Malaya Journal of Matematik, Vol. S, No. 1, 380-385, 2020
https://doi.org/10.26637/MJM0S20/0072

Strong convergence theorems for mixed type asymptotically nonexpansive mappings in hyperbolic spaces

S. Jone Jayashree[1]* and A. Anthony Eldred[2]

Abstract
In this paper, we prove strong convergence theorems for mixed type asymptotically nonexpansive self and nonself mappings namely S_1, S_2 and T_1, T_2 respectively on a uniformly convex Hyperbolic space (X,d,H), Using a two step iterative scheme as follows:

$$x_{n+1} = P(H(S_1^n x_n, T_1(PT_1)^{n-1} y_n, \alpha_n))$$
$$y_n = P(H(S_2^n x_n, T_2(PT_2)^{n-1} x_n, \beta_n))$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1)$ and P is a projection on X.

Keywords
Uniformly convex Hyperbolic space; strong convergence; common fixed point; mixed type asymptotically nonexpansive mapping.

AMS Subject Classification
47H10.

[1] *Department of Mathematics, Holy Cross College, Trichy-620002, Tamil Nadu, India.*
[2] *Department of Mathematics, St.Joseph's College, Trichy-620002, Tamil Nadu, India.*
***Corresponding author**: [1] jone-shree@rediffmail.com; [2] anthonyeldred@yahoo.co.in
Article History: Received **01** February **2020**; Accepted **29** March **2020**

Contents

1. Introduction

For asymptotically nonexpansive self-mappings in Banach spaces[2-16], introduced by Goebel and Kirk [1] in 1972, which is an important generalization of the class of nonexpansive self-mappings, some authors proved weak and strong convergence theorems, which extend and improve this result of Goebel and Kirk in several ways. Intially Goebel and Kirk proved that if K is a nonempty closed subset of a real uniformly convex Banach space E and T is an asymptotically nonexpansive self-mapping of K, then T has a fixed point. Recently, Chidume et al.[10] introduced the concept of asymptotically nonexpansive nonself-mappings which is a generalization of asymptotically nonexpansive self-mapping.

The following iterative scheme was studied in 2003, by Chidume et al.[10]: For a nonempty closed convex subset K of a real uniformly convex Banach space E,

$$\begin{cases} x_1 & \in K \\ x_{n+1} & = P\big((1-\alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}x_n\big) \end{cases} \tag{1.1}$$

for each $n \geq 1$, P is a nonexpansive retraction of E onto K, and $\{\alpha_n\}$ a sequence in $(0,1)$. For an asymptotically nonexpansive nonself-mapping, the authors also proved some strong and weak convergence theorems.

A generalization of (1.1) was given in 2006 by Wang [11] as follows:

$$\begin{cases} x_1 & \in K \\ x_{n+1} & = P\big((1-\alpha_n)x_n + \alpha_n T_1(PT_1)^{n-1}y_n\big) \\ y_n & = P\big((1-\beta_n)x_n + \beta_n T_2(PT_2)^{n-1}x_n\big) \end{cases} \tag{1.2}$$

for each $n \geq 1$, $\{\alpha_n\}$ and $\{\beta_n\}$ are real sequences in $[0,1)$ and $T_1, T_2 : K \to E$ are two asymptotically nonexpansive nonself-mappings. Wang proved some strong and weak convergence theorems for two asymptotically nonexpansive nonself-mappings.

For the iteration process (1.2), recently Guo and Guo[12] proved some new weak convergence theorems.

In this paper we prove some strong convergence theorems on uniformly convex Hyperbolic spaces, by constructing a new iteration scheme of mixed type for each of two asymptotically nonexpansive self-mappings and nonself-mappings, in particular on uniformly convex Hyperbolic spaces.

2. Preliminaries

Definition 2.1. *For any non empty subset K of a real metric space (X,d), any $S : K \to K$ is said to be an asymptotically nonexpansive self-mapping if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that,*

$$d(S^n x, S^n y) \leq k_n d(x,y) \tag{2.1}$$

for all $x,y \in K$ and $n \geq 1$

Definition 2.2. *Any mapping $f : X \to K$, where K is a mapping subset of a real metric space (X,d) is called as an retraction if,*

$$f^2 = f \tag{2.2}$$

Note 2.3. *If f is a retraction then $fy = y$ for all y in the range of f and we also say that K is a retract of X.*

Definition 2.4. *[10] For any nonempty subset K of a real metric space (X,d), let $P : X \to K$ be a nonexpansive retraction of X onto K. Then, $T : K \to X$ is said to be an asymptotically nonexpansive nonself-mapping if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that*

$$d\big(T(PT)^{n-1}x, T(PT)^{n-1}y\big) \leq k_n d(x,y) \tag{2.3}$$

Definition 2.5. *A triplet (X,d,H) is said te be a Hyperbolic metric space if (X,d) is a metric space and $H : X \times X \times [0,1] \to X$ is a function such that, for all $u,v,w,z \in X$ and $\beta, \gamma \in [0,1]$, the following hold:*

$$1.\, d(u, H(u,v,\beta)) \leq (1-\beta)d(z,u) + \beta d(z,v) \tag{2.4}$$

$$2.\, d(H(u,v,\beta)), H(u,v,\gamma) = |\beta - \gamma| d(u,v)$$

$$3.\, H(u,v,\beta) = H(v,u,1-\beta)$$

$$4.\, d(H(u,z,\beta), H(v,w,\beta)) \leq (1-\beta)d(u,v) + \beta d(z,w)$$

Definition 2.6. *A Hyperbolic space (X,d,H) is said to be uniformly convex if for any $r > 0$ and $\varepsilon \in (0,2]$, there exists a $\delta \in (0,1]$ such that for all $u,x,y \in X$,*

$$d\big(H(x,y,\tfrac{1}{2}),u\big) \leq (1-\delta)r \tag{2.5}$$

provided $d(x,u) \leq r, d(y,u) \leq r$ and $d(x,y) \geq \varepsilon r$

Let (X,d,H) be a *uniformly convex Hyperbolic space*, K be a nonempty closed subset of X and $P : X \to K$ be a nonexpansive retraction of X onto K.

Let $S_1, S_2 : K \to K$ be two asymptotically nonexpansive self-mappings and $T_1, T_2 : K \to X$ be two asymptotically nonexpansive nonself-mappings. For $n \geq 1$, we define

$$\begin{cases} x_1 & \in K \\ x_{n+1} & = P\big(H(S_1^n x_n, T_1(PT_1)^{n-1} y_n, \alpha_n)\big) \\ y_n & = P\big(H(S_2^n x_n, T_2(PT_2)^{n-1} x_n, \beta_n)\big) \end{cases} \tag{2.6}$$

$\{\alpha_n\}$ and $\{\beta_n\}$ being two real sequences in $[0,1)$

Note 2.7. *The common fixed points of S_1, S_2, T_1 and T_2 is denoted by F_{cp}, where, $F_{cp} = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$*

Lemma 2.8. *If the condition, $a_{n+1} \leq (1+b_n)a_n + c_n$ is satisfied for any three nonnegative sequences $\{a_n\}, \{b_n\}$ and $\{c_n\}$ and for each $n \geq n_0$, where n_0 is some nonnegative integer with $\sum_{n=n_0}^{\infty} b_n < \infty$ and $\sum_{n=n_0}^{\infty} c_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.*

Lemma 2.9. *Suppose $\{x_n\}$ and $\{y_n\}$ are two sequences of a uniformly convex Hyperbolic space (X,d,H) such that, for $R \in [0,\infty)$,*
$\lim_{n\to\infty} \sup d(x_n,a) \leq R$, $\lim_{n\to\infty} \sup d(y_n,a) \leq R$ and
$\lim_{n\to\infty} d\big(H(x_n,y_n,\alpha_n)\big) = R$ where $\alpha_n \in [a,b]$ with
$0 < a \leq b < 1$, then we have, $\lim_{n\to\infty} d(x_n,y_n) = 0$

3. Main Results

In this section, we consider a uniformly convex hyperbolic space (X,d,H) and prove a strong convergence theorem for X, using the iterative scheme given in (2.6)

Lemma 3.1. *Let (X,d,H) be a real uniformly convex Hyperbolic space and K be a nonempty closed convex subset of X. Let $S_1, S_2 : K \to K$ be two asymptotically nonexpansive selfmappings with $\{k_n^{(1)}\}, \{k_n^{(2)}\} \subset [1,\infty)$ and $T_1, T_2 : K \to X$ be two asymptotically nonexpansive nonself-mappings with $\{l_n^{(1)}\}, \{l_n^{(2)}\} \subset [1,\infty)$ such that, $\sum_{n=1}^{\infty} (k_n^{(i)} - 1) < \infty$ and $\sum_{n=1}^{\infty} (l_n^{(i)} - 1) < \infty$ for $i = 1,2$, respectively and $F_{cp} = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2) \neq \phi$. Let $\{x_n\}$ be a sequence defined by (2.3) where $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[0,1)$. Then,*

$$\lim_{n\to\infty} d(x_n,q) \textit{ exists for any } q \in F_{cp};$$

Proof. (1) Using (2.1) and (2.3) and setting
$h_n = max\{k_n^{(1)}, k_n^{(2)} l_n^{(1)}, l_n^{(2)}\}$

we have,

$$
\begin{aligned}
d(y_n,q) &= d\big(P(H(S_2^n x_n, T_2(PT_2)^{n-1}x_n,\beta_n)),q\big)\\
&\le d\big(H(S_2^n x_n, T_2(PT_2)^{n-1}x_n,\beta_n),q\big)\\
&\le (1-\beta_n)d(q,S_2^n x_n)+\beta_n d(q,T_2(PT_2)^{n-1}x_n)\\
&\le (1-\beta_n)d(S_2^n q,S_2^n x_n)+ \qquad (3.1)\\
&\qquad \beta_n d(T_2(PT_2)^{n-1}q,T_2(PT_2)^{n-1}x_n)\\
&\le (1-\beta_n)h_n d(q,x_n)+\beta_n h_n d(q,x_n)\\
&\le h_n d(q,x_n) \qquad (3.2)
\end{aligned}
$$

Also,

$$
\begin{aligned}
d(x_{n+1},q) &= d\big(P(H(S_1^n x_n, T_1(PT_1)^{n-1}y_n,\alpha_n)),q\big)\\
&= d\big(P(H(S_1^n x_n, T_1(PT_1)^{n-1} \qquad (3.3)\\
&\qquad \big[P(H(S_2^n x_n,T_2(PT_2)^{n-1}x_n,\beta_n))\big],\alpha_n)),q\big)\\
&\le d\big(H(S_1^n x_n, T_1(PT_1)^{n-1} \qquad (3.4)\\
&\qquad [H(S_2^n x_n,T_2(PT_2)^{n-1}x_n,\beta_n)],\alpha_n),q\big)\\
&\le (1-\alpha_n)d(q,S_1^n x_n)+\alpha_n d\big(q,T_1(PT_1)^{n-1} \qquad (3.5)\\
&\qquad [H(S_2^n x_n,T_2(PT_2)^{n-1}x_n,\beta_n)]\big)\\
&\le (1-\alpha)h_n^2 d(q,x_n)+\alpha_n h_n^2 d(q,x_n) \qquad (3.6)\\
&\qquad (\text{using}(3.2))\\
&\le (1+h_n^2-1)d(q,x_n) \qquad (3.7)
\end{aligned}
$$

By the hypothesis, $\sum_{n=1}^{\infty}(k_n^i-1)<\infty$ and $\sum_{n=1}^{\infty}(l_n^i-1)<\infty$ for $i=1,2$. Therefore, $\sum_{n-1}^{\infty}(h_n^2-1)<\infty$, for $i=1,2$.
Using lemma 2.8 , $\lim_{n\to\infty} d(x_n,q)$ exists.

□

Lemma 3.2. *Let (X,d,W) be a real uniformly convex Hyperbolic space and K be a nonempty closed convex subset of X. Let $S_1,S_2:K\to K$ be two asymptotically nonexpansive selfmappings with $\{k_n^{(1)},k_n^{(2)}\}\subset[1,\infty)$ and $T_1,T_2:K\to X$ be two asymptotically nonexpansive nonself-mappings with $\{l_n^{(1)},l_n^{(2)}\}\subset[1,\infty)$ such that $\sum_{n=1}^{\infty}(k_n^{(i)}-1)<\infty$ and $\sum_{n=1}^{\infty}(l_n^{(i)}-1)<\infty$ for $i=1,2$. respectively, and $F_{cp}=F(S_1)\cap F(S_2)\cap F(T_1)\cap F(T_2)\neq\phi$. Let $\{x_n\}$ be the sequence defined by (2.6) and the following conditions hold:*

(i) $\{\alpha_n\}$ and $\{\beta_n\}$ are two real sequences in $[\varepsilon,1-\varepsilon]$ for some $\varepsilon\in(0,1)$

(ii) $d(x,T_iy)\le d(S_ix,T_iy)$ for all $x,y\in K$ and $i=1,2$.

Then $\lim_{n\to\infty} d(x_n,S_ix_n)=\lim_{n\to\infty} d(x_n,T_ix_n)=0$ for $i=1,2$.

Proof. For any given $q\in F_{cp}$, $\lim_{n\to\infty} d(x_n,q)$ exists, by lemma 3.1,
Taking $h_n=max\{k_n^{(1)},k_n^{(2)},l_n^{(1)},l_n^{(2)}\}$
Suppose $\lim_{n\to\infty} d(x_n,q)=c$
By (3.7) and $\sum_{n=1}^{\infty}(h_n^2-1)<\infty$, we have,

$$\lim_{n\to\infty} d(H(S_1^n x_n,T_1(PT_1)^{n-1}y_n,\alpha_n),q)=c$$

and

$$\lim_{n\to\infty}\sup d(S_1^n x_n,q)\le\lim_{n\to\infty}\sup h_n d(x_n,q)=c$$

Taking lim sup on both sides of (3.2) we obtain,

$$\lim_{n\to\infty}\sup d(y_n,q)\le c$$

and so we have,

$$\lim_{n\to\infty}\sup d\big(T_1(PT_1)^{n-1}y_n,q\big)\le\lim_{n\to\infty}\sup h_n d(y_n,q)=c$$

By lemma 2.9 we thus have,

$$\lim_{n\to\infty} d\big(S_1^n x_n,T_1(PT_1)^{n-1}y_n\big)=0 \qquad (3.8)$$

By condition (ii) and from (3.7) we get,

$$d\big(x_n,T_1(PT_1)^{n-1}y_n\big)\le d\big(S_1^n x_n,T_1(PT_1)^{n-1}y_n\big)$$

and thus

$$\lim_{n\to\infty} d\big(x_n,T_1(PT_1)^{n-1}y_n\big)=0 \qquad (3.9)$$

Also

$$d\big(x_n,q\big)\le d\big(x_n,T_1(PT_1)^{n-1}y_n\big)+d\big(T_1(PT_1)^{n-1}y_n,q\big)$$

and

$$d\big(T_1(PT_1)^{n-1}y_n,q\big)\le h_n d\big(y_n,q\big)$$

which implies,

$$d\big(x_n,q\big)\le d\big(x_n,T_1(PT_1)^{n-1}y_n\big)+h_n^n d\big(y_n,q\big)$$

In the above inequality taking infimum on both sides and applying (3.9)
we obtain,

$$\lim_{n\to\infty}\inf d(y_n,q)\ge c$$

and

$$\lim_{n\to\infty}\sup d(y_n,q)\le c$$

Therefore, $\lim_{n\to\infty} d(y_n,q)=c$

Using the arguments in (3.1) and by $\sum_{n=1}^{\infty}(h_n^2-1)<\infty$
we have,

$$\lim_{n\to\infty} d\big(H(S_2^n x_n,T_2(PT_2)^{n-1}x_n,\beta_n),q\big)=c$$

and

$$\lim_{n\to\infty}\sup d(S_2^n x_n, q) \le \lim_{n\to\infty}\sup h_n d(x_n, q) = c$$

Also,

$$\lim_{n\to\infty}\sup d(T_2(PT_2)^{n-1}x_n, q) \le \lim_{n\to\infty}\sup h_n d(x_n, q) = c$$

Applying by Lemma 2.9, again we have,

$$\lim_{n\to\infty} d\big(S_2^n x_n, T_2(PT_2)^{n-1}x_n\big) = 0 \tag{3.10}$$

From condition (ii) and (3.10) we get,

$$d\big(x_n, T_2(PT_2)^{n-1}x_n\big) \le d\big(S_2^n x_n, T_2(PT_2)^{n-1}x_n\big)$$

which means

$$\lim_{n\to\infty} d\big(x_n, T_2(PT_2)^{n-1}x_n\big) = 0 \tag{3.11}$$

$P : X \to K$ is a nonexpansive retraction of X onto K and also $S_2^n x_n = P(S_2^n x_n)$ we get,

$$d\big(y_n, S_2^n x_n\big) \le \beta_n d\big(S_2^n x_n, T_2(PT_2)^{n-1}x_n\big)$$

and hence by (3.5)

$$\lim_{n\to\infty} d\big(y_n, S_2^n x_n\big) = 0 \tag{3.12}$$

Considering,

$$d\big(y_n, x_n\big) \le d\big(y_n, S_2^n x_n\big) + d\big(S_2^n x_n, T_2(PT)^{n-1}x_n\big) + d\big(T_2(PT_2)^{n-1}x_n, x_n\big)$$

By (3.10), (3.11) and (3.12) we have,

$$\lim_{n\to\infty} d(x_n, y_n) = 0 \tag{3.13}$$

We have,

$$d\big(S_1^n x_n, T_1(PT_1)^{n-1}x_n\big) \le d\big(S_1^n x_n, T_1(PT_1)^{n-1}y_n\big) + h_n d(y_n, x_n)$$

By (3.8) and (3.13) we hence have,

$$\lim_{n\to\infty} d\big(S_1^n x_n, T_1(PT_1)^{n-1}x_n\big) = 0 \tag{3.14}$$

$$\text{Therefore, } \lim_{n\to\infty} d\big(x_n, T_1(PT_1)^{n-1}x_n\big) = 0 \tag{3.15}$$

We know that,
$d\big(x_n, T_1(PT_1)^{n-1}x_n\big) \le d\big(S_1^n x_n, T_1(PT_1)^{n-1}x_n\big)$

Considering,

$$\begin{aligned}
&d\big(x_{n+1}, S_1^n x_n\big)\\
&= d\big(P(H(S_1^n x_n, T_1(PT_1)^{n-1}y_n, \alpha_n)), S_1^n x_n\big)\\
&\le (1-\alpha_n) d\big(S_1^n x_n, S_1^n x_n\big) +\\
&\qquad \alpha_n d\big(T_1(PT_1)^{n-1}y_n, S_1^n x_n\big)\\
&\le \alpha_n d\big(S_1^n x_n, T_1(PT_1)^{n-1}y_n\big)
\end{aligned}$$

which implies by (3.8) ,

$$\lim_{n\to\infty} d(x_{n+1}, S_1^n x_n) = 0 \tag{3.16}$$

Also,
$d\big(x_{n+1}, T_1(PT_1)^{n-1}y_n\big) \le d\big(x_{n+1}, S_1^n x_n\big)$
$+ d\big(S_1^n x_n, T_1(PT_1)^{n-1}y_n\big)$
and therefore by (3.8) and (3.16) we have

$$\lim_{n\to\infty} d\big(x_{n+1}, T_1(PT_1)^{n-1}y_n\big) = 0 \tag{3.17}$$

Consider
$d(S_1^n x_n, x_n) \le d\big(S_1^n x_n, T(PT_1)^{n-1}x_n\big) + d\big(T_1(PT_1)^{n-1}x_n, x_n\big)$
By (3.9) and (3.10) we have,

$$\lim_{n\to\infty} d(S_1^n x_n, x_n) = 0$$

Since
$d\big(S_1^n x_n, T_2(PT_2)^{n-1}x_n\big) \le d\big(S_1^n x_n, x_n\big) + d\big(x_n, T_2(PT)^{n-1}x_n\big)$,
we have from(3.6) ,

$$\lim_{n\to\infty} d\big(S_1^n x_n, T_2(PT_2)^{n-1}x_n\big) = 0 \tag{3.18}$$

Also,

$$\begin{aligned}
&d\big(x_{n+1}, T_2(PT_2)^{n-1}y_n\big)\\
&\le d\big(x_{n+1}, S_1^n x_n\big)\\
&\quad + d\big(S_1^n x_n, T_2(PT_2)^{n-1}x_n\big) + d\big(T_2(PT_2)^{n-1}x_n, y_n\big)
\end{aligned}$$

Thus, by (3.8),(3.11) and (3.13),

$$\lim_{n\to\infty} d\big(x_{n+1}, T_2(PT_2)^{n-1}y_n\big) = 0 \tag{3.19}$$

T_1 and T_2 are asymptotically nonexpansive nonself-mappings and we know that
$(PT_i)(PT_i)^{n-2}y_{n-1}, x_n \in K$ for $i = 1,2$. Hence, we have,

$$\begin{aligned}
&d\big(T_i(PT_i)^{n-1}y_{n-1}, T_i x_n\big)\\
&= d\big(T_i[(PT_i)(PT_i)^{n-2}y_{n-1}], T_i(Px_n)\big)\\
&\le h_n d\big((PT_i)(PT_i)^{n-2}y_{n-1}, Px_n\big)\\
&\le h_n d\big(T_i(PT_i)^{n-2}y_{n-1}, x_n\big)
\end{aligned} \tag{3.20}$$

For $i = 1,2$ using (3.12) and (3.14) in (3.15) we obtain,

$$\lim_{n\to\infty} d\big(T_i(PT_i)^{n-1}y_{n-1}, T_i x_n\big) = 0 \tag{3.21}$$

and we take,

$$d\big(x_{n+1},y_n\big) \le d\big(x_{n+1},T_1(PT_1)^{n-1}y_n\big)+d\big(T_1(PT_1)^{n-1}y_n,x_n\big) + d(x_n,y_n)$$

Substituting (3.4),(3.8)and (3.12) in the above inequality, we get,

$$\lim_{n\to\infty} d(x_{n+1},y_n)=0 \tag{3.22}$$

For $i=1,2, d(x_n,T_ix_n)$ can be written as follows,

$$\begin{aligned} d(x_n,T_ix_n) &\le d\big(x_n,T_i(PT_i)^{n-1}x_n\big) \\ &\quad + d\big(T_i(PT_i)^{n-1}x_n,T_i(PT_i)^{n-1}y_{n-1}\big) \\ &\quad + d\big(T_i(PT_i)^{n-1}y_{n-1},T_ix_n\big) \\ &\le d\big(x_n,T_i(PT_i)^{n-1}x_n\big)+h_nd(x_n,y_{n-1}) \\ &\quad + d\big(T_i(PT_i)^{n-1}y_{n-1},T_ix_n\big) \end{aligned}$$

By (3.6),(3.10),(3.16) and(3.17) we have,

$$\lim_{n\to\infty} d(x_n,T_1x_n)=\lim_{n\to\infty} d(x_n,T_2x_n)=0 \tag{3.23}$$

The first part of the theorem is hence proved. We prove the next part of the theorem, ie.,
$\lim_{n\to\infty} d(x_n,S_1x_n)=\lim_{n\to\infty} d(x_n,S_2x_n)=0$
By condition (ii) of the theorem for $i=1,2$ we have,

$$\begin{aligned} d(x_n,S_ix_n) &\le d\big(x_n,T_i(PT_i)^{n-1}x_n\big) \\ &\quad + d\big(T_i(PT_i)^{n-1}x_n,S_ix_n\big) \\ &\quad (or) \\ d(x_n,S_ix_n) &\le d\big(x_n,T_i(PT_i)^{n-1}x_n\big) \\ &\quad + d\big(S_ix_n,T_i(PT_i)^{n-1}x_n\big) \\ &\le d\big(x_n,T_i(PT_i)^{n-1}x_n\big) \\ &\quad + d\big(S_i^nx_n,T_i(PT_i)^{n-1}x_n\big) \end{aligned}$$

Thus by (3.5),(3.6),(3.9) and (3.10) we see that,

$$\lim_{n\to\infty} d\big(x_n,S_1x_n\big)=\lim_{n\to\infty} d\big(x_n,S_2x_n\big)=0 \tag{3.24}$$

Hence the required second part of the theorem is proved. □

Theorem 3.3. *Considering the assumption in lemma 3.2 and if one of S_1,S_2,T_1 and T_2 is completely continuous after that the sequence $\{x_n\}$ defined by 2.6 converges Strongly to a point in F_{cp}.*

Proof. Let S_1 be completely continuous.
By lemma 3.1, $\{x_n\}$ is bounded.
Which means, there is a subsequence $\{S_1x_{n_j}\} of \{S_1x_n\}$ such that $\{S_1x_{n_j}\}$ converges strongly to some $q^*\in K$. Moreover, By lemma 3.2, we have,

$$\lim_{j\to\infty} d(x_{n_j},S_1x_{n_j})=\lim_{j\to\infty} d(x_{n_j},S_2x_{n_j})=0 \text{ and}$$
$$\lim_{j\to\infty} d(x_{n_j},T_1x_{n_j})=\lim_{j\to\infty} d(x_{n_j},T_2x_{n_j})=0$$

which implies that,
$d(x_{n_j},q^*)\le d(x_{n_j},S_1x_{n_j})+d(S_1x_{n_j},q^*)\to 0\ as\ j\to\infty$ and so $x_{n_j}\to q^*\in K$.
Thus ,

$$d(q^*,S_iq^*)=\lim_{j\to\infty} d(x_{n_j},S_ix_{n_j})=0$$

But S_1,S_2,T_1 and T_2 are continuous, for $i=1,2$.
By lemma 3.2, therefore we have

$$d(q^*,T_iq^*)=\lim_{j\to\infty} d(x_{n_j},T_ix_{n_j})=0$$

which implies $q^*\in F(S_1)\cap F(S_2)\cap F(T_1)\cap F(T_2)=F_{cp}$
By lemma 3.1, $\lim_{n\to\infty} d(x_n,q)$ exists for $q\in F_{cp}$
Thus $\lim_{n\to\infty} d(x_n,q^*)$ exists and $\lim_{n\to\infty} d(x_n,q^*)=0$
Hence the proof . □

References

[1] Goebel, K, Kirk, WA, A fixed point theorem for asymptotically nonexpansive mapping, *Proc. Amer. Math. Soc.,* 35(1972), 171–174.

[2] Chang, S.S, Cho, Y.J, Zhou, H.Y, Demi-closed principle and weak convergence problems for asymptotically nonexpansive mappings, *J. Korean Math. Soc.*, 38(2001), 1245–1260.

[3] Chang, S.S, Tan, K.K, Lee, H.W.J, Chan, C.K, On the convergence of implicit iteration process with error for a finite family of asymptotically nonexpansive mappings, *J. Math. Anal. Appl.,* 313(2006), 273–283.

[4] Cho, Y.J, Kang, J.I, Zhou, H.Y, Approximating common fixed points of asymptotically nonexpansive mappings, *Bull. Korean Math. Soc.,* 42(2005), 661–670.

[5] Cho, Y.J, Zhou, H.Y, Guo, G.T, Weak and strong convergence theorems for three-step iterations with errors for asymptotically nonexpansive mappings, *Comput. Math. Appl.,* 47(2004), 707–717.

[6] Chen. W, Guo. W, Convergence theorems for two finite families of asymptotically nonexpansive mappings, *Math. Comput. Model.,* 54(2011), 1311–1319.

[7] Guo, W, Cho, Y.J, On strong convergence of the implicit iterative processes with errors for a finite family of asymptotically nonexpansive mappings, *Appl. Math. Lett.,* 21(2008), 1046–1052.

[8] Schu, J, Weak and strong convergence of fixed points of asymptotically nonexpansive mappings, *Bull. Aust. Math. Soc.,* 43(1991), 153–159.

[9] Zhou, Y.Y, Chang, S.S, Convergence of implicit iterative process for a finite family of asymptotically nonexpansive mappings in Banach spaces, *Numer. Funct. Anal, Optim.*, 23(2002), 911–921.

[10] Chidume, C.E, Ofoeau, E.U, Zegeye, H, Strong and weak convergence theorems for asymptotically nonexpansive mappings, *J. Math. Anal. Appl.,* 280(2003), 364–374.

[11] Wang, L, Strong and weak convergence theorems for common fixed points of nonself asymptotically nonexpansive mappings, *J. Math. Anal. Appl.,* 323(2006), 550–557.

[12] Guo, W, Weak convergence theorems for asymptotically nonexpansive mappings, *J. Math. Anal. Math. Lett.,* 24(2011), 2181–2185.

[13] Pathak, H.K, Cho, Y.J, Kang, S.M, Strong and weak convergence theorems for nonself-asymptotically perturbed nonexpansive mappings, *Nonlinear Anal.,* 70(2009), 1929–1938.

[14] Zhou, H.Y, Cho, Y.J, Kang, S.M, A new iterative algorithm for approximating common fixed points for asymptotically nonexpansive mappings, *Fixed Point Theory Appl.,* 2007, Article ID 64974(2007).

[15] Sun, Z.H, Strong convergence of an implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings, *J. Math. Anal. Appl.,* 286(2003), 351–358.

[16] Falset, J.G, Kaczor, W, Kuczumow, T, Reich, S, Weak convergence theorems for asymptotically nonexpansive mappings and semigroups, *Nonlinear Anal.,* 43(2001), 377–401.

[17] Suanoom Cholatis, Klin-eam Chakkrid, Remark on fundamentally non-expansive mappings in hyperbolic spaces, *J. Nonlinear Sci. Appl.,* 9(2016), 1952–1956.

⋆⋆⋆⋆⋆⋆⋆⋆⋆
ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
⋆⋆⋆⋆⋆⋆⋆⋆⋆