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A study on subdirect irreducibility of the subgroup lattices of the group of 2×2 matrices over Z_3 and Z_5

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Abstract

In this paper we determine subdirect irreducibility of the subgroup lattice of the group of 2×2 matrices over Z_3 and Z_5.

Keywords

Matrix group, subgroups, Lattice, Congruence, subdirect irreducibility.

AMS Subject Classification

03G10.

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Contents

1. Introduction

Let $L(G)$ be the Lattice of Subgroups of G, where G is a group of 2×2 matrices over Z_p having determinant value 1 under matrix multiplication modulo p, where p is a prime number.

Let $\mathscr{G} = \left(\begin{array}{cc} a & b \\ c & d \end{array}\right) : a,b,c,d \in Z_p, ad-bc \neq 0.$

Then $\mathscr{G}$ is a group under matrix multiplication modulo p.

Let $G = \left\{\left(\begin{array}{cc} a & b \\ c & d \end{array}\right) \in \mathscr{G} : ad-bc = 1.\right\}$

Then $\mathscr{G}$ is a subgroup of G.

we have, $o(\mathscr{G}) = p(p^2-1)(p-1)$ and $o(G) = p(p^2-1)$. [6]

2. Preliminaries

Definition 2.1. ***(Poset)*** *A partial order on a non-empty set P is a binary relation $\leq$ on P that is reflexive, anti-symmetric and transitive. The pair $(P,\leq)$ is called a **partially ordered set or poset**. A poset. $(P,\leq)$ is totally ordered if every $x,y\in P$ are comparable, that is either $x\leq y$ or $y\leq x$. A non-empty subset S of P is a chain in P if S is totally ordered by $\leq$.*

Definition 2.2. *Let $(P,\leq)$ be a poset and let $S\subseteq P$. An upper bound of S is an element $x\in P$ for which $s\leq x$ for all $s\in S$. The least upper bound of S is called the **supremum or join** of S.A lower bound for S is an element $x\in P$ for which $x\leq s$ for all $s\in S$. The greatest lower bound of S is called the **infimum or meet** of S.*

Definition 2.3. ***(Lattice)*** *Poset $(P,\leq)$ is called a lattice if every pair x,y elements of P has a supremum and an infimum, which are denoted by $x\vee y$ and $x\wedge y$ respectively.*

Definition 2.4. ***(Atom)*** *An element a is an atom, if $a>0$ and a dual atom, if $a<1$.*

Definition 2.5. *An equivalence relation θ on a lattice L is called a congruence relation on L iff $(a_0,b_0)\in\theta$ and $(a_1,b_1)\in\theta$ imply that $(a_0\wedge a_1, b_0\wedge b_1)\in\theta$ and $(a_0\vee a_1, b_0\vee b_1)\in\theta$.*

Definition 2.6. *The collection of all congruence relations on L, is denoted by Con L.*

Note: Con L with respect to the set inclusion relation becomes an algebraic lattice.[1]

Definition 2.7. *If a lattice L has only two trivial congruence relations, namely ω, the diagonal and $\tau = L\times L$, then L is said to be simple. (e.g. M_3 is simple)*

Definition 2.8. *If Con L contains a unique atom, then we say that L is subdirectly irreducible. (e.g N_5 is subdirectly irreducible)*

We give below the diagrams of $L(G)$ when $P = 3$ and 5.

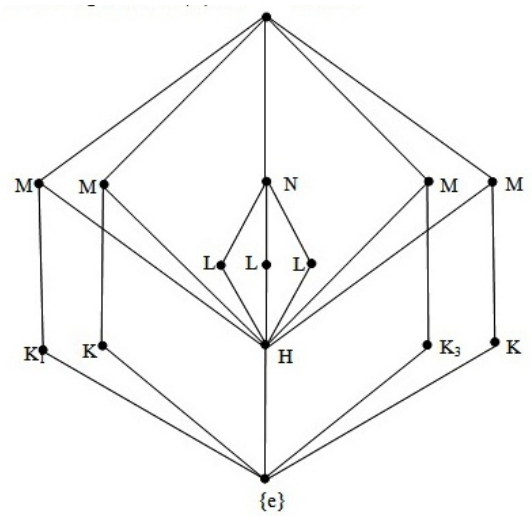

Fig. 2.1 : $L(G)$ when $p = 3$

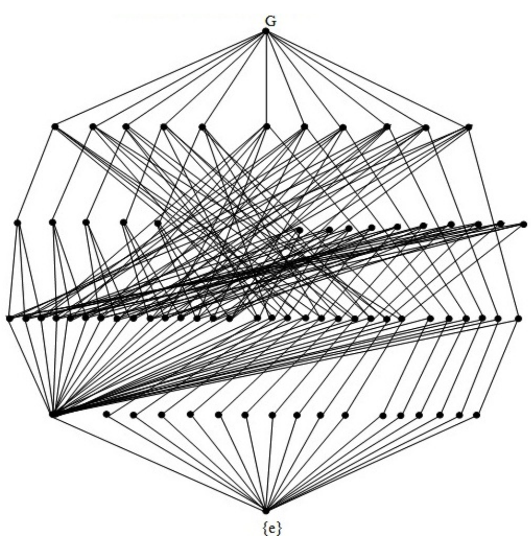

Fig. 2.2 : $L(G)$ when $p = 5$

Row I : (Left to right): S_1 to S_5 and T_1 to T_6.
Row II : (Left to right): P_1 to P_5 and R_1 to R_{10}.
Row III: (Left to right): L_1 to L_{15}, N_1 to N_{10} and Q_1 to Q_6.
Row IV: (Left to right): H_1, K_1 to K_{10} and M_1 to M_6.

3. Subdirect irreducibility of $L(G)$ when $p = 3$ and 5

In the following theorems we consider $L(G)$ when $p = 3$ and 5.

Lemma 3.1. *$\theta(\{e\},H_1)$, the principal congruence generated by $(\{e\},H_1)$ is a proper congruence relation on $L(G)$.*

Proof. When $p = 3$, $L(G)$ is given in figure 2.1. The principal congruence relation generated by $\theta(\{e\},H_1)$ is equal to $\omega \cup \{(\{e\},H_1),(H_1,(\{e\}),(K_1,M_1),(M_1,K_1),(K_2,M_2),(M_2,K_2),(K_3,M_3),(M_3,K_3),(K_4,M_4),(M_4,K_4)\}$, where ω is the diagonal relation on $L(G)$, is a proper congruence relation of $L(G)$. □

Lemma 3.2. *$\theta(A,B) = L(G)\times L(G)$, for all other pairs A and B of elements in $L(G)$.*

Proof.

$$\theta(\{e\},K_1) \\ = \omega\cup\{(\{e\},K_1),(K_1,(\{e\}),(K_2,G),(K_3,G),(\{e\},G),...\} \\ = L(G)\times L(G)$$

Similarly,

$$\theta(\{e\},K_2) = L(G)\times L(G)$$
$$\theta(\{e\},K_3) = L(G)\times L(G)$$
$$\theta(\{e\},K_4) = L(G)\times L(G)$$

$$\theta(\{e\},M_1) \\ = \omega\cup\{(\{e\},M_1),(M_1,(\{e\}),(K_3,G),(K_4,G),(\{e\},G),...\} \\ = L(G)\times L(G)$$

Similarly,

$$\theta(\{e\},M_2) = L(G)\times L(G)$$
$$\theta(\{e\},M_3) = L(G)\times L(G)$$
$$\theta(\{e\},M_4) = L(G)\times L(G)$$

$$\theta(\{e\},N_1) = \omega\cup\{(\{e\},N_1),(N_1,(\{e\}),(K_1,G),(K_2,G), (K_3,G),(K_4,G),(\{e\},G),...\} = L(G)\times L(G)$$

$$\theta(\{e\},L_1) = \omega\cup\{(\{e\},L_1),(L_1,(\{e\}),(K_1,G),(K_2,G), (K_3,G),(K_4,G),(\{e\},G),...\} = L(G)\times L(G)$$

Similarly,

$$\theta(\{e\},L_2) = L(G)\times L(G)$$
$$\theta(\{e\},L_3) = L(G)\times L(G)$$

$$\theta(H_1,G) \\ = \omega\cup\{(H_1,G),(\{e\},K_3),(\{e\},K_4),(\{e\},G),...\} \\ = L(G)\times L(G)$$

$$\theta(H_1,M_1) \\ = \omega\cup\{(H_1,M_1),(L_1,G),(L_2,G),(H_1,G),(\{e\},K_3), (\{e\},K_4),(\{e\},G),...\} = L(G)\times L(G)$$

Similarly,

$$\theta(H_1, M_2) = L(G) \times L(G)$$
$$\theta(H_1, M_3) = L(G) \times L(G)$$
$$\theta(H_1, M_4) = L(G) \times L(G)$$

$$\theta(L_1, L_2) = \omega \cup \{(L_1, L_2), (H_1, N_1), (M_1, G), (M_2, G), (M_3, G), (M_4, G), (H_1, G), (\{e\}, K_3), (\{e\}, K_4), (\{e\}, G), \ldots\} = L(G) \times L(G)$$

Similarly,

$$\theta(L_1, L_3) = L(G) \times L(G)$$
$$\theta(L_2, L_4) = L(G) \times L(G)$$

$$\theta(K_1, K_2) = \omega \cup \{(K_1, K_2), (M_1, G), (M_2, G), (M_4, G), (H_1, G), (\{e\}, K_3), (\{e\}, K_4), (\{e\}, G), \ldots\} = L(G) \times L(G)$$

Similarly,

$$\theta(L_1, L_3) = L(G) \times L(G)$$
$$\theta(L_2, L_3) = L(G) \times L(G)$$

$$\theta(M_1, M_2) = \omega \cup \{(M_1, M_2), (H_1, G), (\{e\}, K_3), (\{e\}, K_4), (\{e\}, G) \ldots\} = L(G) \times L(G)$$

Similarly,

$$\theta(M_1, M_3) = L(G) \times L(G)$$
$$\theta(M_1, M_4) = L(G) \times L(G)$$
$$\theta(M_2, M_3) = L(G) \times L(G)$$
$$\theta(M_2, M_4) = L(G) \times L(G)$$
$$\theta(M_3, M_4) = L(G) \times L(G)$$

Therefore, $\theta(A, B)$ is an improper congruence for all other pairs A and B of elements in $L(G)$. □

Remark 3.3. *For $p = 5$, by similar argument we can prove that the only proper congruence of $L(G)$ is $\theta(\{e\}, H_1)$.*

Theorem 3.4. *$Con(L(G))$ is a 3-element chain when $p = 3$ and 5. In otherwords, $L(G)$ is subdirectly irreducible when $p = 3$ and 5.*

Proof. From Lemma 3.1 and Lemma 3.2 we get the result.

The Hasse diagram of $Con(L(G))$ is as shown below.

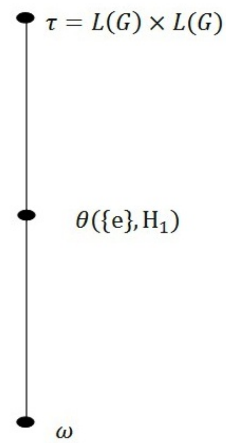

Fig. 3.1: $Con(L(G))$

□

4. Conclusion

In this paper we proved that the subgroup lattices of the group of 2×2 matrices over Z_3 and Z_5, are subdirect irreducibility.

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