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Solving linear system of equations in fuzzy environment

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Abstract
The system of linear equations arises in a wide variety of applications. In this paper, fuzzy linear systems with the aid of generalized triangular fuzzy number have been studied. Some arithmetic operations of the above discussed fuzzy numbers have been reviewed. Also new fuzzy arithmetic operations have been utilized for solving the proposed fuzzy linear system of equations. Finally few relevant illustrations have been given for the justification of the proposed notion.

Keywords
Triangular Fuzzy Number, Generalized Triangular Fuzzy Number, Generalized Triangular Fuzzy Matrix, Fuzzy Linear System.

AMS Subject Classification:
15A06,15A09,15A15,15A29,65F05.

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Contents

1. Introduction

System of linear equations has a wide application, especially, in the field of science and engineering. Because of the widespread importance of linear systems, much research has been devoted to their numerical solution. Excellent algorithms have been developed for the most common types of problems for linear systems. The concepts of fuzzy numbers and its arithmetic operations were developed first time by Zadeh [10]. When the inverse of a square and non- singular matrix whose entries are real numbers in concerned in one of the computing methods in the form of linear equation system. In [9] Yun.et.al generated the triangular fuzzy numbers and calculated the extended algebraic operations between two generalized triangular fuzzy set.

General model for solving a $n \times n$ fuzzy linear system whose co–efficient matrix is crisp and right hand side column is an arbitrary fuzzy number vectors established by Friedman et al in [3]. In [6] Rituparna.et.al discussed the existence of solution of linear equation of non – normal or generalized triangular fuzzy number. The necessary and sufficient condition for the existence of solution are being discussed. In [7,8] Stephen Dinagar.et.al discussed the notion of inverse of fuzzy matrices in a distinct way.

This paper is organized as follows. In section 2, we give some basic preliminaries which are needed for this work. In section 3, Generalized Triangular fuzzy matrix is introduced and Matrix Inversion method for fuzzy linear system is studied with the aid of Generalized Triangular fuzzy number. Relevant numerical examples are illustrated in section 4. Finally the conclusion is also included in section 5.

2. Preliminaries

In this section, some basic definitions and results are studied and recalled the representation of fuzzy numbers.

Definition 2.1. (Fuzzy set)
A fuzzy set is characterized by a membership function mapping the elements of a domain, space or universe of discourse to the unit interval [0, 1].
A fuzzy set A in a universe of discourse X is defined as the following set of pair
$A = \{(x, \mu_A(x)); x \in X\}$

Definition 2.2. (Convex Fuzzy set)
A fuzzy set $A = \{(x,\ \mu_A(x))\} \subseteq X$ *is called convex set in all* A_α *are convex set (i.e) for every element* $x_{1 \in A_\alpha}$ *and* $x_{2 \in A_\alpha}$ *for every* $\alpha \in [0,1]$. $\lambda x_1 + (1-\lambda)x_2 \in A_\alpha$ *for all* $\lambda \in [0,1]$. *Otherwise the fuzzy set is called non – convex fuzzy set.*

Definition 2.3. (Triangular Fuzzy Number)
A fuzzy number represented with three points as follows $\tilde{A} = (a,b,c)$ *is called triangular fuzzy number. This representation is interpreted as the following membership function*

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \le a \\ \frac{x-a}{b-a}, & a < x \le b \\ 1, & x = b \\ \frac{c-x}{c-b}, & b < x \le c \\ 0, & x \ge c. \end{cases}$$

Definition 2.4. (Generalized Triangular Fuzzy Number)
A generalized triangular fuzzy number $\tilde{A} = (a_1, a_2, a_3; w_{\tilde{A}}) > 0$ *if* $a_1 > 0, \tilde{A} \ge 0$, *if* $a_1 \le 0, \tilde{A} < 0$ *if* $a_3 < 0$ *and* $\tilde{A} \le 0$ *if* $a_1 \le 0$.
Let $\tilde{A} = (a_1, a_2, a_3; w_{\tilde{A}})$ *and* $\tilde{B} = (b_1, b_2, b_3; w_{\tilde{B}})$ *be two generalized triangular fuzzy numbers, such that* $a_1 < a_2 < a_3$ *and* $b_1 < b_2 < b_3$ *are real values.*

2.1 Arithmetic Operations on Generalized Triangular Fuzzy Numbers (GTFNs)

The arithmetic operations between generalized triangular fuzzy numbers (GTFNs) are proposed given below. Let us consider $\tilde{A} = (a_1, a_2, a_3; w_{\tilde{A}})$ and $\tilde{B} = (b_1, b_2, b_3; w_{\tilde{B}})$ be two generalized triangular fuzzy numbers, then,

(i)Addition:
$\tilde{A} \oplus \tilde{B} = (a_1, a_2, a_3; w_{\tilde{A}}) \oplus (b_1, b_2, b_3\ ; w_{\tilde{B}})$
$= (a_1 + b_1, a_2 + b_2, a_3 + b_3; min(w_{\tilde{A}}, w_{\tilde{A}})$.

(ii) Subtraction:
$\tilde{A} \ominus \tilde{B} = (a_1, a_2, a_3; w_{\tilde{A}}) \ominus (b_1, b_2, b_3; w_{\tilde{B}})$
$= (a_1 - b_3, a_2 - b_2, a_3 - b_1; min(w_{\tilde{A}}, w_{\tilde{B}}))$.

(iii) Multiplication:
$\tilde{A} \otimes \tilde{B} = (a_1, a_2, a_3; w_{\tilde{A}}) \otimes (b_1, b_2, b_3; w_{\tilde{B}})$
$= (a_1 R(B), a_2 R(B), a_3 R(B); min(w_{\tilde{A}}, w_{\tilde{A}}))$.
Where, $R(B) = \left(\frac{b_1+b_2+b_3}{3}\right)$,
if $R(B) > 0$
$\tilde{A} \otimes \tilde{B} = (a_3 R(B), a_2 R(B), a_1 R(B); min(w_{\tilde{A}}, w_{\tilde{B}}))$,
if $R(B) < 0$

(iv) Division:
$\tilde{A} \oslash \tilde{B} = (a_1, a_2, a_3; w_{\tilde{A}}) \oslash (b_1, b_2, b_3; w_{\tilde{B}})$
$= (a_1 / R(B), a_2 / R(B), a_3 / R(B); min(w_{\tilde{A}}, w_{\tilde{B}}))$
if $R(B) > 0$.
$\tilde{A} \oslash \tilde{B} = (a_3 / R(B), a_2 / R(B), a_1 / R(B); min(w_{\tilde{A}}, w_{\tilde{B}}))$
if $R(B) < 0$.

2.2 Ranking Function

We define a ranking function $R : F(R) \to R$ which maps each fuzzy numbers to real line F(R) represented the set of all generalized fuzzy numbers.
$R(\tilde{A}) = \left(\frac{a_1+a_2+a_3}{3}\right)$, if $\tilde{A} = (a_1, a_2, a_3)$
and $R(\tilde{A}) = \frac{w}{3}(a_1 + a_2 + a_3)$, if $\tilde{A} = (a_1, a_2, a_3; w)$

3. Generalized Triangular Fuzzy Matrix

In this section we proposed new definition of generalized triangular fuzzy matrix and corresponding its matrix operations.

Definition 3.1. (Generalized Triangular Fuzzy Matrix)
A fuzzy matrix $\tilde{A} = (\tilde{a}_{ij}, \tilde{w}_i)$ *of order* $m \times n$ *is said to be generalized triangular fuzzy matrix if the elements of the matrix are generalized triangular fuzzy numbers.*
i.e.,$\tilde{A} = (\tilde{a}_{ij}, \tilde{w}_i)_{mxn}$ *where* $\tilde{a}_{ij} = (\tilde{a}_{ij1}, \tilde{a}_{ij2}, \tilde{a}_{ij3})$.

3.1 Fuzzy Matrix Inversion Method for Linear System

In this section, we define the concept of fuzzy linear system is justify in matrix inversion method with the aid of generalized triangular fuzzy numbers.
Let the system of equations be, The above linear system is

$$(a_{11}^{(1)}, a_{11}^{(2)}, a_{11}^{(3)}; w_1)(x_1', x_2', x_3'; w_1) + \ldots + (a_{1n}^{(1)}, a_{1n}^{(2)}, a_{1n}^{(3)}; w_n)(x_1^{n'}, x_2^{n'}, x_3^{n'}; w_n) = (r_1^{(1)}, r_1^{(2)}, r_1^{(3)}; w_r^{(1)})$$
$$(a_{21}^{(1)}, a_{21}^{(2)}, a_{21}^{(3)}; w_1)(x_1', x_2', x_3'; w_1) + \ldots + (a_{2n}^{(1)}, a_{2n}^{(2)}, a_{2n}^{(3)}; w_n)(x_1^{n'}, x_2^{n'}, x_3^{n'}; w_n) = (r_2^{(1)}, r_2^{(2)}, r_2^{(3)}; w_r^{(2)})$$
$$\vdots$$
$$(a_{n1}^{(1)}, a_{n1}^{(2)}, as_{n1}^{(3)}; w_1)(x_1', x_2', x_3'; w_1) + \ldots + (a_{nn}^{1)}, a_{nn}^{(2)}, a_{nn}^{(3)}; w_n)(x_1^{n'}, x_2^{n'}, x_3^{n'}; w_n) = (r_n^{(1)}, r_n^{(2)}, r_n^{(3)}; w_r^{(n)})$$

represented in the matrix form as;
$\tilde{A}\tilde{x} = \tilde{b}$
The solution of GTFLS will be represented by
$\tilde{x} = A^{-1}\tilde{b}$

Definition 3.2. (Determinant of GTFM) *The Determinant of* $n \times n$ *GTFM* $\tilde{A} = (\tilde{a}_{ij}, \tilde{w}_i)$ *is denoted by* $|\tilde{A}|$ *or det(A) and is defined as follows,*
$|\tilde{A}| = \sum_{q \in s_n} sgnq \prod_{i=1}^{n} (\tilde{a}_i, \tilde{w}_i)(\tilde{q}_i, \tilde{w}_i)$
$= \sum_{q \in s_n} sgnq\,(\tilde{a}_1\tilde{q}_1, min(\tilde{w}_{a_1}, \tilde{w}_{q_1}))(\tilde{a}_2\tilde{q}_2, \min(\tilde{w}_{a_2}, \tilde{w}_{q_2}))\ldots(\tilde{a}_n\tilde{q}_n, min(\tilde{w}_{a_n}, \tilde{w}_{q_n}))$
Where, $(\tilde{a}_i\tilde{q}_i, min(\tilde{w}_{a_i}, \tilde{w}_{q_i}))$ *are generalized triangular fuzzy number (GTFN) and* s_n *denotes the symmetric group of all permutations of the indices* $\{1, 2, 3, \ldots, n\}$ *and* $sgnq = 1$ *or -1 according as the permutation* $q = \begin{pmatrix} 1 & 2 & \ldots & n \\ q(1) & q(2)\ldots & & n \end{pmatrix}$ *is even or odd respectively.*

Definition 3.3. (Adjoint of GTFM)
Let $\tilde{B} = (\tilde{A}_{ij}, \tilde{w}_i)$, *then the transpose of* $\tilde{B}$ *is called the adjoint or adjugate of* $\tilde{A}$ *and is denote by* $adj\tilde{A}$. *i.e.,* $\tilde{B}^T =$

$\left(\tilde{A}_{ij},\tilde{w}_i\right) = adj\tilde{A}$, *where* $(\tilde{A}_{ij},\tilde{w}_i)$ *is the co – factor of every element* $(\tilde{a}_{ij},\tilde{w}_i)$.

Definition 3.4. (Singular GTFM)
Let $\tilde{A} = (\tilde{a}_{ij},\tilde{w}_i)$ *be a square GTFM of order n, then it is said to be singular GTFM if* $|\tilde{A}| = \tilde{0}$.

Definition 3.5. (Non - singular GTFM)
Let $\tilde{A} = (\tilde{a}_{ij},\tilde{w}_i)$ *be a square GTFM of order n, then it is said to be non – singular GTFM if* $|\tilde{A}| \neq \tilde{0}$.

Definition 3.6. (Inverse of GTFM) $\tilde{B}$ *is called the inverse of* $\tilde{A}$ *and it is denoted by* $\tilde{A}^{-1}$. *Then* $\tilde{A}^{-1} = \frac{1}{|\tilde{A}|}(adj\tilde{A})$.

4. Numerical Illustration

In this section two simple examples are given in order to illustrate the proposed method.

Illustration 4.1. *Consider the following fuzzy linear system and solve by fuzzy matrix inversion method.*

$$(1,2,3;0.5)(x_1',x_2'\,x_3';\ w_1)+(2,3,4;0.6)(x_1'',x_2''\,,x_3'';\ w_2)=(4,8,12;0.8)$$
$$(2,4,6;0.5)(x_1',x_2'\,x_3';\ w_1)+(3,5,7;0.7)(x_1'',x_2''\,,x_3'';\ w_2)=(6,10,14;0.9)$$

Solution
The given fuzzy linear system becomes,

$$\begin{pmatrix}(1,\ 2,3;0.5) & (2,3,4;0.6)\\(2,4,6;0.5) & (3,5,7;0.7)\end{pmatrix}\begin{pmatrix}\left(x_1',x_2',x_3';\ w_1\right)\\\left(x_1'',x_2''\,,x_3'';\ w_2\right)\end{pmatrix}=\begin{pmatrix}(4,8,12;0.8)\\(6,10,14;0.9)\end{pmatrix}$$

$\tilde{A}\tilde{x} = \tilde{b}$
Then the solution is, $\tilde{x} = \tilde{A}^{-1}\tilde{b}$
By definition 3.2 $|\tilde{A}| = (-11,-2,7;0.5)$
Since $\tilde{A}$ *is non – singular, then* $\tilde{A}^{-1}$ *exist and*
$\tilde{A}^{-1} = \frac{1}{|\tilde{A}|}(adj\tilde{A})$.

$$adj\tilde{A} = \begin{pmatrix}(3,5,7;0.7) & -(2,3,4;0.6)\\-(2,4,6;0.5) & (1,2,3;0.5)\end{pmatrix}$$

$$\tilde{A}^{-1} = \frac{1}{(-11,-2,7;0.5)}\begin{pmatrix}(3,5,7;0.7) & -(2,3,4;0.6)\\-(2,4,6;0.5) & (1,2,3;0.5)\end{pmatrix}$$

Thus we have, $\tilde{x} = \tilde{A}^{-1}\tilde{b}$

$$\begin{pmatrix}\left(x_1',x_2'\,,x_3';\ w_1\right)\\\left(x_1'',x_2''\,,x_3'';\ w_2\right)\end{pmatrix}=\begin{pmatrix}(-8,5,18;0.5)\\(-7,6,19;0.5)\end{pmatrix}$$

Hence the solution is, $\left(x_1',x_2'\,,x_3';\ w_1\right) = (-8,5,18;0.5)$,
$\left(x_1'',x_2''\,,x_3'';\ w_2\right) = (-7,6,19;0.5)$.

Illustration 4.2. *Consider the following fuzzy linear system and solve by fuzzy matrix inversion method.*

$$(2,3,4;0.5)(x_1',x_2'\,,x_3';\ w_1)-(1,2,3;0.6)(x_1'',x_2''\,,x_3'';\ w_2)+(-1,1,3;0.7)(x_1''',x_2'''\,,x_3''';\ w_3)=(1,2,3;0.8)$$
$$(1,2,3;0.5)(x_1',x_2'\,,x_3';\ w_1)+(2,3,4;0.6)(x_1'',x_2''\,,x_3'';\ w_2)-(-1,1,3;0.6)(x_1''',x_2'''\,,x_3''';\ w_3)=(3,5,7;0.8)$$
$$(-1,1,3;0.6)(x_1',x_2'\,,x_3';\ w_1)+(-1,1,3;0.7)(x_1'',x_2''\,,x_3'';\ w_2)+(-1,1,3,0.8)(x_1''',x_2'''\,,x_3''';\ w_3)=(4,6,8;0.9)$$

Solution
The given fuzzy linear system,

$$\begin{pmatrix}(2,3,4;0.5) & -(1,2,3;0.6) & (-1,1,3;0.7)\\(1,2,3;0.5) & (2,3,4;0.6) & -(-1,1,3;0.6)\\(-1,1,3;0.6) & (-1,1,3;0.7) & (-1,1,3,0.8)\end{pmatrix}\begin{pmatrix}\left(x_1',x_2'\,,x_3';\ w_1\right)\\\left(x_1'',x_2''\,,x_3'';\ w_2\right)\\\left(x_1''',x_2'''\,,x_3''';\ w_3\right)\end{pmatrix}=\begin{pmatrix}(1,2,3;0.8)\\(3,5,7;0.8)\\(4,6,8;0.9)\end{pmatrix}$$

$\tilde{A}\tilde{x} = \tilde{b}$
Then the solution is, $\tilde{x} = \tilde{A}^{-1}\tilde{b}$
$|\tilde{A}| = (8,17,26;0.5)$

Since $\tilde{A}$ *is non – singular, then* $\tilde{A}^{-1}$ *exist and,*
$\tilde{A}^{-1} = \frac{1}{|\tilde{A}|}(adj\tilde{A})$

$$adj\tilde{A} = \begin{pmatrix}(1,4,7;0.6) & (0,3,6;0.6) & (-8,-1,6;0.6)\\-(0,3,6;0.5) & (-1,2,5;0.5) & (0,5,10;0.5)\\(-3,-1,1;0.5) & -(3,5,7;0.5) & (8,13,18;0.5)\end{pmatrix}$$

$$\tilde{A}^{-1} = \frac{1}{(8,\ 17,26;0.5)}\begin{pmatrix}(1,4,7;0.6) & (0,3,6;0.6) & (-8,-1,6;0.6)\\-(0,3,6;0.5) & (-1,2,5;0.5) & (0,5,10;0.5)\\(-3,-1,1;0.5) & -(3,5,7;0.5) & (8,13,18;0.5)\end{pmatrix}$$

Thus we have, $\tilde{x} = \tilde{A}^{-1}\tilde{b}$

$$\begin{pmatrix}\left(x_1',x_2'\,,x_3';\ w_1\right)\\\left(x_1'',x_2''\,,x_3'';\ w_2\right)\\\left(x_1''',x_2'''\,,x_3''';\ w_3\right)\end{pmatrix}=\begin{pmatrix}(-2.7,1,4.7;0.5)\\(-1,2,5;0.5)\\(0.41,3,5.59;0.5)\end{pmatrix}$$

Hence The solution is,
$\left(x_1',x_2'\,,x_3';\ w_1\right) = (-2.7,1,4.7;0.5)$
$\left(x_1'',x_2''\,,x_3'';\ w_2\right) = (-1,2,5;0.5)$
$\left(x_1''',x_2'''\,,x_3''';\ w_3\right) = (0.41,3,5.59;0.5)$.

5. Conclusion

In this article, we investigated the fuzzy linear system of equations with generalized triangular fuzzy numbers involving in fuzzy variables. The fuzzy matrix inversion method is applied to solve the generalized triangular fuzzy linear system. This proposed method is illustrated with numerical illustrations. The main advantage of this proposed notion is that the final result will appear in fuzzified form. The discussed notion can be extended in some other concepts like Crammers rule, Decomposition method and iterative methods and so on in future.

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