Intuitionistic fuzzy 2-metric groups

Sambandan Sivaramakrishnan$^1$* Srinivasan Vijayabalaji$^2$ and Ranganathan Rajasekaran$^3$

Abstract
This paper is an inspiration received from the hybrid structures of metric space and group theory in fuzzy setting namely fuzzy metric group. As a generalization of this structure in intuitionistic fuzzy setting we introduce the notion of intuitionistic fuzzy 2-metric group (IF 2-MG) and study some properties on it.

Keywords
Fuzzy metric space, fuzzy 2-metric space, intuitionistic fuzzy metric space, intuitionistic fuzzy 2-metric space.

AMS Subject Classification
47H10, 54A40, 54H25, 46S40.

1 Introduction .................................. 192
2 Preliminaries .................................. 192
3 Intuitionistic fuzzy 2-Metric groups ................. 193
References ...................................... 194

1. Introduction

Metric space is one of the finest structure in the study of Topology. This structure was generalized to two dimension and $n$-dimension [7]. In their work on statistical metric space, Schweizer and Sklar introduced the novel ideal of triangular norm and triangular co-norm.

Zadeh [22] made a remarkable contribution in defining a new uncertainty theory namely fuzzy set. Though there are several applications of this theory many Mathematicians started working in generalizing concept of various algebraic and topological structures in fuzzy setting.

George and Veeramani [8,9] made significant contribution in fuzzy metric space. They generalized the idea given in [12] using $T$-norm and studied about the topological spaces induced by fuzzy metric.


The above literature impressed the authors to develop the present structure namely intuitionistic fuzzy 2-Metric group as a generalization of intuitionistic fuzzy metric group. Some interesting results on this structure are also provided.

2. Preliminaries

We present some basic theories needed for our new approach. Definition 2.1[9]. Suppose $*$ is mapping from $[0,1] \times [0,1]$ to $[0,1]$ satisfies the below conditions namely,

(T$_1$) $*$ is commutative and associative,
(T$_2$) $*$ is continuous,
(T$_3$) $a * 1 = a, a \in [0,1],$
(T$_4$) $a * b \leq c * d$, whenever $a \leq c$ and $b \leq d$ for each $a, b, c, d \in [0,1]$, then $*$ is called as $T$-norm.

Remark 2.2[9]. Suppose $\circ$ is a mapping from $[0,1] \times [0,1] \rightarrow [0,1]$ satisfying (T$_1$), (T$_2$), (T$_4$) of the above definition with (T$_1$) respectively $a \circ a = a$ for all $a \in [0,1]$, then $\circ$ is called as $T$-co-norm.

Definition 2.3[8,9]. The 3-tuple $(X, \zeta, *)$ is said to be a fuzzy metric space if $X$ is an arbitrary set, * is a continuous $t$-norm
and $M$ is a fuzzy set on $X^2 \times (0, \infty)$ satisfying the following conditions: for all $x, y, z \in X$ and $s, t > 0$:

1. $(FM - 1) \; \psi(x, y, t) > 0$,
2. $(FM - 2) \; \psi(x, y, t) = 1$ if and only if $x = y$,
3. $(FM - 3) \; \psi(x, y, t) = \psi(y, x, t)$,
4. $(FM - 4) \; \psi(x, y, t) \ast \psi(y, z, s) \leq \psi(x, z, t + s)$,
5. $(FM - 5) \; \psi(x, y, \cdot) : [0, \infty) \to [0, 1]$ is continuous.

If $(X, \psi, \ast)$ is a fuzzy metric space, we say that $(\psi, \ast)$ is a fuzzy metric on $X$.

**Example 2.4 [8]**

Let $(X, \psi, \ast)$ be a metric space. Define $a \ast b = \min\{a, b\}$ and $\psi(x, y, t) = \frac{t}{t + d(x, y)}$ for all $x, y \in X$ and $t > 0$.

Then $(X, \psi, \ast)$ is a fuzzy metric space. It is called fuzzy metric space induced by $d$.

**Definition 2.4 [11]**

A 5-tuple $(X, \psi, \ast, \circ, \cdot)$ is said to be an Intuitionistic Fuzzy Metric space (briefly IFMs) if $X$ is an arbitrary set, $\ast$ is a continuous $\ast$-norm, $\circ$ is a continuous $\circ$-conorm and $\psi, \ast, \circ, \cdot$ are fuzzy sets on $X^2 \times [0, \infty]$ satisfying the following conditions: for all $x, y, z \in X$ and $s, t > 0$,

1. $(IFMs - 1) \; \psi(x, y, t) + \psi(y, x, t) \leq 1$,
2. $(IFMs - 2) \; \psi(x, y, t) > 0$,
3. $(IFMs - 3) \; \psi(x, y, t) = 1$ if and only if $x = y$,
4. $(IFMs - 4) \; \psi(x, y, t) = \psi(y, x, t)$,
5. $(IFMs - 5) \; \psi(x, y, t) \ast \psi(y, z, s) = \psi(x, z, t + s)$,
6. $(IFMs - 6) \; \psi(x, y, t) : [0, \infty) \to [0, 1]$ is continuous,
7. $(IFMs - 7) \; \psi(x, y, 0) = 1$.

Then $(\psi, \ast, \circ, \cdot)$ is an intuitionistic fuzzy metric on $X$. The functions $\psi(x, y, t)$ and $\psi(y, x, t)$ denote the degree of nearness and the degree of non-nearness between $x$ and $y$ with respect to $t$, respectively.

**Example 2.5 [11]**

(Induced intuitionistic fuzzy metric space)

Let $(X, \psi, \ast, \circ, \cdot)$ be an intuitionistic fuzzy metric space. Define $a \ast b = ab$ and $a \circ b = \min\{a, a + b\}$ for all $a, b \in [0, 1]$. Let $\psi, \ast, \circ, \cdot$ be fuzzy sets on $X^2 \times (0, 1)$ defined as follows:

$\psi(x, y, t) = \frac{t}{t + d(x, y)}$.

Then $(X, \psi, \ast, \circ, \cdot)$ is an intuitionistic fuzzy metric space. We call this intuitionistic fuzzy metric induced by a metric as standard intuitionistic fuzzy metric.

**Definition 2.6 [11]**

For a non-empty set $X$ with $\ast$ and $\circ$ being $\ast$-norm and $\circ$-norm, a five tuple $(X, \psi, \ast, \circ, \cdot)$ is termed as intuitionistic fuzzy 2-metric space (IF 2-MS), where $\psi$ and $\ast, \circ, \cdot$ are two fuzzy sets defined on $X^3 \times (0, \infty)$ known as membership and non-membership functions respectively if it satisfies the following conditions.

For all $x, y, z \in X$ and $s, t > 0$,

1. $\psi(x, y, z, t) + \psi(y, z, t) \leq 1$,
2. $\psi(x, y, z, t) > 0$,
3. $\psi(x, y, z, t) = 1$ if and only two of $x, y, z$ are equal,
4. $\psi(x, y, z, t) = \psi(y, x, z, t) = \psi(y, z, x, t)$,
5. $\psi(x, y, w, t) \ast \psi(x, w, z, s) \ast \psi(y, w, z, r) \leq \psi(x, y, z, t + s + r)$,
6. $\psi(x, y, z, \cdot) : [0, \infty) \to [0, 1]$ is continuous.

3. **Intuitionistic fuzzy 2-Metric groups**

We generalize the idea of intuitionistic fuzzy metrizable and intuitionistic fuzzy metric group as follows.

**Definition 3.1**

An intuitionistic fuzzy 2-metrizable is a topological space $(X, \psi, \ast, \circ, \cdot)$ such that its topology $\tau_{\psi, \ast, \circ, \cdot}$ induced by $(\psi, \ast, \circ, \cdot)$ coincides
with \( \tau \).

**Remark 3.2.** (i) \((\zeta, \psi)_2\) is compatible with \(\tau\).

(ii) A topological group \(G\) is called intuitionistic fuzzy 2-metrizable if it is an intuitionistic fuzzy 2-metrizable as a topological space.

**Definition 3.2.** The structure \((G, \cdot, \zeta, \psi, *, \odot)\) such that 
\((G, \zeta, \psi, *, \odot)\) is called an intuitionistic fuzzy 2-metric group where \((G, \cdot, \zeta, \psi, *, \odot)\) is an intuitionistic fuzzy metric space and \((G, \cdot, \zeta, \psi)\) being topological group.

**Definition 3.3.** Suppose the following conditions hold.

(i) \(\zeta(x, y, z, t) = \zeta(ax, ay, z, t)\),

(ii) \(\psi(x, y, z, t) = \psi(ax, ay, z, t)\), where \(a, x, y, z \in G\) and \(t > 0\).

Then the intuitionistic fuzzy 2-metric \((\zeta, \psi)_2\) is called as left invariant.

**Definition 3.4.** Suppose the following conditions hold.

(i) \(\zeta(x, y, z, t) = \zeta(xa, ya, z, t)\),

(ii) \(\psi(x, y, z, t) = \psi(xa, ya, z, t)\), where \(a, x, y, z \in G\) and \(t > 0\).

Then \((\zeta, \psi)_2\) is called as right invariant.

**Theorem 3.5.** For a given intuitionistic 2-metrizable topological group \(G\) left (right) invariant intuitionistic fuzzy 2-metric on \(G\), the uniform structure \((G, \mathcal{U}, (\zeta, \psi)_2)\) induced by \((\zeta, \psi)_2\) coincides with the left (right) uniform structure \((G, \mathcal{U})\).

**Proof.** We have proved that the two uniformities have a common base. Let, for each \(n \in \psi\), take a basic neighborhood of \(e\).

\[ K_n = B(e, \frac{1}{n}, \frac{1}{n}) = \{ y : \zeta(e, y, z, \frac{1}{n}) > 1 - \frac{1}{n}, \psi(e, y, z, \frac{1}{n}) < \frac{1}{n} \} \] for all \(z \in G\).

Hence, for \(\mathcal{U}\),

\[ L_n = \{ (x, y) : x^{-1}y \in K_n \} = \{ (x, y) : \zeta(e, x^{-1}y, z, \frac{1}{n}) > 1 - \frac{1}{n}, \psi(e, x^{-1}y, z, \frac{1}{n}) < \frac{1}{n} \} \] for all \(z \in G\).

Since \((\zeta, \psi)_2\) is left invariant, \(L_n = K_n, \forall n \in \psi\).

**Lemma 3.6.** If \((\zeta, \psi)_2\) is a left invariant intuitionistic 2-fuzzy metric on \(G\). Then \(B(e, \frac{1}{n}, \frac{1}{n}) = B(e, \frac{1}{n}, \frac{1}{n})\) is symmetric for all \(n \in \mathbb{N}\) and \(B(x, \frac{1}{n}, \frac{1}{n}) = x \cdot B(e, \frac{1}{n}, \frac{1}{n})\) for all \(n \in \mathbb{N}\).

**Proof.** Let \(x \in B(e, \frac{1}{n}, \frac{1}{n})\). Then \(\zeta(e, x, z, \frac{1}{n}) > 1 - \frac{1}{n}\) and \(\psi(e, x, z, \frac{1}{n}) < \frac{1}{n}\)

\[ \Longrightarrow \zeta(e, x, z, \frac{1}{n}) > 1 - \frac{1}{n} \quad \text{and} \quad \psi(e, x, z, \frac{1}{n}) < \frac{1}{n} \]

\[ \Longrightarrow \zeta(e, x^{-1}y, z, \frac{1}{n}) > 1 - \frac{1}{n} \quad \text{and} \quad \psi(e, x^{-1}y, z, \frac{1}{n}) < \frac{1}{n} \]

\[ \Longrightarrow x^{-1}y \in B(e, \frac{1}{n}, \frac{1}{n}) \]

From the following equalities we see that the second part of the Lemma.

\(B(e, \frac{1}{n}, \frac{1}{n}) = \{ y \in G : \zeta(e, y, z, \frac{1}{n}) > 1 - \frac{1}{n}, \psi(e, y, z, \frac{1}{n}) < \frac{1}{n} \} \) for all \(z \in G\) = \(\{ y \in G : \zeta(e, x^{-1}y, z, \frac{1}{n}) > 1 - \frac{1}{n}, \psi(e, x^{-1}y, z, \frac{1}{n}) < \frac{1}{n} \} = \{ y \in G : x^{-1}y \in B(e, \frac{1}{n}, \frac{1}{n}) \} = \{ y \in G : y \in x \cdot B(e, \frac{1}{n}, \frac{1}{n}) \} \)

**Proposition 3.7.** A topological group approves a compatible left invariant intuitionistic fuzzy 2-metric if and only if it approves a compatible right invariant intuitionistic fuzzy 2-metric.

**Proof.** Let \((\zeta, \psi)_2\) be a compatible left invariant intuitionistic fuzzy 2-metric on a topological group \(G\). Define two functions \(\zeta, \psi\) on \(G \times G \times (0, \infty)\) as follows:

\[ \zeta(x, y, z, t) = \zeta(x^{-1}, y, z, t) \quad \text{and} \quad \psi(x, y, z, t) = \psi(x^{-1}, y, z, t) \]

whenever \(x, y \in G\) and \(t > 0\).

\[ (\zeta, \psi)_2 \]

is a right invariant intuitionistic fuzzy 2-metric on \(G\). Indeed

\[ \zeta(xa, ya, z, t) = \zeta((xa)^{-1}, (ya)^{-1}, z, t) = \zeta(a^{-1}x^{-1}a^{-1}y^{-1}z, t) = \zeta(x^{-1}, y^{-1}, z, t) \]

Similarly, \(\psi(xa, ya, z, t) = \psi(x, y, z, t)\).

Now, we will show that it is compatible one.

Since \(B(e, \frac{1}{n}, \frac{1}{n})\) is symmetric for all \(n \in \psi\), then we have \(B(e, \frac{1}{n}, \frac{1}{n}) = B(e, \frac{1}{n}, \frac{1}{n})\).

Hence, \((B(e, \frac{1}{n}, \frac{1}{n}))_{n \in \psi}\) is a base neighbourhood of \(e\) and that every topological group is homogeneous.

**References**


**********

ISSN(P):2319 – 3786
Malaya Journal of Matematik
ISSN(O):2321 – 5666
**********