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# Convergence of fixed points for closed graph operator satisfies Zamfirescu conditions

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**Abstract**
In this article we establish a unique fixed point of a well known contraction mapping known as Kannan and Chatterjea mapping having a closed graph in a complete metric space. For such mappings, we consider an increasing sequence of subsets of a complete metric space into itself, so that a contraction condition is satisfied.

**Keywords**
Contraction mapping, closed graph, fixed point.

**AMS Subject Classification**

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## Contents



## 1. Introduction

A mapping $T:(M,d)\to(M,d)$ from a metric space $(M,\ d)$ into itself is said to have a closed graph if whenever $x_n\to x_0$ and $Tx_n\to y_0$ for some sequence $(x_n)$ in $M$ and some $x_0,\ y_0$ in $M$, we have $y_0=Tx_0$. C.Ganesa Moorthy and P.Xavier Raj [1] presented results for contraction mappings with variations in Domain. In this paper, we extend the Kannan and Chatterjea techniques to some contraction mapping of a closed graph to establish some fixed point theorems.

**Definition 1.1.** *Zamfirescu [4] introduced the Banach fixed point theorem by combining Banach, Kannan and Chatterjea. i.e., For a mapping $T:E\to E$, there exist real numbers $\alpha,\ \beta,\ \gamma$ satisfying $0<\alpha<1,\ 0<\beta,\ \gamma<\dfrac{1}{2}$ such that for each $x,\ y\in E$ at least one of the following is true*

1. $d(Tx,\ Ty)\le\alpha d(x,\ y)$
2. $d(Tx,\ Ty)\le\beta\left[d(x,\ Tx)+d(y,\ Ty)\right]$
3. $d(Tx,\ Ty)\le\gamma[d(x,\ Ty)+d(y,\ Tx)]$

**Theorem 1.2** ([1], Theorem 2.1)**.** *Let $(M,\ d)$ be a complete metric space. $T:M\to M$ have a closed graph. Let $M_1\subseteq M_2\subseteq M_3\subseteq\cdots$ be subsets of $M$ such that $T(M_i)\subseteq M_{i+1},\ \forall i,\ M=\bigcup_{j=1}^{\infty}M_j$ and $d(Tx,\ Ty)\le k_id(x,\ y)\ \forall x,\ y\in M_i$ where $k_i\in(0,1)$ are constants such that $\sum_{n=1}^{\infty}k_1k_2\ldots k_n<\infty$. Then $T$ has a unique fixed point in $M$.*

**Theorem 1.3** ([1], Theorem 2.6)**.** *Let $T:M\to M$ be a mapping on a complete metric space $(M,\ d)$ with a closed graph. Let $k_i\in(0,1)\ \forall i$ such that $nk_1k_2\ldots k_n\to 0$ as $n\to\infty$ and ${k_n}^n$ does not converge to 1 as $n\to\infty$. Suppose $M_1\subseteq M_2\subseteq M_3\subseteq\ldots$ be subsets of $M$ such that for every i, $T(M_i)\subseteq M_{i+1}$ and $d(Tx,\ Ty)\le k_id(x,\ y)\forall x\in M_i,\ \forall y\in M$. Let $x_1\in\bigcup_{j=1}^{\infty}M_j$. Then the sequence $(T^nx_1)$ converges to a fixed point of $T$ in $M$. If $M=\bigcup_{j=1}^{\infty}M_j$, then $T$ has a unique fixed point in $M$.*

## 2. Main Results

**Theorem 2.1.** *Let $(M,\ d)$ be a complete metric space. $T:M\to M$ have a closed graph. Let $M_1\subseteq M_2\subseteq M_3\subseteq\ldots$ be subsets of $M$ such that $T(M_i)\subseteq M_{i+1},\ \forall i,\ M=\bigcup_{j=1}^{\infty}M_j$ and $d(Tx,\ Ty)\le k_i\left[d(x,\ Tx)+d(y,\ Ty)\right]\ \forall x,\ y\in M_i$ where $k_i\in\left(0,\dfrac{1}{2}\right)$ such that $\sum_{n=1}^{\infty}\dfrac{k_1}{1-k_1}\dfrac{k_2}{1-k_2}\cdots\dfrac{k_n}{1-k_n}<\infty$. Then $T$ has a unique fixed point in $M$.*

*Proof.* Fix $x_1\in M_1$. Define $x_{n+1}=Tx_n=T^nx_1\ \forall n=1,\ 2,\ \ldots$ Then $d(x_2,\ x_1)=d(Tx_1,\ x_1)$

Now,

$$\begin{aligned} d(x_3, x_2) &= d(Tx_2, Tx_1) \\ &\leq k_1 [d(x_2, Tx_2) + d(x_1, Tx_1)] \\ &= k_1 [d(x_1, x_2) + d(x_2, x_3)] \end{aligned}$$

$$d\left(T^2 x_1, Tx_1\right) \leq \frac{k_1}{1-k_1} d(x_1, Tx_1)$$

Similarly,

$$d\left(T^3 x_1, T^2 x_1\right) \leq \frac{k_2}{1-k_2} \frac{k_1}{1-k_1} d(x_1, Tx_1)$$

Ingeneral we get,

$$\begin{aligned} d\left(T^{n+1} x_1, T^n x_1\right) \leq \frac{k_n}{1-k_n} \frac{k_{n-1}}{1-k_{n-1}} \cdots \\ \cdots \frac{k_1}{1-k_1} d(x_1, Tx_1) \end{aligned}$$

For $n > m \geq 1$, we have

$$\begin{aligned} & d(T^m x_1, T^n x_1) \\ & \leq d\left(T^m x_1, T^{m+1} x_1\right) \\ & \quad + d\left(T^{m+1} x_1, T^{m+2} x_1\right) + \cdots \\ & \quad + d\left(T^{n-1} x_1, T^n x_1\right) \\ & \leq \frac{k_m}{1-k_m} \frac{k_{m-1}}{1-k_{m-1}} \cdots \\ & \quad \cdots \frac{k_1}{1-k_1} d(x_1, Tx_1) \\ & \quad + \cdots + \frac{k_{n-1}}{1-k_{n-1}} \frac{k_{n-2}}{1-k_{n-2}} \cdots \\ & \cdots \frac{k_1}{1-k_1} d(x_1, Tx_1) \\ & \leq \sum_{i=m}^{n-1} \frac{k_1}{1-k_1} \frac{k_2}{1-k_2} \cdots \frac{k_i}{1-k_i} d(x_1, Tx_1) \end{aligned}$$

$\therefore (T^n x_1)_{n=1}^{\infty}$ is a Cauchy sequence in $M$. Let it converges to $x^*$in $M$. Then $\left(T^{n+1} x_1\right)_{n=1}^{\infty}$ is also Cauchy and it converges to $x^*$in $M$. Since $T$ has a closed graph in $M$. We should have $Tx^* = x^*$ then $x^*$ is a fixed point of $T$. If $y^*$ is a fixed point of $T$. Then $x^*, y^* \in M_n$ so that,

$$\begin{aligned} 0 \leq d(x^*, y^*) &= d(Tx^*, Ty^*) \\ &\leq k_i [d(x^*, Tx^*) + d(y^*, Ty^*)] = 0 \\ 0 \leq d(x^*, y^*) &\leq 0 \end{aligned}$$

Which implies that $x^* = y^*$. This proves the uniqueness of the fixed point. □

**Example 2.2.** *Let* $k_n = \frac{1}{3}$ , $n = 1, 2, 3, \ldots$. *Then* $\sum_{n=1}^{\infty} \frac{k_1}{1-k_1} \frac{k_2}{1-k_2} \cdots \frac{k_n}{1-k_n} < \infty$.
*Let* $M = \left\{\frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \ldots\right\}$ *and* $d$ *be the usual metric on* $M$. *Let us define* $M_n = \left\{\frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \ldots, \frac{1}{3^{2n+1}}\right\}$ *for each* $n = 1, 2, 3, \ldots$ . *Define* $T : M \to M$ *by*

$$T(x) = \begin{cases} 0, & x = 0 \\ \frac{1}{3^{n+2}}, & x = \frac{1}{3^n} \text{ for } n = 1, 2, 3, \ldots \end{cases}$$

*For* $n > m$, *we have*

$$\begin{aligned} & \left| T\left(\frac{1}{m}\right) - T\left(\frac{1}{n}\right) \right| \\ & = \left| \frac{1}{3^{m+2}} - \frac{1}{3^{n+2}} \right| \\ & = \left| \frac{1}{3^{m+2}} - \frac{1}{3^{n+1}} + \frac{1}{3^{n+1}} - \frac{1}{3^{n+2}} \right| \\ & \leq \frac{1}{3} \left\{ \left| \frac{1}{3^{m+1}} - \frac{1}{3^n} \right| + \left| \frac{1}{3^n} - \frac{1}{3^{n+1}} \right| \right\} \\ & < \frac{1}{3} \left\{ \frac{2}{3^{n+1}} + \left| \frac{1}{3^n} - \frac{1}{3^{n+1}} \right| \right\} \\ & = \frac{1}{3} \left\{ \left| \frac{1}{3^m} - \frac{1}{3^{m+1}} \right| + \left| \frac{1}{3^n} - \frac{1}{3^{n+1}} \right| \right\} \\ & < \frac{1}{3} \left\{ \left| \frac{1}{3^m} - \frac{1}{3^{m+2}} \right| + \left| \frac{1}{3^n} - \frac{1}{3^{n+2}} \right| \right\} \end{aligned}$$

*This verifies the conditions of the theorem 2.1 and the fixed point is 0.*

**Theorem 2.3.** *Let* $(M, d)$ *be a complete metric space.* $T : M \to M$ *have a closed graph. Let* $M_1 \subseteq M_2 \subseteq M_3 \subseteq \ldots$ *be subsets of* $M$ *such that* $T(M_i) \subseteq M_{i+1}$, $\forall i$, $M = \bigcup_{j=1}^{\infty} M_j$ *and* $d(Tx, Ty) \leq k_i [d(x, Ty) + d(y, Tx)]$ $\forall x, y \in M_i$ *where* $k_i \in \left(0, \frac{1}{2}\right)$ *such that* $\sum_{n=1}^{\infty} \frac{k_1}{1-k_1} \frac{k_2}{1-k_2} \cdots \frac{k_n}{1-k_n} < \infty$. *Then* $T$ *has a unique fixed point in* $M$.

*Proof.* Fix $x_1 \in M_1$. Define $x_{n+1} = Tx_n = T^n x_1$ $\forall n = 1, 2, \ldots$ Then $d(x_2, x_1) = d(Tx_1, x_1)$ Now,

$$\begin{aligned} d(x_3, x_2) &= d(Tx_2, Tx_1) \\ &\leq k_1 [d(x_2, Tx_1) + d(x_1, Tx_2)] \\ &= k_1 [d(x_2, x_2) + d(x_1, x_3)] \\ &\leq k_1 [d(x_1, x_2) + d(x_2, x_3)] \end{aligned}$$

$$d\left(T^2 x_1, Tx_1\right) \leq \frac{k_1}{1-k_1} d(x_1, Tx_1)$$

Similarly,

$$d\left(T^3 x_1, T^2 x_1\right) \leq \frac{k_2}{1-k_2} \frac{k_1}{1-k_1} d(x_1, Tx_1)$$

In general we get,

$$d\left(T^{n+1} x_1, T^n x_1\right) \leq \frac{k_n}{1-k_n} \frac{k_{n-1}}{1-k_{n-1}} \cdots \frac{k_1}{1-k_1} d(x_1, Tx_1)$$

For $n > m \geq 1$, we have

$$\begin{aligned}
&d\left(T^m x_1,\ T^n x_1\right)\\
&\leq d\left(T^m x_1,\ T^{m+1} x_1\right) + d\left(T^{m+1} x_1,\ T^{m+2} x_1\right)\\
&\quad + \cdots + d\left(T^{n-1} x_1,\ T^n x_1\right)\\
&\leq \frac{k_m}{1-k_m}\frac{k_{m-1}}{1-k_{m-1}}\cdots\frac{k_1}{1-k_1}\ d\left(x_1,\ T x_1\right)\\
&\quad + \cdots + \frac{k_{n-1}}{1-k_{n-1}}\frac{k_{n-2}}{1-k_{n-2}}\cdots\\
&\quad \cdots\frac{k_1}{1-k_1}\ d\left(x_1,\ T x_1\right)\\
&\leq \sum_{i=m}^{n-1}\frac{k_1}{1-k_1}\frac{k_2}{1-k_2}\cdots\frac{k_i}{1-k_i} d\left(x_1,\ T x_1\right)
\end{aligned}$$

$\therefore \left(T^n x_1\right)_{n=1}^{\infty}$ is a Cauchy sequence in $M$.

Let it converges to $x^*$in $M$. Then $\left(T^{n+1} x_1\right)_{n=1}^{\infty}$is also Cauchy and it converges to $x^*$in $M$.

Since $T$ has a closed graph in $M$. We should have $Tx^* = x^*$ then $x^*$ is a fixed point of $T$.

If $y^*$ is a fixed point of $T$. Then $x^*,\ y^* \in M_n$ so that,

$$\begin{aligned}
0 &\leq d\left(x^*,\ y^*\right) = d\left(T x^*,\ T y^*\right)\\
&\leq k_i\left[d\left(x^*,\ T y^*\right) + d\left(y^*,\ T x^*\right)\right]\\
&= k_i\left[d\left(x^*,\ y^*\right) + d\left(y^*,\ x^*\right)\right]\\
&= 2k_i\ d\left(x^*,\ y^*\right)\\
\left(1-2k_i\right) d\left(x^*,\ y^*\right) &\leq 0
\end{aligned}$$

This implies $d\left(x^*,\ y^*\right) = 0$ as $1 - 2k_i > 0$. Hence $x^*$ is a unique fixed point of $T$. ☐

**Theorem 2.4.** *Let $T : M \to M$ be a mapping on a complete metric space $(M,\ d)$ with a closed graph. Let $k_i \in \left(0, \frac{1}{2}\right)$ $\forall i$ such that $n\ \frac{k_1}{1-k_1}\frac{k_2}{1-k_2}\cdots\frac{k_n}{1-k_n} \to 0$ as $n \to \infty$ and $\left(\frac{k_n}{1-k_n}\right)^n$ does not converge to 1 as $n \to \infty$. Suppose $M_1 \subseteq M_2 \subseteq \ldots$ be subsets of $M$ such that for every i, $T\left(M_i\right) \subseteq M_{i+1}$ and $d\left(Tx,\ Ty\right) \leq k_i\left[d\left(x,\ Tx\right) + d\left(y,\ Ty\right)\right]\ \forall x \in M_i,\ \forall y \in M$. Let $x_1 \in \bigcup_{j=1}^{\infty} M_j$. Then the sequence $\left(T^n x_1\right)$ converges to a fixed point of $T$ in $M$. If $M = \bigcup_{j=1}^{\infty} M_j$, then $T$ has a unique fixed point in $M$.*

*Proof.* Let $x_1 \in M_1$. Let us write $x_{n+1} = Tx_n = T^n x_1\ \forall n = 1,\ 2,\ \ldots$

Then $d\left(T^{n+1} x_1,\ T^n x_1\right) \leq \frac{k_1}{1-k_1}\frac{k_2}{1-k_2}\cdots\frac{k_n}{1-k_n} d(Tx_1, x_1)$ for each $n$.

Hence $d\left(x_{m+1},\ x_m\right) \to 0$ as $m \to \infty$.

For $n > m \geq 1$,

$$\begin{aligned}
&d\left(x_n,\ x_m\right)\\
&\leq d\left(x_n,\ x_{n+1}\right) + d\left(x_{n+1},\ x_{m+1}\right) + d\left(x_m, x_{m+1}\right)\\
&= d\left(T^{n-1} x_1,\ T^n x_1\right) + d\left(T^n x_1, T^m x_1\right)\\
&\quad + d(T^{m-1} x_1, T^m x_1)\\
&\leq \frac{k_1}{1-k_1}\frac{k_2}{1-k_2}\cdots\frac{k_{n-1}}{1-k_{n-1}}\ d\left(T x_1,\ x_1\right)\\
&\quad + \frac{k_m}{1-k_m} d\left(x_n, x_m\right) + \frac{k_1}{1-k_1}\frac{k_2}{1-k_2}\cdots\\
&\quad \cdots\frac{k_{m-1}}{1-k_{m-1}}\ d\left(T x_1,\ x_1\right)\\
&\leq m\frac{k_1}{1-k_1}\frac{k_2}{1-k_2}\cdots\frac{k_{n-1}}{1-k_{n-1}}\ d\left(T x_1,\ x_1\right)\\
&\quad + \left(\frac{k_m}{1-k_m}\right)^m d\left(x_n, x_m\right)\\
&\quad + m\frac{k_1}{1-k_1}\frac{k_2}{1-k_2}\cdots\frac{k_{m-1}}{1-k_{m-1}}\ d\left(T x_1,\ x_1\right)
\end{aligned}$$

So,

$$\begin{aligned}
&d(x_n, x_m)\\
&\leq \left[m\left(\frac{k_1}{1-k_1}\frac{k_2}{1-k_2}\cdots\frac{k_{n-1}}{1-k_{n-1}}\right)\right.\\
&\quad \left. + m\left(\frac{k_1}{1-k_1}\frac{k_2}{1-k_2}\cdots\frac{k_{m-1}}{1-k_{m-1}}\right)\right]\frac{d\left(T x_1,\ x_1\right)}{1-\left(\frac{k_m}{1-k_m}\right)^m}\\
&\leq \left[m\left(\frac{k_1}{1-k_1}\frac{k_2}{1-k_2}\cdots\frac{k_{m-1}}{1-k_{m-1}}\right)\right.\\
&\quad \left. + m\left(\frac{k_1}{1-k_1}\frac{k_2}{1-k_2}\cdots\frac{k_{m-1}}{1-k_{m-1}}\right)\right]\frac{d\left(T x_1,\ x_1\right)}{1-\left(\frac{k_m}{1-k_m}\right)^m}\\
&= \frac{2m\left(\frac{k_1}{1-k_1}\frac{k_2}{1-k_2}\cdots\frac{k_{m-1}}{1-k_{m-1}}\right)}{1-\left(\frac{k_m}{1-k_m}\right)^m}\ d\left(T x_1,\ x_1\right)
\end{aligned}$$

Therefore, $d\left(x_n, x_m\right) \to 0$ as $m, n \to \infty$. Hence $\left(x_n\right)$ is Cauchy. Thus, if $x_n \to x^*$ as $n \to \infty$, $Tx^* = x^*$ because $T$ has a closed graph. The same argument is applicable for the general case $x_1 \in M_n$. ☐

**Theorem 2.5.** *Let $T : M \to M$ be a mapping on a complete metric space $(M,\ d)$ with a closed graph. Let $k_i \in \left(0, \frac{1}{2}\right)$ $\forall i$ such that $n\ \frac{k_1}{1-k_1}\frac{k_2}{1-k_2}\cdots\frac{k_n}{1-k_n} \to 0$ as $n \to \infty$ and $\left(\frac{k_n}{1-k_n}\right)^n$ does not converge to 1 as $n \to \infty$. Suppose $M_1 \subseteq M_2 \subseteq \ldots$ be subsets of $M$ such that for every i, $T\left(M_i\right) \subseteq M_{i+1}$ and $d\left(Tx,\ Ty\right) \leq k_i\left[d\left(x,\ Ty\right) + d\left(y,\ Tx\right)\right]\ \forall x,\ y \in M_i\ ,\ \forall y \in M$. Let $x_1 \in \bigcup_{j=1}^{\infty} M_j$. Then the sequence $\left(T^n x_1\right)$ converges to a fixed point of $T$ in $M$. If $M = \bigcup_{j=1}^{\infty} M_j$, then $T$ has a unique fixed point in $M$.*

*Proof.* The proof follows from the above theorems. □

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