\[ \theta g^{\prime\prime\prime} \]-Open sets in fuzzy topological spaces

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Abstract
In this paper, we introduce a new class of fuzzy generalized open sets called fuzzy \( \theta g^{\prime\prime\prime} \)-open and using these new generalized set, we introduce fuzzy \( \theta g^{\prime\prime\prime} \)-continuous, fuzzy \( \theta g^{\prime\prime\prime} \)-irresolute functions. Some of their properties have been investigated.

Keywords
Fuzzy generalized open sets, fuzzy \( \theta g \)-open sets, fuzzy \( \theta gs \)-open sets, fuzzy \( \theta g^{\prime\prime\prime} \)-open, fuzzy \( \theta g^{\prime\prime\prime} \)-continuous, fuzzy \( \theta g^*s \)-continuous.

AMS Subject Classification
54C10, 54C08, 54C05.

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Article History: Received 10 January 2020; Accepted 01 May 2020 ©2020 MJM.

2. Basic Concepts
Throughout the present paper, \((X, \tau)\) or simply \(X\) mean fuzzy topological space (abbreviated as fts) on which no separation axioms are assumed unless otherwise mentioned. We denote and define the closure and interior for a fuzzy set \(A\) by \(Cl(A) = \bigwedge \{\mu : \mu \geq A, 1 - \mu = \text{int}\}\) and \(Int(A) = \bigvee \{\mu : \mu \leq A, \mu \in \tau\}\). fuzzy \(\theta\)-closure of \(A\) [6] and fuzzy semi-\(\theta\) closure of \(A\) [13] are denoted by \(Cl_\theta(A) = \bigwedge \{cl(\mu) : A \leq \mu, \mu \in \tau\}\) and \(sCl_\theta(A) = \bigwedge \{scl(\mu) : A \leq \mu, \mu \text{ is semi-open in } \tau\}\) respectively.

Definition 2.1. A fuzzy set \(A\) of \((X, \tau)\) is called
(i) fuzzy semi-open [1] if \(A \subseteq Cl(Int(A))\)
(ii) fuzzy \(\alpha\)-open [3] if \(A \subseteq Int(Cl(Int(A)))\)
(iii) fuzzy \(\theta\)-open if \(A = Int_\theta(A)\) [6] and fuzzy \(\theta\)-closed [6] if \(A = Cl_\theta(A)\)
(iv) fuzzy semi-\(\theta\)-open [13] if \(A = sInt_\theta(A)\) and fuzzy semi-\(\theta\)-closed [13] \(A = sCl_\theta(A)\)

The semi closure [11] (respectively \(-CL\) closure [10]) of a fuzzy set \(A\) of \((X, \tau)\) is the intersection of all \(fs\)-closed (respectively \(\alpha\)-closed sets) that contain \(A\) and is denoted by \(sCl(A)\) (respectively \(\alpha Cl(A)\)).

Definition 2.2. A fuzzy set \(A\) of \((X, \tau)\) is called
(i) fuzzy generalized closed (in short, fg-closed)[4] if \(Cl(A) \subseteq H\), whenever \(A \leq H\) and \(H\) is fuzzy open set in \(X\). The complelement of fuzzy \(g\)-closed set is fuzzy \(g\)-open.
(ii) fuzzy $\theta$-generalized closed (in short, $f\theta g$-closed) if $\text{Cl}_\theta(A) \leq H$, whenever $A \leq H$ and $H$ is $f$-open set in $X$.

(iii) fuzzy $\theta$-generalized semi closed (in short, $f\theta g$s-closed) if $\text{Cl}_\theta(A) \leq H$, whenever $A \leq H$ and $H$ is $f$-open set in $X$.

(iv) fuzzy $g^\theta$-closed [8] if $\text{Cl}_\theta(A) \leq H$, whenever $A \leq H$ and $H$ is fuzzy $g$-open in $X$.

(v) fuzzy $\theta g^\theta$-closed [12] if $\text{Cl}_\theta(A) \leq H$, whenever $A \leq H$ and $H$ is fuzzy $\theta g$s-open in $X$.

(vi) fuzzy $\theta g^\theta$s-closed [12] if $\text{Cl}_\theta(A) \leq H$, whenever $A \leq H$ and $H$ is fuzzy $g$-open in $X$.

(vii) fuzzy $g^\theta$-closed [12] if $\text{Cl}_\theta(A) \leq H$, whenever $A \leq H$ and $H$ is fuzzy $\theta g$s-open in $X$.

(viii) fuzzy $g^\theta$s-$\theta$-closed [12] if $\text{Cl}_\theta(A) \leq H$, whenever $A \leq H$ and $H$ is fuzzy $g$-open in $X$.

Theorem 2.3. [12] For any topological space $(X, \tau)$, the following assertions hold

(i) Every fuzzy $\theta$-closed set is fuzzy $g^\theta$-closed set in a fuzzy topological space $(X, \tau)$ but not conversely.

(ii) Every fuzzy $g^\theta$-$\theta$-closed set is fuzzy $g^\theta$s-$\theta$-closed set in a fuzzy topological space $(X, \tau)$ but not conversely.

(iii) Every fuzzy $\theta g^\theta$s-closed set is fuzzy $g^\theta$s-$\theta$-closed set in a fuzzy topological space $(X, \tau)$ but not conversely.

(iv) Every fuzzy $\theta g^\theta$-closed set is fuzzy $\theta g^\theta$s-$\theta$-closed set in a fuzzy topological space $(X, \tau)$ but not conversely.

(v) Every fuzzy $\theta g^\theta$s- closed set is fuzzy $\theta g^\theta$s-$\theta$-closed set in a fuzzy topological space $(X, \tau)$ but not conversely.

Definition 2.4. A function $f : (X, \tau) \to (Y, \sigma)$ is called

1. fuzzy continuous (in short $f$-continuous) if $f^{-1}(V)$ is $f$-open set in $X$, for every fuzzy open set $V$ in $Y$. [5]

2. $fg$-continuous function if $f^{-1}(V)$ is $fg$-closed in $X$, for every fuzzy closed set $V$ in $Y$. [4]

3. $f\theta$-continuous function if $f^{-1}(V)$ is $f\theta$-closed in $X$, for every fuzzy closed set $V$ in $Y$. [13]

3. Fuzzy $\theta g^\theta$-Open Sets In Fuzzy Topological Spaces

Definition 3.1. A fuzzy subset $A$ of a fuzzy topological space $(X, \tau)$ is called fuzzy $\theta g^\theta$-open set in $X$ if $A^c$ is fuzzy $\theta g^\theta$-closed in $(X, \tau)$.

Lemma 3.2. A fuzzy subset $A$ of a fuzzy topological space $(X, \tau)$ is fuzzy $\theta g^\theta$-open if and only if $U \leq \text{Int}_\theta(A)$ whenever $U$ is fuzzy $\theta g$s-closed in $X$ and $U \leq A$.

Proof. Suppose that $U \leq \text{Int}_\theta(A)$ such that $U$ is fuzzy $\theta g$s-closed in $X$ and $U \leq A$. Let $A'$ where $H$ is fuzzy $\theta g$s-open. Then $H' \leq A$ and $H'$ is fuzzy $\theta g$s-closed. Therefore by hypothesis, $H' \leq \text{Int}_\theta(A)$. Since $H' \leq \text{Int}_\theta(A)$, we have $(\text{Int}_\theta(A))' \leq H$. That is $\text{Cl}_\theta(A') \leq H$, because $\text{Cl}_\theta(A') = (\text{Int}_\theta(A))'$. Thus $A'$ is fuzzy $\theta g^\theta$-closed set. Hence $A$ is fuzzy $\theta g^\theta$-open in $X$.

Conversely, suppose that $A$ is fuzzy $\theta g^\theta$-open such that $U \leq A$ and $U$ is fuzzy $\theta g$s-closed in $X$. Then $U' \leq A'$ is fuzzy $\theta g$s-open and $A' \leq U'$. Therefore, by definition of fuzzy $\theta g^\theta$-closedness $\text{Cl}_\theta(A') \leq U'$. Thus $U \leq \text{Int}_\theta(A)$, because $\text{Cl}_\theta(A') = (\text{Int}_\theta(A))'$.

Theorem 3.3. Every fuzzy $\theta$-open set is fuzzy $\theta g^\theta$-open set in a fuzzy topological space $(X, \tau)$.

Proof. Let $A$ be fuzzy $\theta$-open set in $(X, \tau)$, then $A^c$ is fuzzy $\theta$-closed set. Since by Theorem 2.3, every fuzzy $\theta$-closed set is fuzzy $\theta g^\theta$-closed. Therefore $A^c$ is fuzzy $\theta g^\theta$-closed set. Hence $A$ is fuzzy $\theta g^\theta$-open set in fuzzy topological space $X$.

Remark 3.4. The converse of the above proposition need not be true as shown in the following example.

Example 3.5. Let $X = \{a\}$. Fuzzy sets $A$ and $B$ are defined by $A(a) = 0.6; B(a) = 0.7$. Let $\tau = \{0, A, 1\}$. Then $B$ is a fuzzy $\theta g^\theta$-open set but it is not a fuzzy $\theta$-open set in $(X, \tau)$.

Definition 3.6. A fuzzy subset $A$ of a fuzzy topological space $(X, \tau)$ is called fuzzy $\theta g^\theta$s-open set $(g^\theta$-$\theta$-open, $g^\theta s\theta$-open, $g^\theta s\theta$-$\theta$-open) in $X$ if $A$ is fuzzy $\theta g^\theta$s-closed (g$^\theta$-$\theta$-closed, $g^\theta s\theta$-$\theta$-closed) in $(X, \tau)$.

Theorem 3.7. Every fuzzy $\theta g^\theta$s-open set is fuzzy $\theta g^\theta$s-closed set in a fuzzy topological space $(X, \tau)$.

Proof. Let $A$ be a fuzzy $\theta g^\theta$s-open set in $X$. then $A^c$ is fuzzy $\theta g^\theta$s-closed set. Since by Theorem 2.3, every fuzzy $\theta g^\theta$s-closed set is fuzzy $\theta g^\theta$s-closed. Therefore $A^c$ is fuzzy $\theta g^\theta$s-closed set. Hence $A$ is fuzzy $\theta g^\theta$s-closed set in fuzzy topological space $X$.

Remark 3.8. The converse of the above proposition need not be true as shown in the following example.

Example 3.9. Let $X = \{a, b\}$ and the fuzzy sets $A$ and $B$ be defined as follows $A(a) = 0.9; A(b) = 0.8; B(a) = 0.5; B(b) = 0.4$. Let $\tau = \{0, B, 1\}$. Then $A$ is fuzzy $\theta g^\theta$s-open but it is not fuzzy $\theta g^\theta$s-open.

Theorem 3.10. Every fuzzy $\theta g^\theta$s-open set is fuzzy $\theta g^\theta$-open set in a fuzzy topological space $(X, \tau)$. 

Theorem 3.16. Clearly $A$ is fuzzy $g^m$-closed set. Since by Theorem 2.3, every fuzzy $g^m$-closed set is fuzzy $g^m$-$\theta$-closed. Therefore $A^c$ is fuzzy $g^m$-$\theta$-closed set. Hence $A$ is fuzzy $g^m$-$\theta$-open set in fuzzy topological space $X$.

Remark 3.11. The converse of the above proposition need not be true as shown in the following example.

Example 3.12. Let $X = \{a, b\}$ and the fuzzy sets $A$, $B$ and $D$ be defined as follows $A(a) = 0.4; A(b) = 0.4; B(a) = 0.5; B(b) = 0.4; D(a) = 0.5; D(b) = 0.4$. Let $\tau = \{0, A, B, 1\}$. Then $D$ is fuzzy $g^m$-$\theta$-open but it is not not $f g^m$-open.

Theorem 3.13. Every fuzzy $g^m$-$\theta$-open set is fuzzy $g^m$-$\alpha \theta$-open set in a fuzzy topological space $(X, \tau)$.

Proof. Let $A$ be a fuzzy $g^m$-$\theta$-open set in $X$, then $A^c$ is fuzzy $g^m$-$\theta$-closed set. Since by proposition 2.3, every fuzzy $g^m$-$\theta$-closed set is fuzzy $g^m$-$\alpha \theta$-closed. Therefore $A^c$ is fuzzy $g^m$-$\alpha \theta$-closed set. Hence $A$ is fuzzy $g^m$-$\alpha \theta$-open set in fuzzy topological space $X$.

Remark 3.14. The converse of the above proposition need not be true as shown in the following example.

Example 3.15. Let $X = \{a, b\}$ and the fuzzy sets $A$ and $B$ be defined as follows $A(a) = 0.3; A(b) = 0.4; B(a) = 0.3; B(b) = 0.4$. Let $\tau = \{0, A, 1\}$. Then $B$ is fuzzy $g^m$-$\alpha \theta$-open but it is not not $f g^m$-open.

Theorem 3.16. Every fuzzy $g^s$-$s \theta$-open set is fuzzy $g^m$-$s \theta$-open set in a fuzzy topological space $(X, \tau)$.

Proof. Let $A$ be a fuzzy $g^s$-$s \theta$-open set in $X$. then $A^c$ is fuzzy $g^s$-$s \theta$-closed set. Since by proposition 2.3, every fuzzy $g^s$-$s \theta$-closed set is fuzzy $g^m$-$s \theta$-closed. Therefore $A^c$ is fuzzy $g^m$-$s \theta$-closed set. Hence $A$ is fuzzy $g^s$-$s \theta$-open set in fuzzy topological space $X$.

Remark 3.17. The converse of the above proposition need not be true as shown in the following example.

Example 3.18. Let $X = \{a\}$ and the fuzzy sets $A$, $B$ and $D$ be defined as follows $A(a) = 0.6; B(b) = 0.5; D(a) = 0.9$. Let $\tau = \{0, A, B, 1\}$. Then $D$ is fuzzy $g^s$-$s \theta$-open but it is not not $f g^m$-open.

Theorem 3.19. Every fuzzy $g^m$-$\theta$-open set is fuzzy $g^s$-$s \theta$-open set in a fuzzy topological space $(X, \tau)$.

Proof. It is clear from Theorem 3.7 and Theorem 3.16.

Remark 3.20. The converse of the above theorem need not be true as shown in the following example.

Example 3.21. $X = \{a, b\}$. Consider the fuzzy topology $\tau$ as in Example 3.12 where $A$ is defined by $A(a) = 0.1; A(b) = 0.2$. Clearly $A$ is fuzzy $g^s$-$s \theta$-open set but not fuzzy $g^m$-$\theta$-open set.

Theorem 3.22. Every fuzzy $g^m$-$\theta$-open set is fuzzy $g^m$-$\alpha \theta$-open set in a fuzzy topological space $(X, \tau)$.

Proof. Let $A$ be a fuzzy $g^m$-$\theta$-open set in $X$. then $A^c$ is fuzzy $g^m$-$\theta$-closed set. Since by proposition 2.3, every fuzzy $g^m$-$\alpha \theta$-closed set is fuzzy $g^m$-$\alpha \theta$-closed. Therefore $A^c$ is fuzzy $g^m$-$\alpha \theta$-closed set. Hence $A$ is fuzzy $g^m$-$\alpha \theta$-open set in fuzzy topological space $X$.

Remark 3.23. The converse of the above proposition need not be true as shown in the following example.

Example 3.24. Let $X = \{a\}$ and the fuzzy sets $A$, $B$ and $D$ be defined as follows $A(a) = 0.5; B(a) = 0.3; D(a) = 0.4$. Let $\tau = \{0, A, B, 1\}$. Then $D$ is fuzzy $g^m$-$\alpha \theta$-open but it is not not $f g^m$-open.

4. Properties of fuzzy $g^m$-$\theta$-open sets

Theorem 4.1. Let $X$ be a fisure, then the intersection of two fuzzy $g^m$-$\theta$-open sets is fuzzy $g^m$-$\theta$-open set.

Proof. Suppose that $A$ and $B$ are fuzzy $g^m$-$\theta$-closed sets in $X$ and let $H$ be a fuzzy $g^m$-$\theta$-closed set in $X$ such that $H \subseteq \{A \cap B\}$. Then $H \subseteq A$ and $H \subseteq B$. Since $A$ and $B$ are fuzzy $g^m$-$\theta$-open sets, $H \subseteq \int_{\theta}(A)$ and $H \subseteq \int_{\theta}(B)$. Hence $H \subseteq \int_{\theta}(A) \cap \int_{\theta}(B)$. Since $\int_{\theta}(A) \cap \int_{\theta}(B) = \int_{\theta}(A \cap B)$. Therefore $H \subseteq \int_{\theta}(A \cap B)$ whenever $A \cap B \subseteq H$ and $H$ is fuzzy $g^m$-$\theta$-closed set in $X$.

Theorem 4.2. If $A$ is a fuzzy $g^m$-$\theta$-open set in $(X, \tau)$ and $\int_{\theta}(A) \subseteq B \subseteq A$, then $B$ is fuzzy $g^m$-$\theta$-open set in $(X, \tau)$.

Proof. Let $A$ be a fuzzy $g^m$-$\theta$-open set in $(X, \tau)$. Let $F \subseteq B$ where $F$ is a fuzzy $g^m$-$s \theta$-closed set in $X$. Then $F \subseteq A$, since $B \subseteq A$. Since $A$ is fuzzy $g^m$-$\theta$-open set, it follows that $F \subseteq \int_{\theta}(A)$.

Now $B \subseteq \int_{\theta}(A)$ implies $\int_{\theta}(B) \subseteq \int_{\theta}(\int_{\theta}(A)) = \int_{\theta}(A)$. We get, $\int_{\theta}(B) \supseteq F$. Hence, $B$ is fuzzy $g^m$-$\theta$-open set in $(X, \tau)$.

Theorem 4.3. If a fuzzy set $A$ of a fuzzy topological space $X$ is both fuzzy $g^m$-$s \theta$-closed and fuzzy $g^m$-$\theta$-open then it is fuzzy $\theta$-open.

Proof. Suppose that a fuzzy set $A$ of $X$ is both fuzzy $g^m$-$s \theta$-closed and fuzzy $g^m$-$\theta$-open then $A \subseteq \int_{\theta}(A)$. Hence $A$ is fuzzy $\theta$-open in $X$.

Definition 4.4. Let $A$ be a fuzzy set in a fuzzy topological space $X$. Then fuzzy $g^m$-$\theta$-closure and fuzzy $g^m$-$\theta$-interior denoted by $g^m Cl(A)$ and $g^m Int(A)$ and defined respectively as follows:

\[ g^m Cl(A) = \{ F : F \text{ is fuzzy } g^m \text{-closed and } A \subseteq F \} \]

\[ g^m Int(A) = \{ U : U \text{ is fuzzy } g^m \text{-opened and } U \subseteq A \}. \]
Remark 4.5. (a) If \( A \) is fuzzy \( \theta_g'' \)-closed set then \( \theta_g'' \text{Cl}(A) = A \). But the converse not necessarily true. Since the intersection of two fuzzy \( \theta_g'' \)-closed sets may not be a fuzzy \( \theta_g'' \)-closed.

(b) The following results are quite obvious:

\[
\theta_g'' \text{Cl}(A^c) = (\theta_g'' \text{Int}(A))^c
\]

and

\[
\theta_g'' \text{Int}(A^c) = (\theta_g'' \text{Cl}(A))^c.
\]

5. Fuzzy \( \theta_g'' \)-Continuous and Fuzzy \( \theta_g'' \)-Irresolute Functions

Definition 5.1. A function \( f : X \to Y \) is called

(a) fuzzy \( \theta_g'' \)-continuous if \( f^{-1}(F) \) is fuzzy \( \theta_g'' \)-closed in \( X \) for every fuzzy \( \theta_g'' \)-closed set \( F \) in \( Y \).

(b) fuzzy \( \theta_g'' \)-irresolute if \( f^{-1}(F) \) is fuzzy \( \theta_g'' \)-closed in \( X \) for every fuzzy \( \theta_g'' \)-closed set \( F \) in \( Y \).

(c) fuzzy \( g'' \)-\( \theta \)-continuous if \( f^{-1}(F) \) is fuzzy \( g'' \)-\( \theta \)-closed in \( X \) for every fuzzy \( g'' \)-\( \theta \)-closed set \( F \) in \( Y \).

(d) fuzzy \( \theta_g''s \)-continuous if \( f^{-1}(F) \) is fuzzy \( \theta_g''s \)-closed in \( X \) for every fuzzy \( \theta_g''s \)-closed set \( F \) in \( Y \).

Theorem 5.2. A function \( f : X \to Y \) is fuzzy \( \theta_g'' \)-continuous if and only if the inverse image of each fuzzy \( \theta_g'' \)-closed set in \( Y \) is fuzzy \( \theta_g'' \)-closed in \( X \).

Theorem 5.3. Every fuzzy \( \theta \)-continuous function is fuzzy \( \theta_g'' \)-continuous.

Proof. Let \( f : X \to Y \) be a fuzzy \( \theta \)-continuous function. Let \( F \) be fuzzy \( \theta_g'' \)-closed set in \( Y \). Since \( f \) is fuzzy \( \theta \)-continuous, \( f^{-1}(F) \) fuzzy \( \theta \)-closed in \( X \) and every fuzzy \( \theta \)-closed set is fuzzy \( \theta_g'' \)-closed therefore for a fuzzy \( \theta_g'' \)-closed set \( F \) in \( Y \), \( f^{-1}(F) \) is fuzzy \( \theta_g'' \)-closed set in \( X \). Hence \( f \) is fuzzy \( \theta_g'' \)-continuous.

Remark 4.6. The converse of the above theorem need not be true as shown in the following example.

Example 5.5. Let \( X = \{a, b\} = Y \). Fuzzy sets \( A, B, D \) and \( K \) are defined by \( A(a) = 0.6; B(a) = 0.5 \). Consider \( \tau = \{0, A, 1\} \) and \( \sigma = \{0, B, 1\} \). Then \( X, \tau \) and \( Y, \sigma \) are fts. Define \( f : (X, \tau) \to (Y, \sigma) \) be Identity function. Then \( f \) is fuzzy \( \theta_g'' \)-continuous map but not fuzzy \( \theta_g'' \)-continuous, since for fuzzy \( \theta_g'' \)-closed set \( B^* \) in \( Y \), \( f^{-1}(B^*) \) is not fuzzy \( \theta \)-closed but it is fuzzy \( \theta_g'' \)-closed set in \( X, \tau \).

Theorem 5.6. Every fuzzy \( \theta_g'' \)-continuous function is fuzzy \( \theta_g''s \)-continuous.

Proof. Let \( f : X \to Y \) be a fuzzy \( \theta_g'' \)-continuous function. Let \( F \) be fuzzy \( \theta_g'' \)-closed set in \( Y \). Since \( f \) is fuzzy \( \theta_g'' \)-continuous, \( f^{-1}(F) \) fuzzy \( \theta_g'' \)-closed in \( X \) and every fuzzy \( \theta_g'' \)-closed set is fuzzy \( \theta_g''s \)-closed therefore for a fuzzy \( \theta_g'' \)-closed set \( F \) in \( Y \), \( f^{-1}(F) \) is fuzzy \( \theta_g''s \)-closed set in \( X \). Hence \( f \) is fuzzy \( \theta_g''s \)-continuous.

Remark 5.7. The converse of the above theorem need not be true as shown in the following example.

Example 5.8. Let \( X = \{a, b\} = Y \). Fuzzy sets \( A, B, K \) are defined by \( A(a) = 0.5; B(a) = 0.4; B(b) = 0.8 \). Consider \( \tau = \{0, A, 1\} \) and \( \sigma = \{0, B, 1\} \). Then \( X, \tau \) and \( Y, \sigma \) are fts. Define \( f : (X, \tau) \to (Y, \sigma) \) be \( f(a) = b, f(b) = a \). Then \( f \) is fuzzy \( \theta_g''s \)-continuous map but not fuzzy \( \theta_g'' \)-continuous, since for fuzzy \( \theta_g'' \)-closed set \( B^* \) in \( Y \), \( f^{-1}(B^*) = K \) not fuzzy \( \theta_g'' \)-closed but it is fuzzy \( \theta_g''s \)-closed set in \( X, \tau \).

Theorem 5.9. Every fuzzy \( \theta_g'' \)-continuous function is fuzzy \( \theta_g'' \)-\( \theta \)-continuous.

Proof. Let \( f : X \to Y \) be a fuzzy \( \theta_g'' \)-continuous function. Let \( F \) be fuzzy \( \theta_g'' \)-closed set in \( Y \). Since \( f \) is fuzzy \( \theta_g'' \)-continuous, \( f^{-1}(F) \) fuzzy \( \theta_g'' \)-closed in \( X \) and every fuzzy \( \theta_g'' \)-closed set is fuzzy \( \theta_g'' \)-\( \theta \)-closed therefore for a fuzzy \( \theta_g'' \)-closed set \( F \) in \( Y \), \( f^{-1}(F) \) is fuzzy \( \theta_g'' \)-\( \theta \)-closed set in \( X \). Hence \( f \) is fuzzy \( \theta_g'' \)-\( \theta \)-continuous.

Remark 5.10. The converse of the above theorem need not be true as shown in the following example.

Example 5.11. Let \( X = \{a, b\} = Y \). Fuzzy sets \( A, B, D \) and \( K \) are defined by \( A(a) = 0.4; A(b) = 0.6 \). Consider \( \tau = \{0, A, 1\} \) and \( \sigma = \{0, D, 1\} \). Then \( X, \tau \) and \( Y, \sigma \) are fts. Define \( f : (X, \tau) \to (Y, \sigma) \) be \( f(a) = b, f(b) = a \). Then \( f \) is fuzzy \( \theta_g'' \)-continuous function but not fuzzy \( \theta_g'' \)-\( \theta \)-continuous, since for fuzzy \( \theta_g'' \)-closed set \( D^* \) in \( Y \), \( f^{-1}(D) \) is fuzzy \( \theta_g'' \)-\( \theta \)-closed set in \( X, \tau \).

Remark 5.12. The following examples shows that the fuzzy \( \theta_g'' \)-\( \theta \)-continuous function and fuzzy \( \theta_g'' \)-irresolute function are independent concepts.

Example 5.13. Let \( X = \{a, b\} = Y \). Fuzzy sets \( A, B, D \) and \( K \) are defined by \( A(a) = 0.7; A(b) = 0.6 \). Consider \( \tau = \{0, A, 1\} \) and \( \sigma = \{0, B, 1\} \). Then \( X, \tau \) and \( Y, \sigma \) are fts. Define \( f : (X, \tau) \to (Y, \sigma) \) be Identity function. Then \( f \) is fuzzy \( \theta_g'' \)-\( \theta \)-continuous function but not fuzzy \( \theta_g'' \)-irresolute, since for fuzzy \( \theta_g'' \)-\( \theta \)-closed set \( K \) in \( Y, \sigma \), \( f^{-1}(K) \) is not fuzzy \( \theta_g'' \)-closed in \( X, \tau \).

Example 5.14. Let \( X = \{a\} = Y \). Fuzzy sets \( A, B, D \) and \( K \) are defined by \( A(a) = 0.3; A(b) = 0.8 \). Consider \( \tau = \{0, A, 1\} \) and \( \sigma = \{0, D, 1\} \). Then \( X, \tau \) and \( Y, \sigma \) are fts. Define \( f : (X, \tau) \to (Y, \sigma) \) be Identity function. Then \( f \) is fuzzy \( \theta_g'' \)-irresolute function. Then \( f \) is fuzzy \( \theta_g'' \)-irresolute function but not fuzzy \( \theta_g'' \)-\( \theta \)-continuous, since for
fuzzy closed set $D'$ in $f^{-1}(D')$, is not fuzzy $\theta g''$-closed in $(X, \tau)$.

**Theorem 5.15.** Let $f : X \to Y$ and $g : Y \to Z$ be any two functions. Then

(a) $g \circ f : Y \to Z$ is fuzzy $\theta g''$-continuous if $f$ is fuzzy $g''\theta$-continuous and $g$ is $f$-continuous.

(b) $g \circ f : Y \to Z$ is fuzzy $\theta g''$-irresolute if $f$ and $g$ are fuzzy $g\theta$-irresolute functions.

(c) $g \circ f : Y \to Z$ is fuzzy $\theta g''$-irresolute if $f$ is fuzzy $g''\theta$-irresolute and $g$ is fuzzy $g\theta$-continuous functions.

**Proof.** (a) Let $F$ be fuzzy closed set in $Z$. Since $g : Y \to Z$ is fuzzy continuous, $g^{-1}(F)$ is a fuzzy closed in $Y$. Now, $f : X \to Y$ is fuzzy $g''\theta$-continuous, therefore $f^{-1}(g^{-1}(F))$ is a fuzzy $\theta g''$-closed in $X$. Since $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$, then $g \circ f$ is fuzzy $\theta g''$-continuous.

(b) Let $F$ be fuzzy $\theta g''$-closed set in $Z$. Since $g : Y \to Z$ is fuzzy $g''\theta$-continuous, $g^{-1}(F)$ is a fuzzy $\theta g''$-closed in $Y$. Now, $f : X \to Y$ is fuzzy $g\theta$-irresolute, therefore $f^{-1}(g^{-1}(F))$ is a fuzzy $\theta g''$-closed in $X$. Since $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$, then $g \circ f$ is fuzzy $\theta g''$-irresolute.

(c) Let $F$ be fuzzy closed set in $Z$. Since $g : Y \to Z$ is fuzzy $g''\theta$-continuous, $g^{-1}(F)$ is a fuzzy $g''\theta$-closed in $Y$. Now, $f : X \to Y$ is fuzzy $g\theta$-irresolute, therefore $f^{-1}(g^{-1}(F))$ is a fuzzy $\theta g''$-closed in $X$. Since $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$, then $g \circ f$ is fuzzy $\theta g''$-continuous.

**Definition 5.16.** A fuzzy topological spaces $X$ is called fuzzy $T_{\theta g''}$-space if every fuzzy $\theta g''$-closed set in $X$ is fuzzy closed set.

**Theorem 5.17.** Let $X$ be any fuzzy topological space and $Y$ be fuzzy $T_{\theta g''}$-space and $f : X \to Y$ be fuzzy $\theta g''$-irresolute function. Then $f$ is fuzzy $g''\theta$-continuous.

**Proof.** Let $F$ be fuzzy $\theta g''$-closed set in $Y$. Since $Y$ is fuzzy $T_{\theta g''}$-space, $F$ is fuzzy closed set in $Y$ then by hypothesis $f^{-1}(F)$ is fuzzy $\theta g''$-closed set in $X$. Hence $f$ is fuzzy $\theta g''$-continuous.

**Theorem 5.18.** Let $f : X \to Y$ be fuzzy $\theta g''$-continuous and $Y$ be fuzzy $T_{\theta g''}$-space, then $f$ is fuzzy $g''\theta$-irresolute.

**Proof.** Let $F$ be fuzzy $\theta g''$-closed set in $Y$. Then $F$ is fuzzy closed set in $Y$ as $Y$ is fuzzy $T_{\theta g''}$-space. Since $f$ is fuzzy $\theta g''$-continuous, we have $f^{-1}(F)$ is fuzzy $\theta g''$-closed set in $X$. Hence $f$ is fuzzy $\theta g''$-irresolute function.

**Theorem 5.19.** Let $f : X \to Y$ be fuzzy $\theta g''$-continuous and $X$ be fuzzy $T_{\theta g''}$-space, then $f$ is fuzzy continuous.

**Proof.** Let $F$ be fuzzy closed set in $Y$. Since $f$ is fuzzy $\theta g''$-continuous, we have $f^{-1}(F)$ is fuzzy $\theta g''$-closed set in $X$. Then $f^{-1}(F)$ is fuzzy closed set in $X$ as $X$ is fuzzy $T_{\theta g''}$-space. Hence $f$ is fuzzy continuous.

**Theorem 5.20.** If $f : X \to Y$ and $g : Y \to Z$ are fuzzy $\theta g''$-continuous and $Y$ be fuzzy $T_{\theta g''}$-space, then their composition $g \circ f : X \to Z$ is fuzzy $\theta g''$-continuous.

**Proof.** Let $F$ be fuzzy closed set in $Z$. Then $g^{-1}(F)$ is fuzzy $\theta g''$-closed set in $Y$. Since $Y$ fuzzy $T_{\theta g''}$-space, $g^{-1}(F)$ is fuzzy $\theta g''$-closed in $Y$ implies $f^{-1}(g^{-1}(F)) = (g \circ f)^{-1}(F)$ is fuzzy $\theta g''$-closed set in $X$. Hence $g \circ f$ is fuzzy $\theta g''$-continuous.

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ISSN(P):2319 – 3786
Malaya Journal of Matematik
ISSN(O):2321 – 5666
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