Cayley–Hamilton theorem for $\alpha$-upper level partition of fuzzy square matrices

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Abstract
In this paper the new approach of Cayley–Hamilton theorem was done, using the $\alpha$-upper level partition of fuzzy square matrices. For this the characteristic equation of $\alpha$-upper level partition of fuzzy square matrix, fuzzy eigen values and eigen vectors have been derived.

Keywords
Fuzzy matrix, $\alpha$-upper level partition of fuzzy square matrix, characteristic equation, Cayley–Hamilton theorem, eigen values and eigen vectors.

AMS Subject Classification
15B15.

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Article History: Received 01 February 2020; Accepted 29 March 2020

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1. Introduction


An extension of the Cayley–Hamilton theorem for discrete time linear systems with delay has been given in Buslowicz and Kaczorek [2]. Eigen values and Eigen vectors for fuzzy matrix was discussed by Joe Anand and Edal Anand [4]. Cayley–Hamilton theorem for fuzzy matrix is verified by Joe Anand et al. [5].

2. Preliminaries

Definition 2.1. If $A = [a_{ij}]$ be a $m \times n$ matrix and every $a_{ij} \in [0, 1]$ then the matrix A is said to be a fuzzy matrix.

When $m = n$, the number of rows are equal to the number of columns then the matrix A is said to be fuzzy square matrix of order n.

Definition 2.2. The $\alpha$-upper level partition of fuzzy square matrix $A^{(\alpha)}$ is a Boolean matrix denoted by

$$A^{(\alpha)} = a^{(\alpha)}_{ij}$$

such that

$$a^{(\alpha)}_{ij} = a_{ij} \text{ if } a_{ij} \geq \alpha$$

$$= 0 \text{ if } a_{ij} < \alpha$$

where $\alpha \in [0, 1]$.

Definition 2.3. If $A^{(\alpha)}$ be a $\alpha$-upper level partition of a fuzzy square matrix then the characteristic equation of $A^{(\alpha)}$ is defined as $|A^{(\alpha)} - \lambda I| = 0$.

Definition 2.4. The roots of the characteristic equation $|A^{(\alpha)} - \lambda I| = 0$ are called the characteristic roots or eigen values of the matrix $A^{(\alpha)}$. 
Definition 2.5. If \( A^{(\alpha)} \) be a \( \alpha \)-upper level partition of a fuzzy matrix of order \( n \). Then there exists a non-zero vector \( X = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix} \) such that \( A^{(\alpha)} X = \lambda X \), the vector \( X \) is said to be the eigen vector of matrix \( A^{(\alpha)} \) corresponding to the fuzzy eigen value \( \lambda \).

Definition 2.6. Trace of \( \alpha \)-upper level partition of fuzzy square matrix \( A^{(\alpha)} \) is defined as
\[
\text{tr} \left( A^{(\alpha)} \right) = \max \left( a_{ii}^{(\alpha)} \right)
\]

3. Cayley–Hamilton Theorem for \( A^{(\alpha)} \)

Statement:
Every \( \alpha \)-upper level partition of fuzzy square matrix satisfies its own characteristic equation.
This statement is verified by the following numerical example.

Example 3.1. Consider a fuzzy matrix
\[
A = \begin{bmatrix} 0.6 & 0.7 & 0.5 \\ 0.4 & 0.5 & 0.3 \\ 0.2 & 0.1 & 0.8 \end{bmatrix}
\]
Take \( \alpha = 0.5 \).
Therefore, \( A^{(\alpha)} = A^{(0.5)} = \begin{bmatrix} 0.6 & 0.7 & 0.5 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.8 \end{bmatrix} \)
The characteristic equation is given by
\[
A^{(\alpha)} - \lambda I = 0.
\]
i.e.,
\[
\begin{vmatrix} 0.6 - \lambda & 0.7 & 0.5 \\ 0 & 0.5 - \lambda & 0 \\ 0 & 0 & 0.8 - \lambda \end{vmatrix} = 0
\]
Proof. On expanding, the characteristic equation is
\[
\lambda^3 - 1.9\lambda^2 + 1.18\lambda - 0.24 = 0.
\]
To prove: By Cayley–Hamilton theorem,
\[
\left( A^{(\alpha)} \right)^3 - 1.9 \left( A^{(\alpha)} \right)^2 + 1.18 A^{(\alpha)} - 0.24 I = 0
\]
\[
\left( A^{(\alpha)} \right)^2 = \begin{bmatrix} 0.6 & 0.7 & 0.5 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.8 \end{bmatrix}
\]
\[
\left( A^{(\alpha)} \right)^3 = \begin{bmatrix} 0.6 & 0.7 & 0.5 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.8 \end{bmatrix} \left( A^{(\alpha)} \right)
\]
\[
\left( A^{(\alpha)} \right)^3 = \begin{bmatrix} 0.36 & 0.77 & 0.7 \\ 0 & 0.25 & 0 \\ 0 & 0 & 0.64 \end{bmatrix} \left( A^{(\alpha)} \right)
\]
\[
A^{(\alpha)} = \begin{bmatrix} a_{11} & a_{12} & a_{1n} \\ a_{21} & a_{22} & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{nn} \end{bmatrix}
\]
\[
\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\]
If a vector \( X \) is transformed into a scalar multiple of the same vector, i.e., \( X \) is transformed into \( \lambda X \) then, \( Y = \lambda X = A^{(\alpha)} X \).
\( A^{(\alpha)} X - \lambda I X = 0 \) where \( I \) is the unit matrix of order \( n \).
\[
A^{(\alpha)} - \lambda I X = 0
\]

4. Eigen Values and Eigen Vectors of \( \alpha \)-upper level partition of fuzzy square matrix

Consider the linear transformation \( Y = A^{(\alpha)} X \).
In general, this transformation transforms a Column vector \( X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \) into another Column vector \( y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \).
Let \( A^{(\alpha)} \) be the \( \alpha \)-upper level partition of a fuzzy square matrix where,
\[
A^{(\alpha)} = \begin{bmatrix} a_{11}^{(\alpha)} & a_{12}^{(\alpha)} & a_{1n}^{(\alpha)} \\ a_{21}^{(\alpha)} & a_{22}^{(\alpha)} & a_{2n}^{(\alpha)} \\ \vdots & \vdots & \vdots \\ a_{n1}^{(\alpha)} & a_{n2}^{(\alpha)} & a_{nn}^{(\alpha)} \end{bmatrix}
\]
\[
\begin{bmatrix} a_{11}^{(\alpha)} - \lambda & a_{12}^{(\alpha)} & \cdots & a_{1n}^{(\alpha)} \\ a_{21}^{(\alpha)} - \lambda & a_{22}^{(\alpha)} & \cdots & a_{2n}^{(\alpha)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^{(\alpha)} - \lambda & a_{n2}^{(\alpha)} & \cdots & a_{nn}^{(\alpha)} - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
\]
we get

\[
\begin{pmatrix}
(a_{11}^{(\alpha)} - \lambda) & a_{12}^{(\alpha)} & \cdots & a_{1n}^{(\alpha)} \\
a_{21}^{(\alpha)} & (a_{22}^{(\alpha)} - \lambda) & \cdots & a_{2n}^{(\alpha)} \\
\vdots & \vdots & \ddots & \vdots \\
a_{n1}^{(\alpha)} & a_{n2}^{(\alpha)} & \cdots & (a_{nn}^{(\alpha)} - \lambda)
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\vdots \\
x_n
\end{pmatrix}
= 0
\]

This system of equations will have a non-trivial solution, if

\[
|A^{(\alpha)} - \lambda I| = 0.
\]

(i.e.,)

The equation \(|A^{(\alpha)} - \lambda I| = 0\) is said to be the characteristic equation of the transformation. Solving \(|A^{(\alpha)} - \lambda I| = 0\), we get \(n\) roots for \(\lambda\), these roots are called the characteristic roots (or) eigen values of the matrix \(A^{(\alpha)}\).

Corresponding to each value of \(\lambda\), the equation \(A^{(\alpha)}X = \lambda X\) has a non-zero solution vector \(X\). Let \(X_r\) be the non-zero vector satisfying \(A^{(\alpha)}X = \lambda X\). When \(\lambda = \lambda_r\), \(X_r\) is said to be the latent vector (or) eigen vector of a matrix \(A^{(\alpha)}\) corresponding to \(\lambda_r\).

5. Numerical Example

Consider \(\alpha\)-upper level partition of fuzzy square matrix in Example 3.1.

\[
A^{(\alpha)} = \begin{bmatrix}
0.6 & 0.7 & 0.5 \\
0 & 0.5 & 0 \\
0 & 0 & 0.8
\end{bmatrix}
\]

Then eigen values are given by the characteristic equation

\[
|A^{(\alpha)} - \lambda I| = 0.
\]

\[
\begin{vmatrix}
0.6 - \lambda & 0.7 & 0.5 \\
0 & 0.5 - \lambda & 0 \\
0 & 0 & 0.8 - \lambda
\end{vmatrix} = 0
\]

On expanding the characteristic equation of fuzzy matrix \(A^{(\alpha)}\) is

\[
\lambda^3 - 1.9\lambda^2 + 1.18\lambda - 0.24 = 0.
\]

Solving the characteristic equation, the eigen values are, \(\lambda = 0.5, 0.6, 0.8\).

When \(\lambda = 0.5\), the eigen vectors are given by,

\[
0.1x_1 + 0.7x_2 + 0.5x_3 = 0 \\
0.3x_3 = 0 \\
x_3 = 0
\]

When \(\lambda = 0.6\), the eigen vectors are given by,

\[
0.1x_1 + 0.7x_2 = 0 \\
0.1x_1 = -0.7x_2 \\
x_1 = -0.7x_2 \\
x_2 = 0.1
\]

When \(\lambda = 0.8\), the Eigen vectors are given by

\[
\begin{pmatrix}
0.7 \\
0 \\
0
\end{pmatrix}
\]

6. Properties of eigen values and eigen vectors of \(\alpha\)-upper level partition of fuzzy square matrix

Property 6.1. (i) The sum of the fuzzy eigen values of matrix \(A^{(\alpha)}\) is equal to the trace of the fuzzy matrix \(A^{(\alpha)}\).

(ii) Product of the fuzzy eigen values of \(A^{(\alpha)}\) is equal to the determinant of fuzzy matrix \(A^{(\alpha)}\).

Proof. Let \(A^{(\alpha)}\) be the \(\alpha\)-upper level partition of fuzzy square matrix of order \(n\).

The characteristic equation of \(A^{(\alpha)}\) is \(|A^{(\alpha)} - \lambda I| = 0\).

\[
\lambda^n - S_1\lambda^{n-1} + \cdots + (-1)^n S_n = 0,
\]

where \(S_1 = \text{sum of the diagonal elements of } A^{(\alpha)}\)

\[
= \max_a
\]

= Trace of \(A^{(\alpha)}\)

\[
\vdots
\]

\(S_n = \text{Product of the roots}\)

\[
= \lambda_1, \lambda_2, \ldots, \lambda_n
\]

\[
= \text{del of } A^{(\alpha)}
\]

Therefore product of the fuzzy eigen values = \(|A^{(\alpha)}|\).

Example 6.2. Take the fuzzy matrix \(A^{(\alpha)}\) in Example 3.1.

\[
A^{(\alpha)} = \begin{bmatrix}
0.6 & 0.7 & 0.5 \\
0 & 0.5 & 0 \\
0 & 0 & 0.8
\end{bmatrix}
\]

The eigen values are \(\lambda = 0.6, 0.5, 0.8\).

Therefore sum of fuzzy eigen value of \(A^{(\alpha)}\)

\(= \max (0.6, 0.5, 0.8) = 0.8 = \text{Trace of fuzzy matrix of } A^{(\alpha)}\)

Product of the eigen values = \(0.6 \times 0.5 \times 0.8 = 0.24\)

Product of the eigen values = Value of determinant \(A^{(\alpha)}\).
Property 6.3. A fuzzy square matrix $A^{(\alpha)}$ and its transpose $(A^{(\alpha)})^T$ have the same fuzzy eigen values.

Proof. Let $A^{(\alpha)}$ be a fuzzy matrix of order $n$. The characteristic equation of $A^{(\alpha)}$ is

$$
(A^{(\alpha)})^T \begin{vmatrix} A^{(\alpha)} - \lambda I \end{vmatrix} = 0
$$

(6.1)

and

$$
\begin{vmatrix} (A^{(\alpha)})^T - \lambda I \end{vmatrix} = 0
$$

(6.2)

Since the determinant value is unaltered by the interchange of rows and columns.

Therefore $|A^{(\alpha)}| = |(A^{(\alpha)})^T|$

Hence (6.1) and (6.2) are identical.

Therefore fuzzy eigen values of $A^{(\alpha)}$ and $(A^{(\alpha)})^T$ is the same.

Example 6.4. Consider the fuzzy matrix $A^{(\alpha)}$ in Example 3.1.

$$
A^{(\alpha)} = \begin{bmatrix}
0.6 & 0.7 & 0.5 \\
0 & 0.5 & 0 \\
0 & 0 & 0.8
\end{bmatrix}
$$

The eigen values of $A^{(\alpha)}$ are 0.6, 0.5, 0.8

$$
(A^{(\alpha)})^T = \begin{bmatrix}
0.6 & 0 & 0 \\
0.7 & 0.5 & 0 \\
0.5 & 0 & 0.8
\end{bmatrix}
$$

The eigen values of $(A^{(\alpha)})^T$ are 0.6, 0.5, 0.8.

Property 6.5. The fuzzy eigen values of a triangular fuzzy matrix $A^{(\alpha)}$ are just the diagonal elements of the fuzzy matrix $A^{(\alpha)}$.

From example in Property 6.3.

The eigen values are the diagonal elements 0.6, 0.5, 0.8.

Property 6.6. If $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the fuzzy eigen values of a fuzzy matrix $A^{(\alpha)}$ then $(A^{(\alpha)})^m$ has the fuzzy Eigen values $\lambda_1^m, \lambda_2^m, \ldots, \lambda_m^m$ (m being a positive integer).

Proof. Let $\lambda_i$ be the fuzzy eigen values of $A^{(\alpha)}$ and $X_i$ be the corresponding eigen vectors.

Then $A^{(\alpha)}X_i = \lambda_i X_i$

$$(A^{(\alpha)})^2X_i = A^{(\alpha)}(A^{(\alpha)}X_i) = A^{(\alpha)}(\lambda_i X_i) = \lambda_i X_i$$

Similarly, $$(A^{(\alpha)})^3X_i = \lambda_i^3 X_i$$

In general, $$(A^{(\alpha)})^mX_i = \lambda_i^m X_i$$

Hence $\lambda_i^m$ is a fuzzy eigen value of $(A^{(\alpha)})^m$.

The corresponding eigen vector is the same $X_i$.

Consider the example as in 1.

The eigen values of $A^{(\alpha)}$ are $\lambda = 0.6, 0.5, 0.8$.

Therefore the eigen values of

$$(A^{(\alpha)})^3 = (0.6)^3, (0.5)^3, (0.8)^3$$

$$= 0.216, 0.125, 0.512$$

7. Conclusion

In this paper, a new approach of Cayley–Hamilton theorem for $\alpha$-upper level partition of fuzzy square matrix was discussed. Also derived the eigen values and eigen vectors of fuzzy matrix $A^{(\alpha)}$. Moreover the properties of eigen values and eigen vectors are discussed.

References


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ISSN(P):2319 – 3786
Malaya Journal of Matematik
ISSN(O):2321 – 5666
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