Euler wavelet based numerical scheme for the solutions of parabolic partial differential equations

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Abstract
Many physical problems when analysed assumes the form of a partial differential equations. Recently wavelet transforms serves as a very useful tool in solving partial differential equations. In this paper, we proposed an efficient numerical scheme based on Euler wavelets for the solutions of parabolic partial differential equations. Some Illustrative examples are included to demonstrate the validity and applicability of the proposed scheme. Numerical findings of the problems with both initial and boundary conditions shows the efficiency and accuracy of the present scheme.

Keywords
Euler wavelets; parabolic partial differential equations; Collocation method.

AMS Subject Classification
65T60, 97N40, 49K20.

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1. Introduction
Numerous partial differential equations originate to depict a variety of kind of phenomena such as a mathematical model of heat transfer and transport phenomena in a viscous fluid. Analytical methods for solving many physical problems in these cases are very restricted and except for a limited number of these problems, we come across difficulties in finding their analytical solutions. Finite difference and Finite element based numerical methods need a large amount of computation and usually the effect of round-off error causes the loss of accuracy [1, 2].

In recent days, numerical methods via wavelets are tremendously growing much faster[4–10]. Hence, we are able to develop new wavelet method for the solutions of parabolic partial differential equations. Euler wavelet method is one of them because; Euler wavelets are continuous functions in respective domain [12]. Many authors researched the Euler wavelet to solve the ordinary differential and integral equations but there is no literature survey on solutions of parabolic partial differential equations using Euler wavelets, this Impe tus us to study the solution of parabolic partial differential equations using Euler wavelets.

The paper is organized as follows: in Section 2, we describe the basic formulation of the Euler wavelets. Section 3 is devoted to the numerical method for solving the parabolic partial differential equations. In Section 4, two numerical illustrative examples are considered. Finally, a conclusion is expressed in Section 5.

2. Euler wavelets
Euler wavelets are defined as

$$\psi_{i,j}(t) = \begin{cases} 2^{j-i+1} \tilde{P}_j \left(2^{j-i} t - i + 1 \right), & \frac{i-1}{2^{j-i}} \leq t \leq \frac{i}{2^{j-i}} \\
0, & \text{Otherwise} \end{cases} \quad (2.1)$$

with

$$\tilde{P}_j(t) = \begin{cases} 1, & j = 0, \\
\frac{P_j(t)}{\sqrt{\frac{(2j)!}{(j)!}} P_{2j}(0)}, & j > 0 \end{cases} \quad (2.2)$$
where \( j = 0, 1, 2, \ldots, M - 1 \) and \( i = 1, 2, \ldots, 2^{j-1}, l \) is assumed any positive integer. The coefficient \( \sqrt{\frac{(2l-1)^{1/2}}{(2j-1)!}} \) is for normality, the dilation parameter is \( a = 2^{-(l-1)} \) and the translation parameter \( b = (i-1)2^{-(l-1)} \). Here \( P_j(t) \) are the well-known Euler polynomials of order \( j \) which can be defined by means of the following generating functions [5],

\[
\frac{2e^t}{e^t + 1} = \sum_{j=0}^{\infty} P_j(t) \frac{s^j}{j!} \quad (|s| < \pi)
\]

\( P_j \) are \( 2^j P_j(1/2) \) Euler numbers. Also, The Euler polynomials of the first kind for \( j = 0, 1, 2, \ldots \) \( N \) can be constructed from the following relation,

\[
\sum_{j=0}^{\infty} \binom{j}{i} = P(t) + P_j(t) = 2t^j \text{ where } \binom{j}{i} \text{ is a binomial coefficient.}
\]

We are going to find, \( P_0(t) = 1, P_1(t) = t - \frac{1}{2} P_2(t) = t^2 - t, \ldots \) and so on. We have,

\[
\int_0^1 P_j(t)P_i(t)dt = (-1)^{(i-j)} \frac{j!(i+1)!}{(j+i+1)!} P_{j+i+1}(0), j, i \geq 1
\]

Euler polynomials form a complete basis over the interval \([0, 1]\) Furthermore, when \( x = 0 \), we have \( P_0(0) = 1, \ P_1(0) = \frac{1}{2}; P_2(0) = 0, P_3(0) = 0, \ldots \) and so on.

### 3. Euler wavelet based numerical method

In this section, Euler wavelet together with the collocation points to solve parabolic partial differential equations is presented.

Consider the general parabolic partial differential equations of the form:

\[
\frac{\partial U}{\partial t}(x, t) = \delta \frac{\partial^2 U}{\partial x^2}(x, t) + F(x, t)
\]

with the initial and boundary conditions,

\[
U(x, 0) = G(x), U(0, t) = P(t) \quad U(L, t) = Q(t)
\]

where \( \delta \) is known constants, \( G(x), P(t) \text{ and } Q(t) \) are continuous real valued functions. Let us assume that for \( l = 1, \alpha = \frac{1}{2} \) and \( i = 1, \)

\[
\frac{\partial^3 U}{\partial x^2 \partial t}(x, t) \approx B^T \psi(x, t) = \sum_{i=1}^{l} C_{i,j}(x) \psi_i(x) + b_{1,j} \psi_{i+1}(t)
\]

(3.3)

where \( B^T = (c_{1,1}, \ldots, c_{1,j}),(b_{1,1}, \ldots, b_{1,j}) \psi(x, t) = (\psi_{1,1}(t), \psi_{1,2}(x), ..., \psi_{1,j}, ..., \psi_{1,j+1}) \)

integrate eq (3.3) with respect to \( t \) from 0 to \( t \) we get the following,

\[
\frac{\partial^2 U}{\partial x^2}(x, t) = \frac{\partial^2 U}{\partial x^2}(x, 0) + \int_0^t B^T \psi(x, t)dt
\]

(3.4)

Now integrate Eqn.(3.4) with respect to \( x \) from 0 to \( x \) we get,

\[
\frac{\partial U}{\partial x}(x, t) = \frac{\partial U}{\partial x}(0, x) + \int_0^x \int_0^t B^T \psi(x, t)dt \ dx
\]

(3.5)

Now integrate Eqn.(3.5) with respect to \( x \) from 0 to \( x \) we get,

\[
U(x, t) = U(0, t) + \int_0^x \int_0^t \frac{\partial U}{\partial x}(0, t) + \frac{\partial U}{\partial x}(0, 0) + U(x, 0) - U(0, 0) + \int_0^x \int_0^t B^T \psi(x, t)dt \ dx \ dx
\]

(3.6)

Now differentiate Eqn.(3.6) with respect to \( t \), we get,

\[
\frac{\partial U}{\partial t}(x, t) = \frac{\partial U}{\partial t}(0, t) + \frac{\partial^2 U}{\partial x \partial t}(0, t) + \frac{d}{dt} \left( \int_0^x \int_0^t B^T \psi(x, t)dt \ dx \right)
\]

(3.7)

Combining Eqn.(3.7), Eqn.(3.6), Eqn.(3.5) and Eqn.(3.4) in Eqn.(3.1) and collocate the obtained equation at the collocation points \( x_s, t_n = \frac{n+1}{N+1} i = 1, 2, 3, \ldots j \) and \( j \) is any positive integer. Then solve obtained system of equations by suitable solver. We can obtain the Euler wavelet coefficients \( B^T \), and then substitute these obtained Euler wavelet coefficients in Eqn.(3.6), will contribute the Euler wavelet based numerical solution of Eqn.(3.1).

### 4. Illustrative examples

#### Example 1.

Firstly we are considering the parabolic partial differential equation of the form [6],

\[
\frac{\partial U}{\partial t}(x, t) = \frac{\partial^2 U}{\partial x^2}(x, t)
\]

with initial and boundary conditions \( U(x, 0) = \sin(\pi x) \ U(0, t) = 0, \ U(1, t) = 0 \)

Its exact solution is \( U(x, t) = \sin(\pi x)e^{-\pi^2 t} \). Using the Euler wavelet based numerical method we were presented in section 3, and we obtained the approximate solution is given in the form of space-time graph. Fig.1(A) and Fig.1(B) shows us the comparison of numerical and Analytical solution at different values of \( t \).

**Figure 1.** [A] Approximate solutions of the example 1, [B] Exact solutions of the example 1

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**5. Conclusion**

In the present study, the Euler wavelet based numerical scheme is presented for the solution of the parabolic partial differential equation and their initial and boundary conditions. The proposed scheme is tested on two examples and the results are quite satisfactory in comparison with the exact solutions. This new scheme is introduced to determine the numerical solution of parabolic partial differential equation and easy to implement in computer programs and can be extended for higher order also with slight changes in the proposed scheme.

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