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Locating chromatic number of direct product of some graphs

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Abstract

The locating chromatic number of G, denoted $\chi_{Lc}(G)$ is the least r in such a way that G requires a locating colouring with r colours. In this paper, we determine the values of the locating chromatic number of direct product graphs among the graphs, complete graph (K_x), path graph (P_x).

Keywords

Chromatic Number, Locating colouring, Locating Chromatic number, Direct product.

AMS Subject Classification

05C90.

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Contents

1. Introduction

Let $G=(V,E)$ be a linear graph which consists a set of two objects called vertices and edges denoted as $V=\{v_1,v_2,\ldots,v_n\}$ and $E=\{e_1,e_2,\ldots,e_n\}$ in such a way that every edge $e_k, 1\leq k\leq n$, is associated with an unordered pair of vertices, say (v_i,v_j). In this paper we only deal the graphs which are without self-loops and parallel edges. Painting every vertices in the graph using some colours provided different colours to be used to the vertices which are adjacent, this process is called proper colouring and also referred as simply colouring of a graph. A graph is called properly coloured graph if all the vertices of a graph is painted a colour according to the proper colouring. In so many different ways a graph can be properly coloured, in which the minimum number of colours used to colour all the vertices of a graph using proper colouring is called chromatic number. If r different colours required for its proper colouring, and not less, is called r chromatic graph and r is the chromatic number of the graph denoted by $\chi(G)$. Let v_1,v_2 be any two distinct vertices of a graph G, then the distance between v_1 & v_2, denoted as $d(v_1,v_2)$, is the length of the smallest path between them, and v_1 be any vertex of G, P be the subset of the vertex set of G, then the distance between v_1 and P is given by $d(v_1,P)=\min\{d(v_1,v_2)/v_2\in P\}$.

Definition 1.1 ([3]). *Let c be a proper r-coloring of a connected graph G and $\Pi=(P_1,P_2,\ldots,P_r)$ be an ordered partition of $V(G)$ into the resulting classes. For a vertex v_1 of G, the colour code of v_1 with respect to Π is defined to be the ordered r tuple*

$$c_\Pi(v_1)=(d(v_1,P_1),d(v_1,P_2),\ldots,d(v_1,P_r)).$$

If different colour codes have assigned to different vertices of G, then c is called a locating colouring of G. The least number of colours required for locating colouring is called locating chromatic number of G, denoted as $\chi_{LC}(G)$.

The direct product is also referred in many terminologies such as tensor product, cardinal product, weak direct product, relational product, Kronecker product, or conjunction. Direct product was introduced by Whitehead and Russel [1]. Direct product satisfy commutative and associative properties [2]

Definition 1.2 ([2]). *Let $G=(V_1,E_1)$ and $H=(V_2,E_2)$ be any two simple graphs with $V_1=\{v_1,v_2,\ldots,v_i\}$ and $V_2=\left\{v'_1,v'_2,\ldots,v'_j\right\}$. Then the direct product of the graphs G and*

H denoted as $G \times H$ and defined as the Cartesian product of the vertex sets V_1 and V_2 in such a way that any two different vertices (v_1, v_1') and (v_2, v_2') of $G \times H$ are adjacent if $v_1, v_2 \in E_1(G)$ and $v_1', v_2' \in E_2(H)$.

The locating chromatic number idea was initiated by Chartrand et al. [3]. They calculated locating chromatic number for some connected graphs. Also they authorized few bounds for the locating chromatic number for connected graph.

The theorem stated as $\chi_{LC}(H) = x$ iff H is a complete multipartite graph with 3 or more vertices proved by them. From this we will come to know that the locating chromatic number of the complete graph with x vertices is x. That is, $\chi_{LC}(K_x) = x$ provided $x \geq 3$. They also proved that in [1], the locating chromatic number for paths and cycles whose number of vertices are 3 or more as

$$\chi_{LC}(P_x) = 3,\ x \geq 3$$

$$\chi_{LC}(C_x) = \begin{cases} 3, & \text{if } x \text{ is odd} \\ 4, & \text{if } x \text{ is even.} \end{cases}$$

The locating chromatic number for trees, Kneser graphs and amalgamation of stars were determined in [3–5], respectively. Refer [3] to [7] for a detailed interpretation on locating chromatic number.

Proposition 1.3 ([4]). *In a locating colouring of H, the same colour will not be assigned to two colourful vertices. Hence at the maximum rcolourful vertices are there for locating r colouring of H.*

Proposition 1.4. *If the connected graph H having two cliques of order r, then $\chi_{LC}(H) \geq\geq r+1$.*

In this paper, we study the locating number of the mesh $P_x \times P_y$ and $K_x \times P_y$.

2. The Locating Chromatic number of product of two Path graphs

Before getting the value of locating chromatic number of $P_x \times P_y$ and $K_x \times P_y$, we need to get an upper bound for the locating chromatic number of any two graphs H_1 and H_2 where H_1 and H_2 are connected graphs.

Theorem 2.1. *Suppose H_1 and H_2 are any two connected graphs then*

$$\chi_{LC}(H_1 \times H_2) \leq \chi_{LC}(H_1)\, \chi_{LC}(H_2).$$

Proof. Let H_1 be any connected graph with x_1 locating colouring with the colour class $C_1, C_2, \ldots, C_{x_1}$, that is, $x_1 = \chi_{LC}(H_1)$ and let H_2 be any connected graph with x_2 locating colouring with the colour class $C_1', C_2', \ldots, C_{x_2}'$, that is, $x_2 = \chi_{LC}(H_2)$. For every $i \in [x_1]$ and every $j \in [x_2]$, $C_i \times C_j'$ is an independent set in $H_1 \times H_2$. Therefore $\left\{C_i \times C_j' / i \in [x_1], j \in [x_2]\right\}$ of vertices of $H_1 \times H_2$ will be in the colour classes of proper colouring of $H_1 \times H_2$. To verify that the above is a locating colouring, let us consider any two distinct vertices say (a_1, b_1) and (a_2, b_2) in the colour class $C_i \times C_j'$ provided that $a_1 \neq a_2$. Also $d\left(b_1 \times C_j'\right) = d\left(b_2 \times C_j'\right) = 0$, then there exists l such that $d(a_1 \times C_l) \neq d(a_2 \times C_l)$. Therefore

$$\begin{aligned} D\left((a_1, b_1) C_l \times C_j'\right) &= d(a_1, C_l) + d\left(b_1, C_j'\right) \\ &= d(a_1, C_l) + 0 \neq d(a_2 \times C_l) \\ &= d\left((a_2, b_2), C_l \times C_j'\right). \end{aligned}$$

It is very clearly indicating that the above mentioned colouring is a locating colouring.

Suppose H_1 and H_2 are complete graph with two vertices (i.e. K_2), then we have

$$\chi_{LC}(K_2 \times K_2) = \chi_{LC}(C_4) = 4 = \chi_{LC}(K_2) \cdot \chi_{LC}(K_2)$$

Hence the above mentioned inequality holds good. □

Theorem 2.2. *Let x and y be any arbitrary integers and y exceeds 2, x exceeds y then the locating chromatic number of the product of two path graphs with x and y vertices respectively is four, that is, $\chi_{LC}(P_x \times P_y) = 4$ provided $y \geq 2\ \&\ x \geq y$.*

Proof. Let P_x and P_y be any two path graphs with x and y vertices provided $y \geq 2\ \&\ x \geq y$. Clearly there is an induced cycle C_4 available with 3 colours in every proper 3 colouring of the product of two path graphs with x and y vertices. Here there are two vertices on this cycle with the same colour. Hence $\chi_{LC}(P_x \times P_y) \geq 4$. For every $i \in [x]$ and $j \in [y]$, let $u_{i,j}$ be the vertex in the mesh of $P_x \times P_y$ as ith row and jth column.

Now $P_x \times P_y$ is the product of two graphs with the proper 2 – colouring with the colour set $\{c_1, c_2\}$ denoting as C. Let us define the colouring class C' as follows

$C'(u_{1,1}) = c_3, C'(u_{1,y}) = c_4$ and $C'(u_{i,j}) = C(u_{i,j})$, then we have

$$d(u_{i,j}, u_{1,1}) = i + j - 2$$

$$d(u_{i,j}, u_{1,n}) = n + i - (j+1).$$

From the above it is clear that, distinct vertices have distinct colour codes, hence it completes the proof. □

3. The locating Chromatic number of the product of Complete graph and Path graph

Let H be the resulting graph of the product of complete graph with x vertices denoted as K_x and the path graph with y vertices denoted as P_y. Vertices of H can be represented in x by y matrix, that is, H contains x number of rows and y number of columns. Clearly the complete graph with x vertices (K_x) is an isomorphic to the induced subgraph on the vertices of every column and by the same way the path graph with y vertices (P_y) is an isomorphic to the induced subgraph on the vertices of every row. Let $u_{i,j}$ be the vertex of the resulting graph of the product of complete graph with x vertices and the path graph with y vertices which is in the ith row and jth column of the x by y matrix where $i \leq x$ and $j \leq y$. Therefore which is clearly indicates that (i,j) is the colour of $u_{i,j}$.

Theorem 3.1. *Let x and y be the positive integers provided $x \geq 3$ and $y \geq 2$ and H be the resulting graph of the product of complete graph with x vertices and the path graph with y vertices and there is a locating $(x+1)$ colouring then X be its colouring matrix. Then every two successive columns of X has contrasting lost colours.*

Proof. Let us consider the number of vertices of a complete graph be 3 i.e, $x = 3$ and X be the matrix of the resulting graph of the product of complete graph with x vertices and the path graph with y vertices $(K_x \times P_y)$. We need to prove that every two successive columns of X has contrasting lost colours.

By the method of contradiction, there are two colours which are successive say X_j and X_{j+1} of X with the same lost colour, say c_4

Let us assume that $X_j = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}^T$, obviously X_{j+1} is the alternate of X_j then $X_{j+1} = \begin{bmatrix} c_3 & c_1 & c_2 \end{bmatrix}^T$ or otherwise $X_{j+1} = \begin{bmatrix} c_2 & c_3 & c_1 \end{bmatrix}^T$. If $j = 1$ and $X_{j'}$ will be the first column and in row i in which the colour c_4 can be used, then the distance between the vertices $u_{i,1}$ and $u_{i+1,2}$ are equal to the colour class c_4. Which is contradiction to the fact that different colours to be used for same distance vertices. For $j+1 = y$, the above discussed procedure can be followed, not only for that, the same argument can be followed pertaining the colour c_4 have the index more than $j+1$ or otherwise all the indices smaller than j

Therefore assume that there are two indices j_1 and j_2 whereas $j_1 < j, j < j+1$ & $j+1 < j_2$ in such a way that the colour c_4 appears in both the columns $j_1 < j_2$. If $j - j_1 = j_2 - j - 1$, then the distance between the columns $j_1 < k < j_2$. If $j - j_1 = j_2 - j - 1$, then the distance between the columns j and $j+1$ are equal to the colour code c_4. Therefore at the minimum, two vertices of the same distances have the same colour code. Therefore WLG assume that $j - j_1 < j_2 - j - 1$, and the colour code c_4 available in I row of the column X_{j_1} Now the distance between the vertices $u_{3,j}$ and $u_{1,j+1}$ are equal and having the colour code c_4, which is the contradiction to the fact that the same distance having vertices assigning different colour codes.

For $x \geq 4$, then the same kind of approach can be viewed, we can find two vertices with the same distance having the same colour code. Which is contradiction to the fact that the same distance having vertices assigning different colour codes.

Therefore our assumption is wrong. Hence every two successive columns of X has contrasting lost colours. □

Theorem 3.2. *Let $x \geq 3$ and $y \geq 2$ then $\chi_{LC}(K_x \times P_y) = x+1$ provided $x \geq y-1$.*

Proof. Let H be the resulting graph of the product of the graphs complete graph with x vertices and path graph with y vertices. By Proposition (2), we have $\chi_{LC}(H) = x+1$, let us assigning $x+1$ colouring to H. Let $[c_{x+1}\ c_1\ c_2\ \dots\ c_{x-2}\ c_{x+2}]^T$ be the first column vector of the colouring matrix X and the other columns are $[c_1\ c_2\ \dots\ c_{x-1}\ c_x]^T$ and $[c_x\ c_1\ c_2\ \dots\ c_{x-1}]^T$ then the distinct vertices of H have distinct colour codes, that is, no two distinct vertices with the same distance and same colour code, clearly it is locating colouring of H.

Hence $\chi_{LC}(H) = x+1$ or $\chi_{LC}(H) = x+2$. We are in a position to prove that if $\chi_{LC}(H) = x+1$ then $x \geq y-1$.

Let us assume that $\chi_{LC}(H) = x+1$ and X be the corresponding matrix of the locating $x+1$ colouring of H. Clearly it indicates that X contains x rows and one colour will be missing in every column. By Theorem 3.1, every two successive columns of X has contrasting lost colours and also all the columns of X have at the minimum of one full colour, which shows that H has $x+1$ columns at the maximum. More precisely, assume $x \geq y-1$ and proving $\chi_{LC}(H) = x+1$. For $x \in \{c_3, c_4\}$ then E_1 be the matrix of $(K_3 \times P_4)$ and $(K_4 \times P_5)$, then

First column of $E_1 = \begin{bmatrix} c_1 & c_2 & c_3 \end{bmatrix}^T$

Second column of $E_1 = \begin{bmatrix} c_4 & c_1 & c_2 \end{bmatrix}^T$

Third column of $E_1 = \begin{bmatrix} c_2 & c_4 & c_3 \end{bmatrix}^T$

Fourth column of $E_1 = \begin{bmatrix} c_3 & c_1 & c_4 \end{bmatrix}^T$.

And E_2 be the matrix of $(K_4 \times P_5)$, then

First column of $E_2 = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}^T$

Second column of $E_2 = \begin{bmatrix} c_5 & c_3 & c_1 & c_2 \end{bmatrix}^T$

Third column of $E_2 = \begin{bmatrix} c_1 & c_5 & c_2 & c_4 \end{bmatrix}^T$

Fourth column of $E_2 = \begin{bmatrix} c_5 & c_3 & c_4 & c_1 \end{bmatrix}^T$

Fifth column of $E_2 = \begin{bmatrix} c_4 & c_5 & c_2 & c_3 \end{bmatrix}^T$.

From the above classifications of E_1 and E_2, we came to know that distinct columns have distinct missing colours and therefore distinct colour codes are assigned to the vertices which are in same colour. There are absolutely $x+1$ colourful vertices. Therefore this is locating colouring. Hence x belonging to either three or four and by the condition $y \leq x+1$. Hence $\chi_{LC}(K_x \times P_y) = x+1$. □

4. Conclusion

In this paper, we studied colouring of the graph and analyze the locating chromatic number of graphs. In particularly studied the locating chromatic number of the direct product of path graphs and direct product of complete graph with path graph. More precisely, the above said techniques were analyzed through matrix method.

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