A note on the double domination in ladder graph

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Abstract
For any graph $G = (V, E)$ the ladder graph $L_n$ is defined by $L_n = P_n \times K_2$ where $P_n$ is a path with $n$ vertices and $\times$ denotes the Cartesian product and $K_2$ is a complete graph with two vertices. A set $D^d$ of $V[L_n]$ is double dominating set of $L_n$ if for every vertex $v \in V[L_n]$, $|N[v] \cap D^d| = 2$, that is $v$ is in $D^d$ and has at least one neighbor in $D^d$ or $v$ is in $V[L_n] - D^d$ and has at least two neighbors in $D^d$. The double domination number $\gamma_{dd}(L_n)$ in ladder graph $L_n$ is a minimum cardinality of double dominating set. In this paper many sharp bounds on $\gamma_{dd}(L_n)$ are obtained and its exact value for ladder graph were found in terms of parameter of $G$. Also its relationship with other domination parameters is investigated.

Keywords
Domination number, $(1,2)$-Domination number, Upper Pendant domination number, Double domination number.

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1 Introduction

Let $G = (V, E)$ be a simple graph. The open neighbourhood $N(v)$ of the vertex $v$ consists of the set of vertices adjacent to $v$, that is, $N(v) = \{w \in V : vw \in E\}$, and the closed neighbourhood of $v$ is $N[v] = N(v) \cup \{v\}$.

A set $S \subseteq V$ of vertices in a graph $G = (V, E)$ is called a dominating set if every vertex $v \in V$ is either an element of $S$ or is adjacent to an element of $S$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$.

A $(1,2)$-dominating set in a graph $G = (V, E)$ is a set $S$ such that for any vertex $v \in V \setminus D$ there is at least one vertex in $S$ at distance 1 from $v$ and a second vertex in $S$ at distance at most 2 from $v$. The order of the smallest $(1,2)$-dominating set of $G$ is called the $(1,2)$-domination number of $G$ and we denote it by $\gamma_{(1,2)}(G)$.

A dominating set $D$ in $G$ is called a pendant dominating set if $\delta(D) > 0$ contains at least one pendant vertex. The minimum cardinality of a pendant dominating set is called the pendant domination number denoted by $\gamma_{pd}(G)$. The minimal pendant dominating set with maximum cardinality is called the upper pendant dominating set and is denoted by $\Gamma_{pd}(G)$.

A set $D^d$ of $V[G]$ is double dominating set of $G$ if for each vertex $v \in V[G]$, $|N[v] \cap D^d| \geq 2$, that is $v$ is in $D^d$ and has at least one neighbor in $D^d$ or $v$ is in $V[G] - D^d$ and has at least two neighbors in $D^d$. The size of a smallest double dominating set is called the double domination number of $G$ and it is denoted by $\gamma_{dd}(G)$.

A Cartesian product of two graphs $G$ and $H$ is the graph $K = G \times H$ has $V(K) = V(G) \times V(H)$ and vertices $(u_1, v_1)$ and $(u_2, v_2)$ in $V(K)$ are adjacent if and only if either $u_1 = u_2$ and $v_1v_2 \in E(H)$ or $v_1 = v_2$ and $u_1u_2 \in E(G)$.

The ladder graph $L_n$ is defined by $L_n = P_n \times K_2$ where $P_n$ is a path with $n$ vertices and $\times$ denotes the Cartesian product and $K_2$ is a complete graph with two vertices. For notation and terminology we follow [1, 4]. For the survey on the area of domination in ladder graph have been considered and investigated in [5, 6, 9–12].

The concepts of double domination in graphs with its
many variations have been actively investigated in [2, 3, 7, 8]. In this paper we determine sharp double domination number and relation with domination number, (1, 2)- domination number and upper pendant domination number of ladder graph.

The following theorems will be useful in the proof of our results.

Theorem 1.1. [9]: For a ladder graph \( L_n \), \( (1, 2) \)- domination number is \( n \). That is, \( \gamma(1,2)(L_n) = n \).

Theorem 1.2. [9]: For a ladder graph \( L_n \) with \( n \) even, \( \gamma(L_n) = n \).

Theorem 1.3. [9]: For a ladder graph \( L_n \) with \( n \) odd, \( \gamma(L_n) = n - 1 \).

Corollary 1.4. [10]: Let \( G \) be ladder graph. Then \( \Gamma_{pe}(G) = n \).

## 2. Main Results

Theorem 2.1. Let \( L_n \) be a ladder graph of order \( 2n \) with \( n = 2k + 2, k = 0, 1, 2, 3, \ldots \), then

\[
\gamma_{dd}(L_n) = n + 1.
\]

**Proof.** Let \( L_n \) be a ladder graph and \( n \) be an even number.

Let \( V(L_n) = \{(1, 1), (1, 2), (1, 3), \ldots, (1, n), (2, 1), (2, 2), (2, 3), \ldots, (2, n)\} \) be the set of vertices of ladder graph \( L_n \).

Let \( A = \{(1, 1), (1, 3), (1, 5), (1, 7), \ldots, (1, n)\} \) be the set of vertices of ladder graph contain \( k + 2 \) vertices and \( B = \{(2, 1), (2, 3), (2, 5), (2, 7), \ldots, (2, n - 1)\} \) be the set of vertices of ladder graph contain \( k + 1 \) vertices.

Let \( D^{k} = \{(1, 1), (1, 3), (1, 5)(1, 7), \ldots, (1, n), (2, 1), (2, 3), (2, 5), (2, 7), \ldots, (2, n - 1)\} = A \cup B \) be the minimal double dominating set of \( L_n \) such that any vertex \( (x, y) \in V(L_n) - D^{k} \) is adjacent to at least two vertices of \( D^{k} \) and

\[
|N((x, y)) - D^{k}| = 2.
\]

Clearly \( D^{k} \) itself is a \( \gamma_{dd} \)-set of \( L_n \).

Therefore it follows that

\[
|D^{k}| = |A| + |B| = k + 2 + k + 1.
\]

\( \Rightarrow 2k + 3, \) but \( 2k = n - 2 \).

\( \Rightarrow n - 2 + 3 = n + 1. \)

Hence

\[
\gamma_{dd}(L_n) = n + 1.
\]

Theorem 2.2. Let \( L_n \) be a ladder graph of order \( 2n \) with \( n = 2k + 1, k = 0, 1, 2, 3, \ldots \), then

\[
\gamma_{dd}(L_n) = n + 1.
\]

**Proof.** Let \( L_n \) be a ladder graph and \( n \) be an odd number.

Let \( V(L_n) = \{(1, 1), (1, 2), (1, 3), \ldots, (1, n), (2, 1), (2, 2), (2, 3), \ldots, (2, n)\} \) be the set of vertices of ladder graph \( L_n \).

Let \( C = \{(1, 1), (1, 3), (1, 5), (1, 7), \ldots, (1, n)\} \) be the set of vertices of \( L_n \) contain \( k + 1 \) vertices and

\[ D = \{(2, 1), (2, 3), (2, 5), (2, 7), \ldots, (2, n - 1)\} \]

be the set of vertices of \( L_n \) contain \( k + 1 \) vertices.

Let \( D^{k} = \{(1, 1), (1, 3), (1, 5)(1, 7), \ldots, (1, n), (2, 1), (2, 3), (2, 5), (2, 7), \ldots, (2, n)\} = A \cup B \) be the minimal double dominating set of \( L_n \) such that \( (x, y) \in V(L_n) - D^{k} \) is adjacent to at least two vertices of \( D^{k} \) and

\[
|N((x, y)) - D^{k}| = 2.
\]

Clearly \( D^{k} \) itself is a \( \gamma_{dd} \)-set of \( L_n \).

Therefore it follows that

\[
|D^{k}| = |C| + |D| = k + 1 + k + 1.
\]

\( \Rightarrow 2k + 2, \) but \( 2k = n - 1 \).

\( \Rightarrow n - 2 + 3 = n + 1. \)

Hence

\[
\gamma_{dd}(L_n) = n + 1.
\]

Theorem 2.3. Let \( L_n \) be a ladder graph of order \( 2n \), then

\[
\gamma_{dd}(L_n) + \text{diam}(L_n) = 2n + 1.
\]

**Proof.**

\[
\text{dist}((u, v), (x, y)) = \text{diam}(L_n)
\]

\[
= \{(1, 1), (1, 2), (1, 3)(1, 4), \ldots, (1, n), (2, n)\}
\]

the longest path between any two distinct vertices \( (u, v), (x, y) \in V(L_n) \) such that contains \( n + 1 \) vertices that is \( n + 1 - 1 = n \) edges.

It clear that

\[
|D^{k}| + |\text{dist}((u, v), (x, y))| = n + 1 + n = 2n + 1.
\]

Hence

\[
\gamma_{dd}(L_n) + \text{diam}(L_n) = 2n + 1.
\]

Theorem 2.4. Let \( L_n \) be a ladder graph of order \( 2n \), then

\[
\gamma_{dd}(L_n) = 2p - q - 1.
\]

**Proof.** Let \( D^{k} \) be a double dominating set of \( L_n \).

Since \( V(L_n) - D^{k} \) is disconnected.

\[
q = |V(L_n) - D^{k}| + |V(L_n)| - 1.
\]

Hence the result.
3. Relation Between Domination Number, (1,2)-Domination Number, Upper Pendent Domination Number and Double Domination Number of Ladder Graph

Theorem 3.1. Let \( L_n \) be a ladder graph of order \( 2n \) with \( n = 2k + 2, k = 0, 1, 2, 3, \ldots \), then
\[
\gamma_{dd}(L_n) + \gamma(L_n) = 2n + 1.
\]

Proof. The result follows from Theorem 2.1 and Theorem 1.2.

Theorem 3.2. Let \( L_n \) be a ladder graph of order \( 2n \) with \( n = 2k + 1, k = 0, 1, 2, 3, \ldots \), then
\[
\gamma_{dd}(L_n) + \gamma(L_n) = 2n.
\]

Proof. The result follows from Theorem 2.2 and Theorem 1.3.

Theorem 3.3. Let \( L_n \) be a ladder graph of order \( 2n \). Then the following is equivalent
\[
\gamma_{dd}(L_n) + \gamma(1,2)(L_n) = 2n + 1.
\]

Proof. The result follows from Theorem 2.1, 2.2 and Theorem 1.1.

Theorem 3.4. Let \( L_n \) be a ladder graph of order \( 2n \). Then the following is equivalent
\[
\gamma_{dd}(L_n) + \Gamma_{pe}(G) = 2n.
\]

Proof. The result follows from Theorem 2.1, 2.2 and Corollary 1.4.

4. Algorithm for to Find a Minimal Double Dominating Set

Theorem 4.1. The following algorithm computes the minimum double dominating set, in a ladder graph \( L_n \) when \( n \) is even.

Input: \( L_n, n \) even

Step 1: Select two vertices from first rung.
Step 2: select two vertices from alternative rung.
Step 3: Repeat the step 2 up to second last rung.
Step 4: Select one vertex from the last rung.
Step 5: Stop when all alternating rungs are executed.
Step 6: Collect all the vertices in \( D^d \).
Output: \( D^d \) is the double dominating set of cardinality \( n + 1 \).

For \( L_2 \),

\[
(1, 1) \quad (2, 1)
(1, 2) \quad (2, 2)
\]

Therefore
\[
\gamma_{dd}(G) = 3, \gamma(G) = 2
\]
and
\[
\gamma(1,2) = 2.
\]

For \( L_3 \),

\[
(1, 1) \quad (2, 1)
(1, 2) \quad (2, 2)
(1, 3) \quad (2, 3)
\]

Therefore
\[
\gamma_{dd}(G) = 4, \gamma(G) = 2
\]
and
\[
\gamma(1,2) = 3.
\]

5. Conclusion

In this paper, we established the relation between double domination number, domination number, (1, 2)-domination number and upper pendent domination number of a ladder graph and also presented good sharp bounds on the double domination number of a Ladder graph.

References

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