Convergence of fixed points for closed graph operator satisfies Zamfirescu conditions

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Abstract
In this article we establish a unique fixed point of a well known contraction mapping known as Kannan and Chatterjea mapping having a closed graph in a complete metric space. For such mappings, we consider an increasing sequence of subsets of a complete metric space into itself, so that a contraction condition is satisfied.

Keywords
Contraction mapping, closed graph, fixed point.

AMS Subject Classification

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1. Introduction

A mapping \( T : (M, d) \to (M, d) \) from a metric space \((M, d)\) into itself is said to have a closed graph if whenever \( x_n \to x_0 \) and \( T x_n \to y_0 \) for some sequence \((x_n)\) in \( M \) and some \( y_0, y_0 \) in \( M \), we have \( y_0 = T x_0 \). C.Ganesa Moorthy and P.Xavier Raj [1] presented results for contraction mappings with variations in Domain. In this paper, we extend the Kannan and Chatterjea techniques to some contraction mapping of a closed graph to establish some fixed point theorems.

Definition 1.1. Zamfirescu [4] introduced the Banach fixed point theorem by combining Banach, Kannan and Chatterjea. i.e., For a mapping \( T : E \to E \), there exist real numbers \( \alpha, \beta, \gamma \) satisfying \( 0 < \alpha < 1, 0 < \beta, \gamma < \frac{1}{2} \) such that for each \( x, y \in E \) at least one of the following is true

1. \( d(Tx, Ty) \leq \alpha d(x, y) \)
2. \( d(Tx, Ty) \leq \beta [d(x, Tx) + d(y, Ty)] \)
3. \( d(Tx, Ty) \leq \gamma [d(x, Ty) + d(y, Tx)] \)

Theorem 1.2 ([1], Theorem 2.1). Let \((M, d)\) be a complete metric space. \( T : M \to M \) has a closed graph. Let \( M_1 \subseteq M_2 \subseteq M_3 \subseteq \ldots \) be subsets of \( M \) such that \( T(M_i) \subseteq M_{i+1} \), \( \forall i, M = \bigcup_{j=1}^{\infty} M_j \) and \( d(Tx, Ty) \leq k_i d(x, y) \) \( \forall x, y \in M_i \), where \( k_i \in (0, 1) \) are constants such that \( \sum_{i=1}^{\infty} k_i^2 \ldots k_n < \infty \). Then \( T \) has a unique fixed point in \( M \).

Theorem 1.3 ([1], Theorem 2.6). Let \( T : M \to M \) be a mapping on a complete metric space \((M, d)\) with a closed graph. Let \( k_i \in (0, 1) \) \( \forall i \) such that \( nk_i k_2 \ldots k_n \to 0 \) as \( n \to \infty \) and \( k_n^m \) does not converge to 1 as \( n \to \infty \). Suppose \( M_1 \subseteq M_2 \subseteq M_3 \subseteq \ldots \) be subsets of \( M \) such that for every \( i, T(M_i) \subseteq M_{i+1} \) and \( d(Tx, Ty) \leq k_i d(x, y) \) \( \forall x \in M_i, \forall y \in M \). Let \( x_1 \in \bigcup_{j=1}^{\infty} M_j \). Then the sequence \((T^n x_1)\) converges to a fixed point of \( T \) in \( M \). If \( M = \bigcup_{j=1}^{\infty} M_j \), then \( T \) has a unique fixed point in \( M \).

2. Main Results

Theorem 2.1. Let \((M, d)\) be a complete metric space. \( T : M \to M \) has a closed graph. Let \( M_1 \subseteq M_2 \subseteq M_3 \subseteq \ldots \) be subsets of \( M \) such that \( T(M_i) \subseteq M_{i+1} \), \( \forall i, M = \bigcup_{j=1}^{\infty} M_j \) and \( d(Tx, Ty) \leq k_i [d(x, Tx) + d(y, Ty)] \) \( \forall x, y \in M_i \) where \( k_i \in (0, \frac{1}{2}) \) such that \( \sum_{i=1}^{\infty} \frac{k_1}{1-k_1} \frac{k_2}{1-k_2} \ldots \frac{k_n}{1-k_n} \to 0 \). Then \( T \) has a unique fixed point in \( M \).

Proof. Fix \( x_1 \in M_1 \). Define \( x_{n+1} = T x_n = T^n x_1 \) \( \forall n = 1, 2, \ldots \) Then \( d(x_2, x_1) = d(Tx_1, x_1) \).
Now,
\[ d(x_1, x_2) = d(Tx_2, Tx_1) \leq k_1 [d(x_2, x_1) + d(x_1, T x_1)] = k_1 [d(x_2, x_1) + d(x_1, x_2)] \]

Similarly,
\[ d(T^2 x_1, Tx_1) \leq \frac{k_1}{1 - k_1} d(x_1, Tx_1) \]

In general, we get,
\[ d(T^{n+1} x_1, T^n x_1) \leq \frac{k_n}{1 - k_0} \frac{k_{n-1}}{1 - k_1} \cdots \frac{k_1}{1 - k_1} d(x_1, Tx_1) \]

For \( n > m \geq 1 \), we have
\[
\begin{align*}
\sum_{i=m}^{n-1} & k_1 \frac{k_2}{1 - k_1} \frac{k_3}{1 - k_2} \cdots \frac{k_i}{1 - k_{i-1}} d(x_1, T x_1) \\
& \leq \frac{n-1}{1 - k_0} \frac{1}{1 - k_1} \frac{1}{1 - k_2} \cdots \frac{1}{1 - k_{n-1}} d(x_1, T x_1)
\end{align*}
\]
\( \therefore \ (T^n x_1)_{n=1}^{\infty} \) is a Cauchy sequence in \( M \). Let it converges to \( x^* \) in \( M \). Then \( (T^n x_1)_{n=1}^{\infty} \) is also Cauchy and it converges to \( x^* \) in \( M \). Since \( T \) has a closed graph in \( M \). We should have \( T x^* = x^* \) then \( x^* \) is a fixed point of \( T \). If \( y^* \) is a fixed point of \( T \). Then \( x^*, y^* \in M \), so that,
\[ 0 \leq d(x^*, y^*) = d(Tx^*, Ty^*) \leq k_1 [d(x^*, T x^*) + d(y^*, T y^*)] = 0 \]
\[ 0 \leq d(x^*, y^*) \leq 0 \]

Which implies that \( x^* = y^* \). This proves the uniqueness of the fixed point.

**Example 2.2.** Let \( k_n = \frac{1}{3^n}, n = 1, 2, 3, \ldots \). Then
\[ \sum_{n=1}^{\infty} \frac{k_1}{1 - k_1} \frac{k_2}{1 - k_2} \cdots \frac{k_n}{1 - k_n} < \infty. \]

Let \( M = \left\{ 1, \frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \ldots \right\} \) and \( d \) be the usual metric on \( M \). Let us define \( M_n = \left\{ 1, \frac{1}{3}, \frac{1}{3^2}, \ldots, \frac{1}{3^{2n+1}} \right\} \) for each \( n = 1, 2, 3, \ldots \). Define \( T : M \to M \) by

\[ T(x) = \begin{cases} 0, & x = 0 \\ \frac{1}{3^{n+2}}, & x = \frac{1}{3^n} \text{ for } n = 1, 2, 3, \ldots \end{cases} \]

For \( n > m \), we have
\[
\begin{align*}
|T\left(\frac{1}{m}\right) - T\left(\frac{1}{n}\right)| &= \frac{1}{3^{m+2}} - \frac{1}{3^{n+2}} \\
&= \frac{1}{3^{m+2}} - \frac{1}{3^{n+1}} + \frac{1}{3^{n+1}} - \frac{1}{3^{n+2}} \\
&\leq \frac{1}{3} \left\{ \frac{1}{3^{m+1}} - \frac{1}{3^n} + \frac{1}{3^n} - \frac{1}{3^{n+1}} \right\} \\
&< \frac{1}{3} \left\{ \frac{1}{3^{m+1}} + \frac{1}{3^n} - \frac{1}{3^{n+1}} \right\} \\
&< \frac{1}{3} \left\{ \frac{1}{3^{m+1}} + \frac{1}{3^n} - \frac{1}{3^{n+2}} \right\}
\end{align*}
\]

This verifies the conditions of the theorem 2.1 and the fixed point is 0.

**Theorem 2.3.** Let \( (M,d) \) be a complete metric space. \( T : M \to M \) have a closed graph. Let \( M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots \) be subsets of \( M \) such that \( T(M_i) \subseteq M_{i+1} \), \( \forall i \). Let \( M = \bigcup_{i=1}^{\infty} M_i \) and \( d(Tx, Ty) \leq k_i [d(x, Ty) + d(y, Tx)] \). \( \forall x, y \in M_i \) where \( k_i \in (0, \frac{1}{2}) \). such that \( \sum_{i=1}^{\infty} \frac{k_1}{1 - k_1} \frac{k_2}{1 - k_2} \cdots \frac{k_n}{1 - k_n} < \infty \). Then \( T \) has a unique fixed point in \( M \).

**Proof.** Fix \( x_1 \in M_i \). Define \( x_{n+1} = T x_n \) \( \forall n = 1, 2, \ldots \). Then \( d(x_2, x_1) = d(Tx_1, x_1) \).

\[ d(x_3, x_2) = d(Tx_2, T x_1) \leq k_1 [d(x_2, T x_1) + d(x_1, T x_2)] = k_1 [d(x_2, x_1) + d(x_1, x_2)] \leq k_1 [d(x_1, T x_1)] \]

\[ d(T^2 x_1, T x_1) \leq \frac{k_1}{1 - k_1} d(x_1, T x_1) \]

Similarly,
\[ d(T^3 x_1, T^2 x_1) \leq \frac{k_2}{1 - k_2} \frac{k_1}{1 - k_1} d(x_1, T x_1) \]

In general, we get,
\[ d(T^{n+1} x_1, T^n x_1) \leq \frac{k_n}{1 - k_n} \frac{k_{n-1}}{1 - k_{n-1}} \cdots \frac{k_1}{1 - k_1} d(x_1, T x_1) \]
For \( n > m \geq 1 \), we have
\[
\begin{align*}
d(T^n x_1, T^n x_1) & \leq d(T^{m+1} x_1, T^{m+2} x_1) + \cdots + d(T^{n-1} x_1, T^n x_1) \\
& \leq \frac{k_m}{1-k_m} \frac{k_{m-1}}{1-k_{m-1}} \cdots \frac{k_1}{1-k_1} d(x_1, T x_1) \\
& \quad + \cdots + \frac{k_{n-1}}{1-k_{n-1}} \frac{k_{n-2}}{1-k_{n-2}} \cdots \frac{k_1}{1-k_1} d(x_1, T x_1) \\
& \leq \sum_{i=m}^{n-1} \frac{k_i}{1-k_1} \frac{k_i}{1-k_2} \cdots \frac{k_i}{1-k_i} d(x_1, T x_1)
\end{align*}
\]
\( : (T^n x_1)_{n=1}^{\infty} \) is a Cauchy sequence in \( M \).

Let it converges to \( x' \in M \). Then \( (T^n x_1)_{n=1}^{\infty} \) is also Cauchy and it converges to \( x' \in M \).

Since \( T \) has a closed graph in \( M \). We should have \( T x' = x' \)
then \( x' \) is a fixed point of \( T \).

If \( y' \) is a fixed point of \( T \). Then \( x', y' \in M \), so that,
\[
0 \leq d(x', y') = d(T x', T y')
\leq k_1 [d(x', T y') + d(y', T x')]
= k_1 [d(x', y') + d(y', x')]
= 2k_1 d(x', y')
(1 - 2k_1) d(x', y') \leq 0
\]
This implies \( d(x', y') = 0 \) as \( 1 - 2k_1 > 0 \). Hence \( x' \) is a unique fixed point of \( T \).

\[\square\]

**Theorem 2.4.** Let \( T : M \to M \) be a mapping on a complete metric space \((M, d)\) with a closed graph. Let \( k_i \in \left(0, \frac{1}{2}\right)\)
\( \forall i \) such that \( n \frac{k_1}{1-k_1} \frac{k_2}{1-k_2} \cdots \frac{k_n}{1-k_n} \to 0 \) as \( n \to \infty \) and \((\frac{k_n}{1-k_n})^n\) does not converge to 1 as \( n \to \infty \). Suppose \( M_1 \subseteq M_2 \subseteq \ldots \) be subsets of \( M \) such that for every \( i \), \( T(M_i) \subseteq M_{i+1} \) and \( d(Tx, Ty) \leq k_i [d(x, Tx) + d(y, Ty)] \) \( \forall x \in M_i, \forall y \in M \).

Let \( x_1 \in \bigcup_{j=1}^{\infty} M_j \). Then the sequence \((T^n x_1)_{n=1}^{\infty}\) converges to a fixed point of \( T \) in \( M \). If \( M = \bigcup_{j=1}^{\infty} M_j \), then \( T \) has a unique fixed point in \( M \).

**Proof.** Let \( x_1 \in M_1 \). Let us write \( x_{n+1} = T x_n = T^n x_1 \) \( \forall n = 1, 2, \ldots \)
Then \( d(T^{n+1} x_1, T^n x_1) \leq \frac{k_1}{1-k_1} \frac{k_2}{1-k_2} \cdots \frac{k_n}{1-k_n} d(Tx_1, x_1) \)
for each \( n \).

Hence \( d(x_{m+1}, x_m) \to 0 \) as \( m \to \infty \).

\[\square\]

For \( n > m \geq 1 \),
\[
\begin{align*}
d(x_n, x_m) & \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{m+1}) + d(x_m, x_{m+1}) \\
& = d(T^{m+1} x_1, T^{m+2} x_1) + d(T^{m+1} x_1, T^{m+2} x_1) \\
& \quad + d(T^{m+1} x_1, T^n x_1) \\
& \leq k_1 \frac{k_1}{1-k_1} \frac{k_2}{1-k_2} \cdots \frac{k_{n-1}}{1-k_{n-1}} d(T x_1, x_1) \\
& \quad + \frac{k_1}{1-k_1} \frac{k_2}{1-k_2} \cdots \frac{k_{n-1}}{1-k_{n-1}} d(T x_1, x_1) \\
& \quad + \cdots + \frac{k_1}{1-k_1} \frac{k_2}{1-k_2} \cdots \frac{k_{n-1}}{1-k_{n-1}} d(T x_1, x_1) \\
& \quad + \cdots + \frac{k_1}{1-k_1} \frac{k_2}{1-k_2} \cdots \frac{k_{n-1}}{1-k_{n-1}} d(T x_1, x_1) \\
& \leq m \left( \frac{k_1}{1-k_1} \frac{k_2}{1-k_2} \cdots \frac{k_{n-1}}{1-k_{n-1}} \right) d(T x_1, x_1) \\
& \quad + m \left( \frac{k_1}{1-k_1} \frac{k_2}{1-k_2} \cdots \frac{k_{n-1}}{1-k_{n-1}} \right) d(T x_1, x_1) \\
& \quad + \cdots + m \left( \frac{k_1}{1-k_1} \frac{k_2}{1-k_2} \cdots \frac{k_{n-1}}{1-k_{n-1}} \right) d(T x_1, x_1)
\end{align*}
\]
So,
\[
\begin{align*}
d(x_n, x_m) & \leq m \left( \frac{k_1}{1-k_1} \frac{k_2}{1-k_2} \cdots \frac{k_{n-1}}{1-k_{n-1}} \right) d(T x_1, x_1) \\
& \quad + m \left( \frac{k_1}{1-k_1} \frac{k_2}{1-k_2} \cdots \frac{k_{n-1}}{1-k_{n-1}} \right) d(T x_1, x_1) \\
& \quad + \cdots + m \left( \frac{k_1}{1-k_1} \frac{k_2}{1-k_2} \cdots \frac{k_{n-1}}{1-k_{n-1}} \right) d(T x_1, x_1)
\end{align*}
\]
Therefore, \( d(x_n, x_m) \to 0 \) as \( m, n \to \infty \). Hence \( (x_n) \) is Cauchy.

Thus, if \( x_n \to x' \) as \( n \to \infty \), \( T x' = x' \) because \( T \) has a closed graph. The same argument is applicable for the general case \( x_1 \in M_n \).

**Theorem 2.5.** Let \( T : M \to M \) be a mapping on a complete metric space \((M, d)\) with a closed graph. Let \( k_i \in \left(0, \frac{1}{2}\right)\)
\( \forall i \) such that \( n \frac{k_1}{1-k_1} \frac{k_2}{1-k_2} \cdots \frac{k_n}{1-k_n} \to 0 \) as \( n \to \infty \) and \((\frac{k_n}{1-k_n})^n\) does not converge to 1 as \( n \to \infty \). Suppose \( M_1 \subseteq M_2 \subseteq \ldots \) be subsets of \( M \) such that for every \( i \), \( T(M_i) \subseteq M_{i+1} \) and \( d(Tx, Ty) \leq k_i [d(x, Tx) + d(y, Ty)] \) \( \forall x \in M_i, \forall y \in M \).

Let \( x_1 \in \bigcup_{j=1}^{\infty} M_j \). Then the sequence \((T^n x_1)_{n=1}^{\infty}\) converges to a fixed point of \( T \) in \( M \). If \( M = \bigcup_{j=1}^{\infty} M_j \), then \( T \) has a unique fixed point in \( M \).

\[\square\]
Proof. The proof follows from the above theorems.

References