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A study on some lattice theoretic identities of the subgroup lattice of 2×2 matrices over Z_7

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Abstract

In this paper we verify the lattice theoretic properties like modularity, distributivity, upper semi modularity and super modularity in the subgroup lattice of the group of 2×2 matrices over Z_7.

Keywords

Matrix group, subgroups, Lattice, Lattice identities.

AMS Subject Classification

03G10.

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Contents

1. Introduction

Let $L(G)$ be the Lattice of Subgroups of G, where G is a group of 2×2 matrices over Z_p having determinant value 1 under matrix multiplication modulo p, where p is a prime number.

Let $\mathscr{G} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a,b,c,d \in Z_p, ad-bc \neq 0.$

Then $\mathscr{G}$ is a group under matrix multiplication modulo p.

Let $G = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathscr{G} : ad-bc = 1. \right\}$

Then $\mathscr{G}$ is a subgroup of G.

we have, $o(\mathscr{G}) = p(p^2-1)(p-1)$ and $o(G) = p(p^2-1)$. [6]

2. Preliminaries

Definition 2.1. ***(Poset)*** *A partial order on a non-empty set P is a binary relation $\leq$ on P that is reflexive, anti-symmetric and transitive. The pair $(P,\leq)$ is called a partially ordered set or poset. A poset. $(P,\leq)$ is totally ordered if every $x,y \in P$ are comparable, that is either $x \leq y$ or $y \leq x$. A non-empty subset S of P is a chain in P if S is totally ordered by $\leq$.*

Definition 2.2. *Let $(P,\leq)$ be a poset and let $S \subseteq P$. An upper bound of S is an element $x \in P$ for which $s \leq x$ for all $s \in S$. The least upper bound of S is called the **supremum or join** of S.A lower bound for S is an element $x \in P$ for which $x \leq s$ for all $s \in S$. The greatest lower bound of S is called the **infimum or meet** of S.*

Definition 2.3. ***(Lattice)*** *Poset $(P,\leq)$ is called a lattice if every pair x,y elements of P has a supremum and an infimum, which are denoted by $x \vee y$ and $x \wedge y$ respectively.*

Definition 2.4. ***(Covering Relation)*** *In the poset $(P,\leq)$, a covers b or b is covered by a (in notation, $a > b$ or $b < a$) if and only if $b < a$ and for no $x, b < x < a$.*

Definition 2.5. ***(Atom)*** *An element a is an atom, if $a > 0$ and a dual atom, if $a < 1$.*

Definition 2.6. ***(Modular Lattice)*** *A lattice L is said to be modular if whenever $a \leq c$ in L, $a \vee (b \wedge c) = (a \vee b) \wedge c$ for all $b \in L$. In other words, a lattice L is said to be modular if $(a \wedge c) \vee (b \wedge c) = [a \wedge c) \vee b] \vee c$ for all $a,b,c \in L$.*

Definition 2.7. ***(Supermodular)*** *A lattice L is said to be* ***supermodular*** *if it satisfies the following identity* $(a \vee b) \wedge (a \vee c) \wedge (a \vee d) = a \vee [b \wedge c \wedge (a \vee d)] \vee [c \wedge d \wedge (a \vee b)] \vee [b \wedge d \wedge (a \vee c)]$ *for all* $a, b, c, d \in L$.

Definition 2.8. ***(Semi-supermodular)*** *A lattice L is said to be* ***semi-supermodular*** *if it satisfies the following identity* $(a \vee x_1) \wedge (a \vee x_2) \wedge (a \vee x_3) \wedge (a \vee x_4) = a \vee [x_1 \wedge x_2 \wedge (a \vee x_3) \wedge (a \vee x_4)] \vee [x_1 \wedge x_3 \wedge (a \vee x_2) \wedge (a \wedge x_4)] \vee [x_1 \wedge x_4 \wedge (a \vee x_2) \wedge (a \vee x_3)] \vee [x_2 \wedge x_3 \wedge (a \vee x_1) \wedge (a \vee x_4)] \vee [x_2 \wedge x_4 \wedge (a \vee x_1) \wedge (a \vee x_3)] \vee [x_3 \wedge x_4 \wedge (a \vee x_1) \wedge (a \vee x_2)]$ *for all* a, x_1, x_2, x_3, x_4 *in L.*

Definition 2.9. ***(Distributive lattice)*** *A Lattice L is said to be* ***distributive*** *if* $a \vee (b \wedge c) = [(a \vee b) \wedge (a \vee c)]$ *for all* $a, b, c \in L$.

Definition 2.10. ***(General Disjointness)*** *A lattice L with 0 satisfies* ***the general disjointness*** *property* ***(GD)*** *if* $x \wedge y = 0$ *and* $(x \vee y) \wedge z = 0$ *implies that* $x \wedge (y \vee z) = 0$, *for* $x, y \in L$.

We give below the structures of some lower intervals of $L(G)$ when $p = 7$

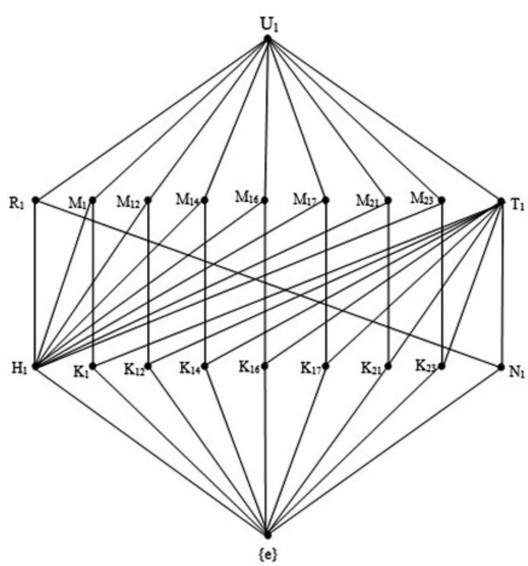

Fig. 2.1: The Interval $[\{e\}, U_1]$

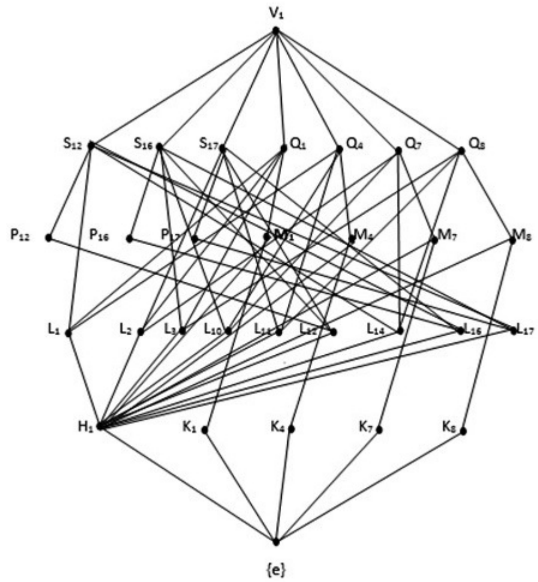

Fig. 2.2: The Interval $[\{e\}, V_1]$

3. Lattice identities in the subgroup lattice of the group of 2×2 matrices over Z_7

Lemma 3.1. *$L(G)$ is non-modular if $p = 7$.*

Proof. when $p = 7$,

From fig. 2.1 and 2.2 we take three subgroups $K_1, N_1, N_2 \in L(G)$

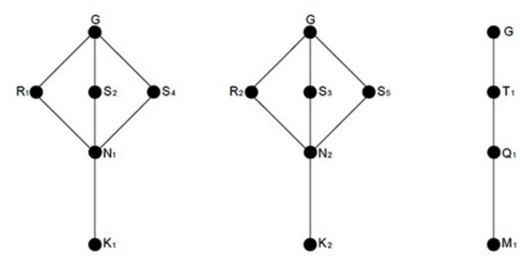

Fig: 3.1

Now, $K_1 \vee (N_1 \wedge N_2) = K_1 \vee \{e\} = K_1$.

But, $(K_1 \vee N_1) \wedge N_2 = T_1 \wedge N_2 = \{e\}$.

Therefore, $K_1 \vee (N_1 \wedge N_2) \neq (K1 \vee N_1) \wedge N_2$.

Therefore, $L(G)$ is not modular when $p = 7$. □

Lemma 3.2. *$L(G)$ is not distributive if $p = 7$.*

Proof. when $p = 7$,

From fig. 2.1 and 2.2 we take three subgroups K_{14}, K_{16}, K_{17} in $L(G)$

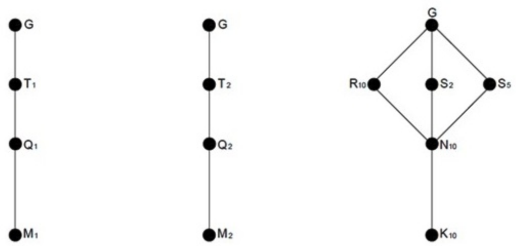

Fig: 3.2

$K14 \vee (K_{16} \wedge K_{17}) = K_{14} \vee \{e\} = K_{14}$.

But, $(K_{14} \vee N_{16}) \wedge (K_{14} \vee N_{17}) = T_1 \wedge T_1 = T_1$.

Therefore, $K_{14} \vee (K_{16} \wedge K_{17}) \neq (K_{14} \vee K_{16}) \wedge (K_{14} \vee K_{17})$.

Hence $L(G)$ is not distributive, when $P = 7$.

□

Lemma 3.3. *$L(G)$ is not upper semi modular if $p = 7$.*

Proof. When $p = 7$,

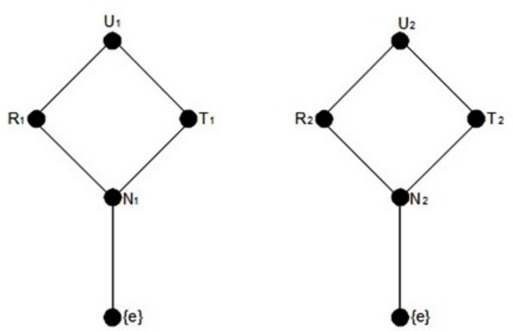

Fig: 3.3

We observe, fig. 2.1 and 2.2, $N_1 \wedge N_2 = \{e\}$ which is covered by N_1 while $N_1 \vee N_2 = G$.

which does not cover N_2.

Therefore, $L(G)$ is not upper semi modular when $p = 7$. □

Lemma 3.4. *$L(G)$ is not super modular if $p = 7$.*

Proof. When $p = 7$, we chose four subgroups $K_2, N_1, N_2 \& N_3$ in $L(G)$ such that $(K_2 \vee N_1) \wedge (K_2 \wedge N2) \wedge (K_2 \vee N_3) = G \wedge G \wedge G = G$.

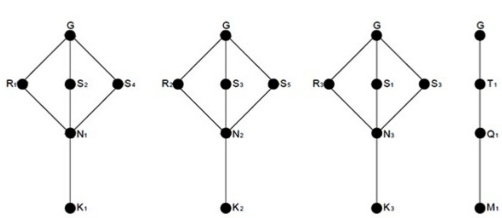

Fig: 3.4

But $K_2 \vee [N_1 \wedge N_2 \wedge (K_2 \wedge N_3)] \vee [N_1 \wedge N_3 \wedge (K_2 \wedge N_2)] \vee [N_2 \wedge N_3 \wedge (K_2 \vee N_1)] = K_2 \vee [N_1 \wedge N_2 \wedge G] \vee [N_1 \wedge N_3 \wedge G] \vee [N_2 \wedge N_3 \wedge G] = K_2 \vee \{e\} \vee \{e\} \vee \{e\} = K_2 \neq G$

Therefore, $L(G)$ is not super modular when $p = 7$. □

4. Conclusion

In this paper, we proved that the modularity, distributivity, upper semi modularity and super modularity in the subgroup lattice of the group of 2×2 matrices over Z_7.

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