Radio labeling to the transformation of cycle

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Abstract
In this paper we have studied the radio labeling to the transformation of cycle. Our special interest is to find a general formula to label the vertices and find the radio number of the graph.

Keywords
Radio Labeling, Complete Graph, Transformation graph, Radio number.

AMS Subject Classification
05C78.

1 Introduction
Graph transformation is one of the graph operation in graph theory. For each graph G, we can obtain eight graphical transformations. Radio Labeling is a recent developing area to label the vertices of the graph in a graph theory. Radio Labeling is use for X-ray crystallography, coding theory, network security, network addressing, channel assignment process, social network analysis such as connectivity, scalability, routing, computing and cell biology etc.

2 Definitions

Definition 2.1. A graph G(V,E) is the collection of non empty set of points / vertices and the relation between the points called edges.

Definition 2.2. Cycle is a two regular graph with n vertices and is denoted by Cn.

Definition 2.3. A complete graph is a simple undirected graph in which every pair of distinct vertices are connected by a unique edge. Complete graph with m vertices is denoted by Kn.

Definition 2.4. Labeling is an assignment to the vertices of the graph and is denoted by f(V).

Definition 2.5. The distance between any pair of vertices is denoted by d(u,v) is the shortest u – v path in G.

Definition 2.6. Let G be a graph and v be a vertex of G. The eccentricity of the vertex v is the maximum distance from v to any other vertex of G.
That is e(v) = max {d(v,w) : w ∈ V(G)}.

Definition 2.7. The diameter of G is the maximum eccentricity among the vertices of G. Which is denoted by diam(G) = max {e(v) : v ∈ V(G)}.

Definition 2.8. The span of f is defined as max{|f(u) − f(v)| : u, v ∈ V(G)}.

Definition 2.9. A radio labeling of G is an assignment of non negative integers to the vertices of G satisfying,
|f(u) − f(v)| ≥ diam(G) + 1 − dc(u,v) ∀ u, v ∈ V(G).
The radio number denoted by rn(G) is the minimum span of a radio labeling of G.

Definition 2.10. Let G = (V(G),E(G)) be a graph and Ghrq = (V(G)∪E(G),E(Ghrq)); h,r,q = ± be a transformation of G where h deals with an adjacent between any two vertices in G, r deals with an adjacent between any two edges in G and q deals with an incident of the edges to corresponding vertices in G. u,v ∈ V(G)∪E(G). u and v are adjacent in G if
(i) u,v ∈ V(G),u and v are adjacent in Ghrq if h = +; u and v are not adjacent in Ghrq if h = −.
(ii) \( u, v \in E(G), \) \( u \) and \( v \) are adjacent in \( G^{hrq} \) if \( r = + \); \( u \) and \( v \) are not adjacent in \( G^{hrq} \) if \( r = - \).

(iii) \( u \in V(G), v \in E(G), u \) and \( v \) are adjacent in \( G^{hrq} \) if \( q = + ; u \) and \( v \) are not adjacent in \( G^{hrq} \) if \( q = - \).

### 3. Theorems

**Theorem 3.1.** Let \( C_n : n > 3 \) be a cycle graph with \( n \) vertices and \( G = C_n^{++} \) be the transformation of \( C_n \) with \( 2n \) vertices then

\[
\text{rn}(G) = \begin{cases} 
2n - 1 & ; \quad 3 < n \leq 5 \\
\left\lfloor \frac{n}{2} \right\rfloor + 2(n + 2) & ; \quad 5 < n < 8 \\
3n - 1 & ; \quad 8 \leq n 
\end{cases}
\]

**Proof.** Let \( C_n : n > 3 \) be a cycle with \( n \) vertices and \( n \) edges. The vertex set of \( C_n \) is \( V(C_n) = \{ v_i / 1 \leq i \leq n \} \) and the edge set of \( C_n \) is \( E(C_n) = \{ e_i = v_i v_{i+1}, e_n = v_1 v_n / 1 \leq i \leq n \} \) such that \( v_i \) is adjacent with \( v_{i-1} \) and \( v_{i+1} \). The vertex \( v_i \) is adjacent with \( v_1 \) and \( v_{n-1} \).

Let \( G = C_n^{++} \) be a transformation of \( C_n \) with \( |V(G)| = 2n \). The vertex set of \( G \) is \( V(G) = V(C_n) \cup E(C_n) \). The adjacency in \( G \) is as follows.

\[
N(v_i) = \{v_{i-1}, v_{i+1}, e_{i-1}, e_i\} \\
N(v_1) = \{v_2, v_n, e_1\} \\
N(v_n) = \{v_1, v_{n-1}, e_{n-1}, e_n\} \\
N(e_i) = \{v_i, v_{i+1}, e_i; |i-j| \neq 1 \text { and } n - 1\}
\]

**Case: 1**

Let \( G = C_n^{++}; 3 < n \leq 5 \).

The longest path of \( C_n^{++} \) is

(i) \( e_i - e_j \) if \( |i-j| = 1 \) and \( |i-j| = n - 1 \)

(ii) \( v_i - e_j \) if \( |i-j| \neq 0, 1 \) and \( |i-j| = n - 1 \)

(iii) \( v_i - v_j \) if \( |i-j| \neq 1 \) and \( |i-j| \neq n - 1 \)

Take any one of the longest path

\[ e_i - e_j \] if \( |i-j| = 1 \) and \( |i-j| = n - 1 \)

The \( e_i - e_j \) path through the vertex \( e_{i+3} \).

\[ d(e_i-e_j) = 2. \]

From (i), (ii) and (iii), \( \text{diam}(G) = 2. \)

Define \( f : V(G) \rightarrow N \cup \{0\} \)

The vertex set of \( G \) can be labeled as follows.

\[
f(v_i) = \begin{cases} 
\left\lfloor \frac{3}{2} \right\rfloor & \text{if } i = 3k - 2 \quad ; \quad 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \\
\left\lfloor \frac{3}{2} \right\rfloor + 2 & \text{if } i = 3k - 1 \quad ; \quad 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \\
\left\lfloor \frac{3}{2} \right\rfloor + 4 & \text{if } i = 3k \quad ; \quad 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor 
\end{cases}
\]

\[
f(e_i) = \left\lfloor \frac{n}{2} \right\rfloor + 2(i+2) \quad ; \quad 1 \leq i \leq n.
\]

By the above labeling \( f(v_i) = 0 \) is minimum and

\[
f(e_i) = \left\lfloor \frac{n}{2} \right\rfloor + 2(n+2)
\]

Therefore, \( \text{rn}(G) = 2n - 1 \).

**Case: 2**

Let \( G = C_n^{++}; n > 5 \).

The longest path of \( C_n^{++} \) is

(a) \( v_1 - v_{\left\lceil \frac{n+1}{2} \right\rceil} \) if \( n \) is even.

(b) \( v_1 - v_{\left\lfloor \frac{n}{2} \right\rfloor} \) or \( v_1 - v_{\left\lceil \frac{n+2}{2} \right\rceil} \) if \( n \) is odd.

Take any one of the longest path.

\[ (a) \quad v_1 - v_{\left\lceil \frac{n+1}{2} \right\rceil} \] if \( n \) is even.

The path \( v_1 - v_{\left\lceil \frac{n+1}{2} \right\rceil} \) passing through the vertices \( e_1 \) and \( e_{\left\lceil \frac{n+1}{2} \right\rceil} \).

\[ \Rightarrow \text{diam}(G) = 3. \]

**Step: 1**

Let \( C_n : 5 < n < 8 \) be a cycle and \( G = C_n^{++}; 5 < n < 8 \) be a transformation of \( C_n \).

\[
f(v_i) = \begin{cases} 
\left\lceil \frac{3}{2} \right\rceil & \text{if } i = 3k - 2 \quad ; \quad 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \\
\left\lceil \frac{3}{2} \right\rceil + 2 & \text{if } i = 3k - 1 \quad ; \quad 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor \\
\left\lceil \frac{3}{2} \right\rceil + 4 & \text{if } i = 3k \quad ; \quad 1 \leq k \leq \left\lfloor \frac{n}{2} \right\rfloor 
\end{cases}
\]

\[
f(e_i) = \left\lceil \frac{n}{2} \right\rceil + 2(i+2) \quad ; \quad 1 \leq i \leq n.
\]

By the above labeling \( f(v_i) = 0 \) is minimum and

\[
f(e_i) = \left\lfloor \frac{n}{2} \right\rfloor + 2(n+2)
\]

Therefore, \( \text{rn}(G) = \left\lceil \frac{n}{2} \right\rceil + 2(n+2). \)

**Step: 2**

Let \( C_n; n \geq 8 \) be a cycle graph with \( n \) vertices and \( n \) edges. \( G = C_n^{++} \) be a transformation of \( C_n \).
Theorem 3.3. Let \( C_n; n > 3 \) be a cycle graph with \( n \) vertices and \( G = C_n^{++} \) be the transformation of \( C_n \) with \( 2n \) vertices then 
\[
 f(v_i) = \begin{cases} 
 \left[ \frac{n}{2} \right] &; i = 3k - 2; \quad 1 \leq k \leq \left[ \frac{n}{2} \right] \\
 \left[ \frac{n}{2} \right] + 2(n + 2) &; 5 < n < 8 \\
 3n - 1 &; 8 \leq n
\end{cases}
\]
\[
f(e_i) = n + 2i - 1; \quad 1 \leq i \leq n.
\]

By the above labeling, 
\[
f(v_i) = 0 \text{ is minimum and } f(e_i) = 3n - 1 \text{ is maximum for any value of } i.
\]

Therefore, \( rn(G) = 3n - 1 \).

Hence the proof.

Corollary 3.2. Let \( C_n; n > 3 \) be a cycle graph with \( n \) vertices and \( G = C_n^{--} \) be the transformation of \( C_n \) with \( 2n \) vertices then 
\[
 rn(G) = \begin{cases} 
 2n - 1 &; 3 < n \leq 5 \\
 \left[ \frac{n}{2} \right] + 2(n + 2) &; 5 < n < 8 \\
 3n - 1 &; 8 \leq n
\end{cases}
\]

Proof. Let \( G = C_n^{--} \) be a transformation of \( C_n \) with \( 2n \) vertices. 
From \( C_n^{--} \), we interchange \( v_i \) to \( e_{n-i+1} \) and \( e_j \) is convert to \( v_{n-j+1} \). 
Then we get the graph \( G = C_n^{--} \). 
The adjacency in \( G = C_n^{--} \) is as follows: 
\[
 N(e_i) = \{e_{i-1}, e_{i+1}, v_{i+1}, v_i\} \\
 N(e_1) = \{e_{2}, e_n, v_1, v_2\} \\
 N(e_n) = \{e_{1}, e_{n-1}, v_1, v_n\} \\
 N(v_i) = \{e_i, e_{i-1}, v_j; |i - j| \neq 1 \text{ and } n - 1\}
\]

The above graphs \( C_n^{--} \) and \( C_n^{++} \) are isomorphic. 

Therefore, \( rn(C_n^{--}) = rn(C_n^{++}) \)
\[\Rightarrow rn(G) = \begin{cases} 
 2n - 1 &; 3 < n \leq 5 \\
 \left[ \frac{n}{2} \right] + 2(n + 2) &; 5 < n < 8 \\
 3n - 1 &; 8 \leq n
\end{cases}
\]

Hence the proof.

Theorem 3.3. Let \( C_n; n > 3 \) be a cycle graph with \( n \) vertices and \( n \) edges. 
Let \( G = C_n^{++} \) be a transformation of \( C_n \) with \( 2n \) vertices then \( rn(G) = 2n - 1 \). 

Proof. Let \( C_n; n > 3 \) be a cycle with \( n \) vertices and \( n \) edges. 
The vertex set of \( C_n \) is \( V(C_n) = \{v_i/1 \leq i \leq n\} \) and the edge set of \( C_n \) is \( E(C_n) = \{e_i = v_{i+1}v_i, e_n = v_{1}v_n/1 \leq i \leq n\} \) such that \( v_i \) is adjacent with \( v_{i-1} \) and \( v_{i+1} \). 
The vertex \( v_i \) is adjacent with \( v_2 \) and \( v_n \). 
The vertex \( v_n \) is adjacent with \( v_1 \) and \( v_{n-1} \).

Let \( G = C_n^{--} \) be a transformation of \( C_n \) with \( |V(G)| = 2n \). 
The vertex set of \( G \) is \( V(G) = V(C_n) \cup E(C_n) \). The adjacency in \( G \) is as follows.
\[
 N(v_i) = \{e_{i-1}, e_i, v_j; |i - j| \neq 1 \text{ and } n - 1\} \\
 N(e_i) = \{v_i, v_{i+1}, e_j; |i - j| \neq 1 \text{ and } n - 1\}
\]
The longest path of \( C_n^{--} \) is 
\[
(A)v_i - v_j \quad \text{if} \quad |i - j| = 1, n - 1. \\
(B)e_i - e_j \quad \text{if} \quad |i - j| = 1, n - 1. \\
(C)v_i - e_j \quad \text{if} \quad i - j \neq 1, 0, n - 1.
\]

Take any one of the longest path.
\[\Rightarrow v_i - v_j \text{ if } i - j \neq 1, n - 1.
\]
The path \( v_i - v_j \) passing through the vertex \( v_{i+1} \). 
Therefore, \( (A), (B) \) and \( (C) \), \( diam(G) = 2 \)

Define \( f: V(G) \to N \cup \{0\} \)
\[
f(v_i) = i - 1; 1 \leq i \leq n. \\
f(e_i) = n + i - 1; 1 \leq i \leq n.
\]

By the above labeling, 
\[
f(v_i) = 0 \text{ is minimum and } f(e_i) = 2n - 1 \text{ is maximum for any value of } i.
\]

Therefore, \( rn(G) = 2n - 1. \)

Hence the proof.

Theorem 3.4. Let \( C_n; n > 3 \) be a cycle graph with \( n \) vertices and \( n \) edges. 
Let \( G = K_m + C_n^{bry} \); \( h, r = \pm d = + \) where \( K_m \) be a complete graph with \( m \) vertices and \( C_n^{bry} \) be a transformation of \( C_n \) with \( 2n \) vertices then \( rn(G) = 2(m + n) - 1. \)

Proof. Let \( K_m \) be a complete graph with \( m \) vertices.
The vertex set of \( K_m \) is \( V(K_m) = \{u_i/1 \leq i \leq m\} \)
Let \( C_n; n > 3 \) be a cycle with \( n \) vertices and \( n \) edges. 
The vertex set of \( C_n \) is \( V(C_n) = \{v_i/1 \leq i \leq n\} \).
The edge set of \( C_n \) is \( E(C_n) = \{e_i = v_{i+1}v_i, e_n = v_{1}v_n/1 \leq i \leq n\} \) such that \( v_i \) is adjacent with \( v_{i-1} \) and \( v_{i+1} \). 
The vertex \( v_i \) is adjacent with \( v_2 \) and \( v_n \). The vertex \( v_n \) is adjacent with \( v_1 \) and \( v_{n-1} \).
Let $C_n^{hrq}$ be a transformation of $C_n$ with $|V(C_n^{hrq})| = 2n$.
The vertex set of $C_n^{hrq}$ is $V(C_n^{hrq}) = V(C_n) \cup E(C_n)$.

The adjacency in $K_m + C_n^{--}$ is as follows.

The adjacency in $K_m + C_n^{--}$ is as follows.

$$N(u_i) = \{ u_j; j \neq t, v_i, e_i \}$$

$$N(v_i) = \{ u_t, e_{i-1}, e_i, v_j; |i - j| \neq 1\text{and}n - 1 \}$$

$$N(e_i) = \{ u_t, v_i, v_{i+1}, e_j; |i - j| \neq 1\text{and}n - 1 \}$$

By the above adjacency,

$$diam(K_m + C_n^{--}) = 2$$

The adjacency in $K_m + C_n^{+++}$ is as follows.

$$N(u_i) = \{ u_j; j \neq t, v_i, e_i \}$$

$$N(v_i) = \{ u_t, v_{i-1}, v_{i+1}, e_{i-1}, e_i \}$$

$$N(v_1) = \{ u_t, v_2, v_n, e_1, e_n \}$$

$$N(v_n) = \{ u_t, v_1, v_{n-1}, e_{n-1}, e_n \}$$

$$N(e_i) = \{ u_t, e_{i-1}, e_{i+1}, v_{i+1}, v_i \}$$

$$N(e_1) = \{ u_t, e_2, e_n, v_1, v_2 \}$$

$$N(e_n) = \{ u_t, e_1, e_{n-1}, v_1, v_n \}$$

By the above adjacency,

$$diam(K_m + C_n^{+++}) = 2$$

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$$N(v_i) = \{ u_t, v_{i-1}, v_{i+1}, e_{i-1}, e_i \}$$

$$N(v_1) = \{ u_t, v_2, v_n, e_1, e_n \}$$

$$N(v_n) = \{ u_t, v_1, v_{n-1}, e_{n-1}, e_n \}$$

$$N(e_i) = \{ u_t, e_{i-1}, e_{i+1}, v_{i+1}, v_i \}$$

$$N(e_1) = \{ u_t, e_2, e_n, v_1, v_2 \}$$

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$$N(e_i) = \{ u_t, e_2, e_n, v_1, v_2 \}$$

$$N(e_n) = \{ u_t, e_1, e_{n-1}, v_1, v_n \}$$

By the above adjacency,

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By the above adjacency,

$$diam(K_m + C_n^{+++}) = 2$$

Define $f : V(G) \rightarrow N \cup \{0\}$

$$f(v_i) = 2(i - 1); 1 \leq i \leq n.$$  

$$f(e_i) = 2(n + i) - 5; i = 1, 2.$$  

$$f(e_i) = 2(i - 2) - 1; 3 \leq i \leq n.$$  

$$f(u_0) = 2n - 1 + 2t; 1 \leq t \leq m.$$  

By the above labeling,

$$f(v_i) = 0$$ is minimum and 
$$f(u_i) = 2n - 1 + 2m$$ is maximum for any value of $i$ and $t$.

Therefore, $rn(G) = 2(m + n) - 1$.

Hence the proof.

\[\square\]

References


