2 + q^2)\]

4. **Conclusion**

In this paper, we partition the vertices and edges of the finite abelian group $Z_{pq}$, where $p < q$ and $p \neq q$ based on the degree of the vertices. The generators of the group is calculated from the vertex partition. Also zagreb indices are calculated with the help of power graph.

**References**


