Complementary nil detour eccentric domination number of a graph

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Abstract
In this paper, the complementary nil detour eccentric domination number of a graph is defined. Few results on complementary nil detour eccentric domination numbers are obtained. The relationship among complementary nil domination number, complementary nil eccentric domination number, detour eccentric domination number and complementary nil detour eccentric domination numbers are discussed. Few theorems related to the above said numbers are stated and proved.

Keywords
Complementary nil domination number, Complementary nil eccentric domination number, Complementary nil detour eccentric domination number.

AMS Subject Classification
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1 Introduction

For any graph $G = (V, E)$ where $V$ is the set of vertices and $E$ is the set of edges. The order and size of $G$ are $n = |V|$ and $m = |E|$ respectively. The length of the shortest $x – y$ path joining is the distance $d(x,y)$. The detour distance is the length of the longest $x – y$ path in graph $G$ and is denoted by $D(x,y)$. For any vertex $x$ in $G$, $e_D(x) = \max \{D(x,y) : y \in V\}$ is called detour eccentricity of $x$. If $e_D(x) = D(x,y)$ then $y$ is called a detour eccentric vertex of $x$ in $G$. The detour radius $R$ and detour diameter $D$ of $G$ are denoted and defined by $R_D = \min \{e_D(y) : y \in V\}$ and $D_D = \max \{e_D(y) : y \in V\}$ respectively. The vertex $y$ in $G$ is called detour central vertex if $e_D(y) = rad(G)$. The detour eccentric set of a vertex $y$ in $G$ is defined as $E_D(y) = \{x \in V : D(x,y) = e_D(y)\}$. A set $D \subseteq V$ is said to be a eccentric set if for all $y$ in $V$ and $e_D(y) = D_D$ is an eccentric vertex of at least one $x$ in $D$. A set $D \subseteq V$ is said to be a detour eccentric set if for all $y$ in $V$ and $D_D$ is a detour eccentric vertex of at least one $x$ in $D$.

A set $D \subseteq V$ is said to be a dominating set in $G$, if every vertex in $V - D$ is dominated by a few vertex in $D$. $\gamma(G) = \min \{|D|/D \text{ is a dominating set}\}$ is called the domination number of $G$. If $D_1 \subseteq V$ in a graph $G$ is a dominating set and a detour set then $D_1$ is called a detour dominating
set (DD-set). \( \gamma_{Ded}(G) = \min\{|D_1|/D_1 \text{ is a detour dominating set}\} \) is called a detour domination number. If \( D_2 \subseteq V \) is a dominating set and every \( v \in V - D_2 \) there exist at least one eccentric vertex \( v \in D_2 \) then \( D_2 \) is called an eccentric dominating set (ED-set). \( \gamma_{Ded}(G) = \min\{|D_2|/|D_2| \text{ is an eccentric dominating set} \} \) is called an eccentric domination number. If \( D_3 \subseteq V \) is an ED-set and every \( v \in V - D_3 \), there exists at least one detour eccentric vertex \( v \in D_3 \) then \( D_3 \) is called a detour eccentric dominating set (DED-set).

**Definition 3.1.** A DED-set \( C \subseteq V \) in a graph \( G \) is said to be a complementary nil detour eccentric dominating set (CNED-set) if \( V - C \) is not a DED-set. A CNED-set is called minimal if there is no set \( C' \subset C \) is a CNED-set. \( \gamma_{cn-Ded}(G) = \min\{|C|/C \text{ is a CNED-set}\} \) is called complementary nil detour eccentric domination number. \( \Gamma_{cn-Ded}(G) = \max\{|C|/C \text{ is a CNED-set}\} \) is called an upper complementary nil detour eccentric domination number. \( \gamma_{cn-Ded}-set \) is called minimum CNED-set. Similarly, \( \gamma_{ed-set}, \gamma_{Ded-set}, \gamma_{cnd-set}, \gamma_{cned-set} \).

**Remark 3.2.** If \( C \) is a CNED-set, then \( V - C \) not contain a dominating set or detour eccentric set but not both.

**Example 3.3.**

From the figure 3.1 \( C = \{x_3, x_4, x_5, x_6, x_7\} \) is a minimum CNED-set.

\( C_1 = \{x_3, x_4, x_5\} \) is a minimum ED-set.

\( C_2 = \{x_1, x_4, x_6, x_7\} \) is a minimum DED-set.

\( C_3 = \{x_4, x_5, x_6, x_3\} \) is a minimum CND-set.

\( C_4 = \{x_3, x_4, x_5, x_6\} \) is a minimum CNED-set.

Therefore,

\( \gamma_{ed} = 3, \gamma_{Ded} = 4, \gamma_{cnd} = 4, \gamma_{cned} = 3, \gamma_{cn-Ded} = 5 \).

\( \gamma_{cned} < \gamma_{cnd} < \gamma_{cn-Ded} \text{ and } \gamma_{ed} < \gamma_{Ded} \).

**Example 3.4.**

From the figure 3.2 \( C = \{x_1, x_2, x_3\} \) is a minimum CND, CNED and CNED-set.
\[ C_1 = \{x_1, x_2, x_3\} \text{ is a minimum ED-set, a minimum DED-set, a minimum CNED-set and a minimum CNED-set.} \]

Therefore,

\[ \gamma_{ed} = 2, \gamma_{ded} = 2, \gamma_{ned} = 3, \gamma_{cned} = 3, \gamma_{cned} = 3. \]

\[ \gamma_{ned} = \gamma_{ned} = \gamma_{cned} \text{ and } \gamma_{ned} = \gamma_{ned}. \]

**Observation 3.5.** In a graph \( G \)

1. \( \gamma_{ned}(G) \leq \gamma_{cned}(G) \)
2. \( \gamma_{cned}(G) \leq \Gamma_{cned}(G) \)
3. Every CNDED-set contains at least 2 vertices.
4. There is no relationship between detour eccentric domination number and complementary nil detour eccentric domination number.
5. \( \gamma_{cned} \) is not exists for complete graph.

Since, \( G = K_n \) then rad\(_D\) = diam\(_D\) = \( n - 1 \).

Hence any vertex \( x \) in \( K_n \) dominates all other vertices and is also a detour eccentric point of other vertices.

Hence, complementary nil detour eccentric domination does not exist in complete graph.

**Theorem 3.6.** For any tree \( T_n, n \geq 3 \), \( \gamma_{cned} \leq n - \Delta(T_n) + 1 \).

**Proof.** In any tree \( T_n, V - S \) is a DED-set of vertices adjacent to a vertex with maximum degree. Therefore, \( |V - S| + 1 \) is a CNDED-set and hence

\[ \gamma_{cned} \leq n - \Delta(T_n) + 1. \]

Therefore,

\[ \gamma_{cned} \leq n - \Delta(T_n) + 1 = \min\{m, n\} + 1. \]

**Theorem 3.7.** For any path \( P_n \) graph,

\[ \gamma_{ned}(P_n) = \gamma_{ned}(P_n) = \frac{n}{2} \text{ or } \frac{n}{2} + 1 \]

where \( n \) is even and \( n > 4 \).

**Proof.** By theorem 2.1(2), we obtain the result.

**Theorem 3.8.** For any complete bipartite graph,

\[ \gamma_{cned}(K_{m,n}) = \gamma_{ned}(K_{m,n}) = \min\{m, n\} + 1, \]

for \( m, n \geq 2, m \geq n \).

**Proof.** By theorem 2.3 (2), we obtain the result.

**Theorem 3.9.** For any star graph,

\[ \gamma_{ned}(K_{1,n}) = \gamma_{ned}(K_{1,n}) = 2, n \geq 2. \]

**Proof.** By theorem 2.3 (1), we obtain the result.

**Theorem 3.10.** If \( G \) is not a complete graph, then \( 2 \leq \gamma_{ned} \leq n - 1 \).

**Proof.** By the observation 3.5 (3) lower bound obtained and every CNDED-set may be contain maximum of \( n - 1 \) vertices. Therefore upper bound also obtained.

Now, the bounds are sharp, Since

\[ \gamma_{cned}(G) = 2 \Leftrightarrow G = K_{1,n} \]

and since

\[ \gamma_{ned}(G) = n - 1 \Leftrightarrow G = (K_{1,n}). \]

\[ \square \]

### 4. Bounds for Complementary Nil Detour Eccentric Domination Number

In this section, the complementary nil detour eccentric domination numbers are obtained for few standard graphs and theorems related to the above said concepts are stated and proved.

**Theorem 4.1.** (i) \( \gamma_{ned}(K_{1,n}) = 2, n \geq 2 \).

(ii) \( \gamma_{ned}(W_n) = 4, n \geq 5 \).

**Proof.** We

(i) If \( G = K_{1,n} \) and let \( C = \{x, y\} \) and \( y \) be a central vertex. Then the central vertex dominates all vertices which are in \( V - C \) and \( x \) is a DED-set. If \( C \) is a CNDED-set, then \( V - C \) not contain a dominating set or detour eccentric set or both. Hence \( \gamma_{ned}(K_{1,n}) = 2, n \geq 2 \).

(ii) If \( G = W_n \), for \( n \geq 5 \), and let \( x \) be a central vertex of \( G \), then \( x \) is a detour eccentric point of all other vertices of \( G \) and \( \{x\} \) is also a dominating set. Hence, \( \{x\} \) is a minimum DED-set. The CNDED-set, not in \( V - C \) or detour eccentric set. Hence, \( \gamma_{ned}(W_n) = 4, n \geq 5 \).

\[ \square \]

**Theorem 4.2.** Let \( C \) be a CNDED-set of a graph \( G \). Then \( C \) is minimal if and only if for each \( x \in C \) one of the following conditions is true.

(i) \( x \) is an isolated vertex of \( C \) or \( x \) is the eccentric point set of \( C \).

(ii) There exists some \( y \) in \( V - C \), such that \( N(y) \cap C = \{x\} \) or \( E_D(y) \cap C = \{x\} \).

(iii) \( V - \{C - \{x\}\} \) is a DED-set.

**Proof.** Let us take \( C \) be a minimal. CNDED-set, we prove this theorem by method is contradiction if there exists a vertex \( x \in C \) such that \( x \) does not satisfy any of the following conditions (i), (ii) and (iii).

(i) Suppose \( x \) is adjacent to all the vertices in \( C \) and all the detour eccentric vertices of \( x \) in \( C \). Hence, \( C - \{x\} \) is a DED-set and detour eccentric point set of \( G \), which is contradiction to the minimality of \( C \).
Theorem 4.4. If $G$ is of radius two, then

\[ \gamma_{cn-Ded}(G) \leq \Delta(G) + 1. \]

Proof. Let the radius of a graph $G$ be 2. Let $x \in V(G)$ be such that maximum $\deg(x)(\Delta(G))$. Now take $C = \{x\} \cup N(x) = N[x]$. Every vertex in $V - C$ is adjacent to elements of $N(x)$ and are detour eccentric to $x$. This implies that $C$ is a DED-set, and $V - C = V - N(x)$, this $V - C$ has no dominating set. Since $x$ cannot be dominated by any element of $-C$.

Therefore, $C$ is a CNDED-set. Hence

\[ \gamma_{cn-Ded}(G) \leq \Delta(G) + 1. \]
All the non-pendent vertices form a dominating set but not a detour dominating set. Therefore all the pendent vertices form a DED-set.

Hence,

\[ \gamma_{cn-Ded}(G) = \frac{q}{2} + 1. \]

References


