Some results on intuitionistic L-fuzzy metric spaces

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Abstract
In this paper, some known results of ordinary metric spaces including Baire’s theorem for intuitionistic L-fuzzy metric spaces are stated and proved.

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Intuitionistic L-Fuzzy Metric Spaces, F-bounded, Dense.

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1 Introduction


In this paper, the complete intuitionistic L-fuzzy metric space is introduced. Baire’s theorem for intuitionistic L-fuzzy metric spaces are stated and proved.

2 Preliminaries

Definition 2.1. A 5-tuple \((X, M, N, *, \diamond)\) is said to be an intuitionistic fuzzy metric space if \(X\) is an arbitrary set, * is a continuous τ-norm, \(\diamond\) is a continuous-τ-conorm and \(M, N\) are fuzzy sets on \(X^2 \times [0, \infty)\) satisfying the conditions:

- \(M(x, y, t) + N(x, y, t) \leq 1\) for all \(x, y \in X\) and \(t > 0\)
- \(M(x, y, 0) = 0\) for all \(x, y \in X\)
- \(M(x, y, t) = 1\) for all \(x, y \in X\), and \(t > 0\) if and only if \(x = y\)
- \(M(x, y, t) = M(y, x, t)\) for all \(x, y \in X\) and \(t > 0\)
- \(M(x, y, t) \ast M(y, z, s) \leq M(x, z, t + s)\) for all \(x, y, z \in X\) and \(s, t > 0\)
- \(M(x, y, \cdot) : [0, \infty) \rightarrow [0, \infty]\) is left continuous, for all \(x, y \in X\)
- \(\lim_{t \to +\infty} M(x, y, t) = 1\) for all \(x, y \in X\) and \(t > 0\)
- \(N(x, y, 0) = 1\) for all \(x, y \in X\)
- \(N(x, y, t) = 0\) for all \(x, y \in X\) and \(t > 0\) if and only if \(x = y\)
- \(N(x, y, t) = N(y, x, t)\) for all \(x, y \in X\), and \(t > 0\)
- \(N(x, y, t) \ast N(y, z, s) \leq N(x, z, t + s)\) for all \(x, y, z \in X\) and \(s, t > 0\)
- \(N(x, y, \cdot) : [0, \infty) \rightarrow [0, 1]\) is right continuous, for all \(x, y \in X\)
- \(\lim_{t \to +\infty} N(x, y, t) = 0\) for all \(x, y \in X\).
The functions $M(x,y,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between $x$ and $y$ w.r.t. $t$ respectively.

**Definition 2.2.** A 5-tuple $(X, M, N, *, \circ)$ is said to be an intuitionistic $L$-fuzzy metric space if $X$ is an arbitrary set, $*$ is a continuoust-norm, $\circ$ is a continuoust-conorm and $M, N$ are fuzzy sets on $X^2 \times [0, \infty)$ satisfying the conditions:

1. $M(x, y, t) + N(x, y, t) \leq 1$,
2. $M(x, y, 0) = 0$
3. $M(x, y, t) = 1$ if and only if $x = y$;
4. $M(x, y, t) = M(y, x, t)$
5. $M(x, y, t) * M(y, z, s) \leq M(x, z, t+s)$,
6. $M(x, y, \cdot) : [0, \infty) \to L$ is left continuous
7. $\lim_{t \to \infty} M(x, y, t) = 1$,
8. $N(x, y, 0) = 1$,
9. $N(x, y, t) = 0$ if and only if $x = y$;
10. $N(x, y, t) = N(y, x, t)$
11. $N(x, y, t) \circ N(y, z, s) \geq N(x, z, t+s)$,
12. $N(x, y, \cdot) : [0, \infty) \to L$ is right continuous,
13. $\lim_{t \to \infty} N(x, y, t) = 0$.

The functions $M(x, y, t) : X^2 \times [0, \infty) \to L$ and $N(x, y, t) : X^2 \times [0, \infty) \to L$ denote the degree of nearness and the degree of non-nearness between $x$ and $y$ w.r.t. respectively, satisfying $M(x, y, t) \leq K(N(x, y, t))$, $K : L \to L$ is an unary involute order preserving operation.

**Example 2.3.** Define $t$-norm $a * b = \min\{a, b\}$ and $t$-co-norm $a \circ b = \max\{a, b\}$ and for all $x, y \in X$ and $t > 0$,

$$M(x, y, t) = \frac{t}{t + x}$$
$$N(x, y, t) = \frac{x}{t + x}$$

Then $(X, M, N, *, \circ)$ is an intuitionistic $L$-fuzzy metric space.

**Proof.** Consider an open ball $B(x, r, t)$. Now, $y \in B(x, r, t)$ implies that $M(x, y, t) > 1 - r$ and $N(x, y, t) < r$.

Since $M(x, y, t) > 1 - r$ and $N(x, y, t) < r, 0 < t_0 < t$, such that $M(x, y, t_0) > 1 - r$ and $N(x, y, t_0) < r$.

Let $r_0 = M(x, y, t_0) > 1 - r$ and $N(x, y, t_0) < r$.

Since $r_0 > 1 - r$ and $r_0 < r$, we can find $s, 0 < s < 1$, such that $r_0 > 1 - s > 1 - r, r_0 < 1 - s < r$.

Now for a given $r_0$ and $s$, such that $r_0 > 1 - s$.

We can find $r_1, 0 < r_1 < 1$, such that $r_0 * r_1 \geq 1 - s$.

Now consider the ball $B(y, 1 - r_1, t - t_0)$.

We claim $B(y, 1 - r_1, t - t_0) \subset B(x, r, t)$.

Now $z \in B(y, 1 - r_1, t - t_0)$ implies that $M(y, z, t - t_0) > 1 - r_1, N(y, z, t - t_0) < r_1$.

Therefore

$$M(x, z, t) \geq M(x, y, t_0) * M(y, z, t - t_0)$$
$$\geq r_0 * r_1$$
$$\geq 1 - s > 1 - r$$

Similarly,

$$N(x, z, t) \leq N(x, y, t_0) \circ N(y, z, t - t_0)$$
$$\leq r_0 \circ r_1$$
$$\leq 1 - s < r$$

Therefore $z \in B(x, r, t)$ and hence $B(y, 1 - r_1, t - t_0) \subset B(x, r, t)$.

**Theorem 3.2.** Every intuitionistic $L$-fuzzy metric space is Hausdorff.

**Proof.** Let $(X, M, N, *, \circ)$ be an intuitionistic $L$-fuzzy metric space.

Let $x, y$ be two distinct points of $X$.

Then $0 < M(x, y, t) < 1, 0 < N(x, y, t) < 1$.

Let $M(x, y, t) = r, N(x, y, t) = 1 - r$ for some $r, 0 < r < 1$.

For each $r_0, r_0 < r < 1$, we can find an $r_1$ such that $r_1 * r_1 \geq r_0$.

Now consider the open balls $B(x, 1 - r_1, \frac{r}{2})$ and $B(y, 1 - r_1, \frac{r}{2})$.  

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**3. Topology Induced By an Intuitionistic L-Fuzzy Metric Space**

**Theorem 3.1.** Every open ball in an intuitionistic $L$-fuzzy metric space $(X, M, N, *, \circ)$ is an open set.
Clearly, $B(x, 1 - r_1, \frac{t}{2}) \cap B(y, 1 - r_1, \frac{t}{2}) = \emptyset$.

For if there exists $z \in B(x, 1 - r_1, \frac{t}{2}) \cap B(y, 1 - r_1, \frac{t}{2})$;

Then
\[
\begin{align*}
  r &= M(x, y; t) \geq M(x, z, \frac{t}{2}) + M(x, y, \frac{t}{2}) \\
  r_1 
\end{align*}
\]

Hence $A$ have $A$ be a compact subset of $X$.

Let $x \in A$. Then $x \in B(x, r, t)$ and $y \in B(y, r, t)$ for some $i, j$.

Therefore $M(x, x_i, t) > 1 - r$ and $N(x, x_i, t) < r$

$M(y, x_j, t) < 1 - r$ and $N(y, x_j, t) > 1 - r$

Now, let $\alpha_m = \min \{M(x_i, x_j, t) : 1 \leq i, j \leq n\}$,

$\alpha_n = \max \{N(x_i, x_j, t) : 1 \leq i, j \leq n\}$.

Then $\alpha_m, \alpha_n > 0$.

Now
\[
\begin{align*}
  M(x, y, 3t) &\geq M(x, x_i, t) + M(x_i, y, t) + M(x_j, y, t) \\
  &\geq (1 - r) + (1 - r) + \alpha_m. \\
  N(x, y, 3t) &\leq N(x, x_i, t) + N(x_i, y, t) \\
  &\leq r \circ \alpha_n \\
  &\leq (1 - r) \circ (1 - r) \circ \alpha.
\end{align*}
\]

Taking $t' = 3t$ and $(1 - r) \circ (1 - r) \circ \alpha > 1 - s, 0 < s < 1$, we have

$M(x, y, t') > s, N(x, y, t') < 1 - s, \text{ for all } x, y \in A$.

Hence $A$ is $F$-bounded.

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**Definition 3.4.** An intuitionistic $L$-fuzzy metric space in which every Cauchy sequence is convergent is called a complete intuitionistic $L$-fuzzy metric space.

**Theorem 3.5.** Let $(X, M, N, \ast, \circ)$ be a complete intuitionistic $L$-fuzzy metric space. Then the intersection of a countable number of dense open sets is dense.

**Proof.** Let $X$ be the given complete intuitionistic $L$-fuzzy metric space.

Let $B_0$ be a nonempty open set.

Let $D^1, D^2, D^3, \ldots$, be dense open sets in $X$.

Since $D_1$ is dense in $X$, $B_0 \cap D_1 \neq \emptyset$.

Let $x_1 \in B_0 \cap D_1$. Since $B_0 \cap D_1$ is open, there exists $0 < r_1 < 1, t > 0$ such that $B(x_1, r_1, t) \subset B_0 \cap D_1$.

Choose $r_1 < r_1$ and $t_0 = \min \{t_1, 1\}$ such that $B(x_1, r_1, t_1) \subset B_0 \cap D_1$.

Let $B_1 = B(x_1, r_1', t_1')$. Since $D_2$ is dense in $X$, $B_0 \cap D_2 \neq \emptyset$.

Let $x_2 \in B_0 \cap D_2$.

Since $B_0 \cap D_2$ is open, there exists $0 < r_2 < \frac{1}{2}$ and $t_2 > 0$ such that $B(x_2, r_2, t_2) \subset B_1 \cap D_2$.

Choose $r_2 < r_2$ and $t_2 = \min \{t_2, \frac{1}{2}\}$ such that $B(x_2, r_2, t_2) \subset B_1 \cap D_2$.

Let $B_2 = B(x_2, r_2', t_2')$. Similarly proceeding by induction we can find an $x_n \in B_{n-1} \cap D_n$.

Since $B_{n-1} \cap D_n$ is open, there exists $0 < r_n < \frac{1}{n}$ and $t_n > 0$ such that $B(x_n, r_n, t_n) \subset B_{n-1} \cap D_n$.

Choose $r_n < r_n, t_n = \min \{t_n, \frac{1}{n}\}$ such that $B(x_n, r_n, t_n) \subset B_{n-1} \cap D_n$.

Let $B_n = B(x_n, r_n', t_n')$.

Now we claim that $\{x_n\}$ is a Cauchy sequence. For a given $t > 0, \varepsilon > 0$ choose $n_0$ such that $\frac{1}{n_0} < t$ and $\frac{1}{n_0} < \varepsilon$. Then for $n \geq n_0, m \geq n$.
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\[ M(x_n, x_m, t) \geq M(x_n, x_m, \frac{1}{n}) \]
\[ \geq 1 - \left( \frac{1}{n} \right) \]
\[ \geq 1 - \epsilon. \]
\[ N(x_n, x_m, t) \leq N(x_n, x_m, \frac{1}{n}) \]
\[ \leq \left( \frac{1}{n} \right) \]
\[ \leq \epsilon. \]

Therefore \( \{x_n\} \) is a Cauchy sequence. Since \( X \) is complete, the sequence \( \{x_n\} \) converges to \( x \) in \( X \). But \( x_k \in B[x_n, r'_n, t'_n] \) for all \( k \geq n \) and \( B[x_n, r'_n, t'_n] \) is a closed set.

Hence \( x \in B[x_n, r'_n, t'_n] \subseteq B_{n-1} \cap D_n \) for all \( n \).

Therefore \( B_0 \cap (\cap_{n=1}^m D_n) \neq \phi \).

Hence \( \cap_{n=1}^m D_n \) is dense in \( X \). \( \Box \)

4. Conclusion

In this paper, we have proved every open ball is an open set and Baire’s theorem for intuitionistic L-fuzzy metric spaces. Also, any complete intuitionistic L-fuzzy metric space cannot be represented as the union of a sequence of nowhere dense sets and hence it is not of first category.

References

[8] B. D. Pant and S Chouhan, fixed point theorem for occasionally weakly compatible mapping in Menger Space, Mathematika Vesnikcs. (Accepted for Publication).