A note on mixed super quasi Einstein manifold

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Abstract
Mixed super quasi Einstein manifold (\(MS(QE)\)) is a generalization of Einstein manifold. In this paper we have studied some geometric properties of \(MS(QE)\). Also we have studied \(MS(QE)\) satisfying some curvature restriction and obtained the form of Riemannian curvature tensor. We have studied conformally flat and conformally conservative \(MS(QE)\). We have deduced a necessary condition for a \(MS(QE)\) to be conformally conservative. Some basic properties of \(MS(QE)\) on viscous fluid \(MS(QE)\) spacetimes are discussed. We have proved that if a viscous fluid \(MS(QE)\) spacetime admitting heat flux obeys Einstein equation with a cosmological constant then none of the energy density and isotropic pressure of the fluid can be a constant.

Keywords
Mixed super quasi-Einstein manifold, conformally flat, conformally conservative, viscous fluid, heat flux, cosmological constant, energy density, isotropic pressure.

AMS Subject Classification
Primary 53C50, 53C25, 53B30; Secondary 53C80, 53B50.

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1. Introduction

Let \(U_s = \{ x \in M : S \neq g, atx \} \), where \(S\) and \(r\) are respectively the Ricci tensor and scalar curvature of a Riemannian manifold \((M^n,g)\), \((n \geq 3)\). Then the manifold is said to be a quasi Einstein [4] manifold if on \(U_s\), we have

\[
S - ag = bA \otimes A,
\]

where \(A\) is a 1-form on \(U_s\) and \(a, b\) are some functions on \(U_s\). It is clear that the 1-form \(A\) as well as the function \(b\) are non zero at every point on \(U_s\). From the above definition it follows that every Einstein manifold is quasi- Einstein. The scalars \(a\) and \(b\) are known as the associated scalars of the manifold. Also the 1-form \(A\) is called the associated 1-form of the manifold defined by \(g(X,U) = A(X)\) for any vector field \(X; U\) being a unit vector field, called the generator of the manifold. Such an \(n\)-dimensional quasi Einstein manifold is denoted by \((QE)\).

There are many generalization of \((QE)\); in literature([1], [2], [3], [4], [5], [7]). One of them is mixed super quasi-Einstein manifold introduced by A. Bhattacharya, M. Tarafdar and D. Deb Nath [2]. According to them a non flat Riemannian manifold is said to be mixed super quasi-Einstein manifold if it satisfies the condition

\[
S(X,Y) = ag(X,Y) + bA(X)A(Y) + cb(X)B(Y) + d[A(X)B(Y) + A(Y)B(X)] + eD(X,Y),
\]

where \(a, b, c, d, e\) are real valued functions on \((M^n,g)\) of which \(b \neq 0, c \neq 0, d \neq 0, e \neq 0\) and \(A, B\) are two non zero 1-forms such that

\[
g(X,U) = A(X), g(X,V) = B(X), g(U,U) = 1, g(V,V) = 1, g(U,V) = 0, D\text{ is a symmetric tensor of type (0,2) with zero trace such that } D(X,U) = 0 \forall X \in \chi(M).\]

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Let us suppose that in a \( MS(QE)_n \) the generator \( U \) is parallel vector field. Then \( \nabla_X U = 0 \) \( \forall X \). So \( R(X,Y)U = 0 \) and \( S(X,U) = 0 \) \( \forall X \)

From (1.1), \( 0 = (a+b)A(X) + dB(X) \) \( \forall X \)

Putting \( X = X \) we obtain \( d = 0 \). Again putting \( X = U \) we obtain \( a + b = 0 \). Hence we have the following

**Theorem 3.1.** If the generator \( U \) of a \( MS(QE)_n \) is a parallel vector field then either \( d = 0 \) or \( a + b = 0 \).

**Theorem 3.2.** In a \( MS(QE)_n \) \( QU, V \) are orthogonal iff \( d = 0 \).

**Proof.** \( S(U,V) = d \) i.e., \( g(QU,V) = d \), which is 0 if and only if \( d = 0 \). Hence the theorem.

**Theorem 3.3.** In a \( MS(QE)_n \) \( QV, V \) are orthogonal iff \( a + c + eD(V,V) = 0 \).

**Proof.** \n
\[
S(V,V) = a + c + eD(V,V) \quad \text{i.e.,} \quad g(QV,V) = a + c + eD(V,V).
\]

So \( g(QV,V) = 0 \), iff \( a + c + eD(V,V) = 0 \).

Hence the theorem.

**Theorem 3.4.** An \( MS(QE)_n \) is a \( P(GQE)_n \) if either of the vector field is a parallel vector field.

**Proof.** If the vector field \( U \) is a parallel vector field, then we have \( \nabla_X U = 0 \) \( \forall X \).So \( R(X,Y)U = 0 \) and eventually \( S(X,U) = 0 \) \( \forall X \)

From (1.1), \( 0 = (a+b)A(X) + dB(X) \) \( \forall X \)

Putting \( X = V \) we obtain \( d = 0 \), i.e the manifold is \( P(GQE)_n \) [6].

Again if the vector field \( V \) is parallel then \( R(X,Y)V = 0 \), consequentially \( S(Y,V) = 0 \), i.e \( aB(Y) + cB(Y) + d[A(Y)] + eD(V,Y) = 0 \).

Putting \( Y = U \) we get \( d = 0 \), i.e the manifold is \( P(GQE)_n \) . Hence the theorem.
Theorem 3.5. In a $MS(QE)_n$ 0 is an eigen value of $L$ in the direction of the eigen vector $U$, i.e. $LU = 0$, where $L$ is the symmetric endomorphism of the tangent space at any point of the manifold corresponding to the structure tensor $D$.

Proof. We have $g(LX, Y) = D(X, Y) \forall X, Y \in \mathcal{X}(M)$. Putting $X = U$, we get, $g(LU, Y) = D(U, Y) = 0 \forall Y$. So $LU = 0$ i.e 0 is an eigen value of $L$ in the direction of $U$. $\Box$

We now consider a compact orientable $MS(QE)_n$ $(n > 2)$ without boundary. From (1.1) we get,

$$
S(X, X) = ag(X, X) + ba(A(X)A(X) + cB(X)B(X) + d[A(X)B(X) + A(X)B(X)] + eD(X, X),
$$

(3.1)

Let us assume that $\theta_\alpha$ be the angle between $U$ and any vector $X$, $\theta$ be the angle between $V$ and any vector field $X$ then

$$
\cos \theta_\alpha = \frac{g(X, U)}{g(X, X)^2}, \cos \theta = \frac{g(X, V)}{g(X, X)^2}
$$

Further we assume that $\theta_\alpha \geq \theta$, then we have $\cos \theta_\alpha \geq \cos \theta$, i.e., $g(X, U) \geq g(X, V)$. Therefore,

$$
S(X, X) \geq [a + b + c + 2d][g(X, U)]^2,
$$

when $a, b, c, d, e, D(X, X)$ are positive.

Definition 3.6. A vector field $H$ in a Riemannian manifold $(M^n, g)$ $(n > 2)$ is said to be harmonic [8] if $d\tau = 0$ and $\delta\tau = 0$ where $\tau(X) = g(X, H) \forall X$.

It is known from a compact orientable Riemannian manifold the following relations holds $\int_M [S(X, X) - \frac{1}{2}(d\tau)^2 + (\nabla X)^2 - (\delta\tau)^2]dv = 0$, for any vector field $X$ where $dv$ denotes the volume element of $M$. Now let $X \in \mathcal{X}(M)$ be harmonic vector field then $\int_M [S(X, X) + (\nabla X)^2]dv = 0$ for any $X$. Hence if each $a, b, c, d, e, D(X, X)$ is positive then $\int_M [(a + b + c + 2d)g(X, U)^2 + (\nabla X)^2]dv \geq 0$, by virtue of $a + b + c + 2d > 0$, $g(X, U) = 0$ and $\nabla X = 0$ for any vector field $X$. This follows that $X$ is orthogonal to $U$ and $X$ is a parallel vector field. Similarly if $\theta_\alpha \geq \theta_\alpha$, assuming as before it can be shown $g(X, V) = 0$ and $\nabla X = 0$ for any vector field $X$. Thus we have the following theorem

Theorem 3.7. In a compact orientable $MS(QE)_n$ $(n > 2)$ without boundary any harmonic vector field $X$ is parallel and orthogonal to one of the generators of the manifold which makes greatest angle with vector $X$ provided $a, b, c, d, e, D(X, X)$ are positive scalars.

Let us now investigate whether a $MS(QE)_n$ $(n > 2)$ is projectively flat or not.

Theorem 3.8. A $MS(QE)_n$ $(n > 2)$ cannot be projectively flat.
Theorem 4.2. A necessary condition for a $MS(QE)_n$ to be conformally conservative is 

$$ (d((n-2)a+(2n-3)b+c)(V) = 2(n-1)(dd)(U) $$

Proof. A Riemannian manifold is said to be conformally conservative if the divergence of its conformal curvature tensor is zero, i.e.,

$$ \frac{1}{2(n-1)} [dr(X)g(Y,Z)-(dr)(Z)g(X,Y)]. $$

Now putting $X = Y = U$ and $Z = V$, in above we get,

$$ (dV)(U) - d(a+b)(V) = \frac{1}{2(n-1)} [n(da)(V) + (db)(V) + (dc)(V)]. $$

On simplification,

$$ (dd)(U) - d(a+b)(V) = \frac{1}{2(n-1)} [n(da)(V) + (db)(V) + (dc)(V)], $$

or

$$ 2(n-1)(dd)(U) - 2(n-1)d(a+b)(V) = -[n(da)(V) + (db)(V) + (dc)(V)], $$

or

$$ 2(n-1)(dd)(U) = (d((n-2)a+(2n-3)b+c)(V). $$

Hence the theorem.

Theorem 5.1. In a $MS(QE)_n$, $n \geq 3$ the following results hold.

\[ R(V, U, U, V) = L_a, \]  
\[ D(R(U, U), V) = 0, \]  
\[ L_a = \frac{D(R(U, U), V)}{D(U, V)}, \]  

provided $D(U, V) \neq 0$.

Proof. We consider $MS(QE)_n$. Then we have

$$ S(R(X,Y)Z,W) + S(Z,R(X,Y)W) = L_a[g(Y,Z)S(X,W) - g(X,Z)S(Y,W) + g(Y,W)S(Z,X) - g(X,W)S(Y,Z)]. $$

or

$$ b[A(R(X,Y)Z)A(W) + A(Z)A(R(X,Y)W)] + c[B(R(X,Y)Z)B(W) + B(Z)B(R(X,Y)W)] + d[A(R(X,Y)Z)B(W) + A(W)B(R(X,Y)Z) + A(Z)B(R(X,Y)W)] + e[D(R(X,Y)Z,W) + D(Z,R(X,Y)W)] = L_a[b\{g(Y,Z)A(X)A(W) - g(X,Z)A(Y)A(W) + g(Y,W)A(A)A(X) - g(X,W)A(A)A(Z) + c\{g(Y,Z)B(X)B(W) - g(X,Z)B(Y)B(W) + g(Y,W)B(Z)B(X) - g(X,W)B(Z)B(Y)] + d\{g(Y,Z)[A(X)B(W) + A(W)B(X)] - g(X,Z)[A(Y)B(W) + A(W)B(Y)] + g(Y,W)[A(A)B(X) + A(X)B(Z)] - g(X,W)[A(A)B(Z) + A(Z)B(Y)]\} + e\{g(Y,Z)(D(X,W) - g(X,Z)D(Y,W) + g(Y,W)D(X,Z) - g(X,W)D(Y,Z)]]. $$

Putting $Z = U$ and $W = V$ in (5.7), we get

$$ b[R(X,Y,V,U)] = L_a[A(X)B(Y) - A(Y)B(X)] + c[R(X,Y,U,V)] - L_a[a(A)B(X) - A(X)B(Y)] + e[D(R(X,Y)U,V)] - L_a[A(Y)D(X,V) - A(X)D(Y,V)] = 0. $$

Putting $Z = W = U$ in (5.7) we get

$$ d[R(X,Y)U,V]) = L_a[A(Y)B(X) - A(X)B(Y)] = 0. $$

Since, $d \neq 0$ we get

$$ R(X,Y)U,V) - L_a[A(Y)B(X) - A(X)B(Y)] = 0. $$}

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Similarly, if we take \( Z = W = V \) in (5.7) we get,
\[
d[R(X, Y)V, V] - L_s[A(Y)B(X)] - e[D(R(X, Y)V, V) - L_s[B(Y)D(X, V) - B(X)D(Y, V)]] = 0.
\]
(5.11)

Using (5.9) we get
\[
e[D(R(X, Y)V, V) - L_s[B(Y)D(X, V) - B(X)D(Y, V)]] = 0.
\]
Since \( e \neq 0 \), we have
\[
D(R(X, Y)V, V) - L_s[B(Y)D(X, V) - B(X)D(Y, V)] = 0.
\]
(5.12)

Putting \( X = V, Y = U \) in (5.10) we get (5.3). Again putting \( X = V, Y = U \) in (5.12) we get (5.4). Using (5.12) in (5.11) we get
\[
D(R(X, Y)U, V) - L_s[A(Y)D(X, V) - A(X)D(Y, V)] = 0.
\]
(5.13)

Putting \( X = U, Y = V \) in above we get (5.5).


Let \((M^n, g)\) be a connected semi-Riemannian viscous fluid spacetime admitting heat flux and satisfying Einstein's equation with a cosmological constant \( \lambda \). Also let \( U \) be the unit timelike velocity vector field, \( V \) be the unit heat flux vector and \( D \) be the anisotropic pressure tensor of the fluid. The we have
\[
g(U, U) = -1, g(V, V) = 1, g(U, V) = 0
\]
(6.1)

\[
D(X, Y) = D(Y, X), Tr.D = 0, D(X, U) = 0 \forall X.
\]
(6.2)

Let
\[
g(X, U) = A(X), g(X, V) = B(X) \forall X.
\]
(6.3)

Also let \( T \) be the energy-momentum tensor of type (0,2) describing the matter distribution of such fluid and it be of the following form
\[
T(X, Y) = \rho g(X, Y) + (\sigma + p)A(X)A(Y) + B(X)B(Y) + A(X)B(Y) + A(Y)B(X) + D(X, Y),
\]
where \( \sigma, p \) are the energy density and isotropic pressure respectively. General relativity flows from Einstein equation given by
\[
S(X, Y) = -\frac{r}{2}g(X, Y) + \lambda g(X, Y) = kT(X, Y),
\]
(6.5)

for all vector fields \( X, Y \). \( S \) is the Ricci tensor of type (0,2) and \( r \) is the scalar curvature, \( \lambda \) is a cosmological constant. Thus by virtue of (6.4) above equation can be written as
\[
S(X, Y) = -\frac{r}{2}g(X, Y) + \lambda g(X, Y) = k|pg(X, Y) + (\sigma + p)A(X)A(Y) + B(X)B(Y) + \{A(X)B(Y) + A(Y)B(X)\} + D(X, Y)|.
\]
(6.6)

Putting this in (1.1) we get
\[
\frac{2kp - 2\lambda + 2a + b + c}{2}g(X, Y) = [b - k(\sigma + p)]A(X)A(Y) + (c - k)B(X)B(Y) + (d - k)[A(X)B(Y) + A(Y)B(X)] + (e - k)D(X, Y).
\]
(6.7)

Putting \( X = U, Y = V \) in above we get \( d = k \)

Putting \( X = U, Y = U \) we get
\[
\sigma = \frac{2a + 3b + c - 2\lambda}{2k},
\]
(6.8)
or,
\[
\sigma = \frac{2a + 3b + c - 2\lambda}{2d}.
\]
(6.9)

Again contracting (6.6) we get
\[
r - 2r + 4\lambda = k|3p - \sigma + 1|,
\]
(6.10)
or,
\[
p = \frac{6\lambda - 6a + b - c - 2d}{6d}.
\]
(6.11)

Hence we can state the following

**Theorem 6.1.** If a viscous fluid MS\((QE)_\lambda\) spacetime admitting heat flux obeys Einstein equation with cosmological constant then none of the energy density and isotropic pressure of the fluid can be a constant.

Now if the associated scalars \( a, b, c, d \) are constants with \( d > 0 \), then from (6.8) and (6.9) \( \sigma, p \) are constants. Since \( \sigma > 0, p > 0 \) we have from (6.8) and (6.9) we get \( \lambda < \frac{2a + 3b + c}{2} \) and \( \lambda > \frac{6a - b + c - 2d}{6} \). And hence
\[
\frac{6a - b + c - 2d}{6} < \lambda < \frac{2a + 3b + c}{2}.
\]
(6.12)

Thus we have the following

**Theorem 6.2.** If a viscous fluid MS\((QE)_\lambda\) spacetime admitting heat flux obeys Einstein equation with cosmological constant \( \lambda \), then \( \lambda \) obeys the above inequality.
References


