Generalized regular open sets

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Abstract
In this paper, we introduce new Generalization of Regular open (briefly, GR-open) sets. This new class shows stronger properties in general topological spaces that mean GR-open sets exists in between the class of regular open sets and the class of open sets. Also, we investigate GR-neighbourhood, GR-interior and GR-closure properties.

Keywords
Regular open sets, g-closed sets, GR-open sets, GR-neighbourhood, GR-interior, GR-closure.

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1. Introduction

Regular open sets have been introduced and investigated by Stone [9]. Levine [5, 6], Cameron [3], Velicko [10] and Sheik John [8] have discussed g-closed sets, semiopen sets, regular semiopen sets, δ-open sets and ω-open sets respectively. Maitra [7] introduced the concept of g-closure and g-interior.

We introduce a new class of sets called Generalization of Regular open sets in topological spaces. Which mainly exists between the class of regular open sets and the class of open sets. Also, we discuss the new class of sets called Generalization of Regular closed sets and studied GR-neighbourhood, GR-interior and GR-closure properties.

2. Preliminaries

Throughout this paper, (P, τ), (Q, σ) and (R, η) or simply P, Q and R always denote the topological spaces on which no separation axioms are assumed unless explicitly stated. Int(M), cl(M) denote the topological spaces on which no separation axioms are assumed unless explicitly stated. Int(M), cl(M) denote the interior of M, closure of M in P respectively. P − M or M c denotes the complement of M in P.

We recall the following definitions and results.

Definition 2.1. A subset M of a topological space P is called

1. Regular open [9], if M = int(cl(M)) and regular closed if cl(int(M)) = M.
2. Semi open [5], if M ⊆ int(cl(M)) and semi-closed[2] if int(cl(M)) ⊆ M.
3. Regular semi open set [3] if there is a regular open set U such that U ⊆ M ⊆ cl(U).
4. π-open [4], if A is a finite union of regular open sets. The complement of π-open set is called the π-closed set.

Definition 2.2. A subset M of a topological space P is called

1. Generalized closed (briefly g-closed)[7] if cl(M) ⊆ U whenever M ⊆ U and U is open in P.
2. Weakly closed (briefly ω-closed)[8] if cl(M) ⊆ U whenever M ⊆ U and U is semi-open in P.
3. $\delta$-closed[10] if $M = cl_\delta(M)$, where $cl_\delta(M) = \{x \in P : int(cl(U)) \cap M \neq \emptyset, U \in M\}$.

The complement of above all closed sets are their respective open sets in the same topological space $P$.

**Definition 2.3.** Let $P$ be any topological space and $M \subseteq P$, then the $g$-closure[7] of $M$ is the intersection of all $g$-closed sets in $P$ containing $M$. The $g$-closure of $M$ is denoted by $g-cl(M)$.

**Definition 2.4.** Let $P$ be a topological space and $M \subseteq P$, then the $g$-interior[7] of $M$ is the union of all $g$-open sets in $P$ contained in $M$. The $g$-interior of $M$ is denoted by $g-int(M)$.

**Lemma 2.5.** Let $P$ be any topological space and $M$ and $N$ are subsets of $P$. Then following properties holds

1. $g-cl(M \cap N) \subseteq g-cl(M) \cap g-cl(N)$.
2. $g-int(M) \cup g-int(N) \subseteq g-int(M \cup N)$.

**Definition 2.6.** A map $f : P \to Q$ is said to be completely continuous[1] if $f^{-1}(M)$ is regular closed set of $P$ for every closed set $M$ of $Q$.

### 3. GR-open sets and their properties

We introduce GR-open sets and investigate some of relationships between existed classes.

**Definition 3.1.** A subset $M$ of space $P$ is called Generalized Regular open (briefly, GR-open) set if $M = int(g-cl(M))$. We denote the class of sets as $GRO(P)$.

Firstly we have to prove the existence of new class GR-open sets in topological spaces.

**Theorem 3.2.** Every regular open set is GR-open set.

**Proof.** Let $M$ be a regular open set in $P$. To prove that $M$ is GR-open in $P$. We know that $M \subseteq g-cl(M) \subseteq cl(M)$ that is $int(M) \subseteq int(g-cl(M)) \subseteq int(cl(M))$. As $M$ is regular open, $M = int(cl(M))$ and $int(M) = M$. Hence $M \subseteq int(g-cl(M)) \subseteq int(cl(M)) = M$. Thus $int(g-cl(M)) = M$. Therefore $M$ is GR-open in $P$.

The converse of above theorem need not be true.

**Example 3.3.** Let $P = \{1, 2, 3, 4\}$ with the topology on it $\tau = \{P, \emptyset, \{1\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$. Then sets $\{2\}, \{1, 2\}$ are GR-open sets but not regular open sets in $P$.

**Theorem 3.4.** Every GR-open set is open set.

**Proof.** Let $M$ be a GR-open set in $P$. That is $M = int(g-cl(M))$. As interior of any subset of $P$ is an open set, therefore $M$ is an open in $P$.

The converse of above theorem need not be true.

**Example 3.5.** Let $P = \{1, 2, 3, 4\}$ with the topology on it $\tau = \{P, \emptyset, \{1\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$. Then the set $\{1, 2, 3\}$ is open set but not GR-open in $P$.

**Remark 3.6.** From Theorem 3.4, we know that every GR-open set is a open set but not conversely. Also from Levine[6] we know that every open set is semiopen but not conversely. Hence every GR-open set is a semiopen set but not conversely.

**Remark 3.7.** From Theorem 3.4, we know that every GR-open set is a open set but not conversely. Also from Sheik John[8] we know that every open set is $\omega$-open set but not conversely. Hence every GR-open set is a $\omega$-open set but not conversely.

**Remark 3.8.** From Theorem 3.4, we know that every GR-open set is a open set but not conversely. Also from Levine[43] we know that every open set is $g$-open but not conversely. Hence every GR-open set is a $g$-open set but not conversely.

**Remark 3.9.** The following example shows that GR-open sets are independent of $\pi$-open sets, $\delta$-open sets and regular semiopen sets.

**Example 3.10.** Let $P = \{1, 2, 3, 4\}$ with topology on it $\tau = \{P, \emptyset, \{1\}, \{1, 2\}, \{1, 4\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$. Then

1. closed sets in $P$ are $P, \emptyset, \{5\}, \{4, 5\}, \{1, 4, 5\}, \{2, 3, 5\}, \{2, 3, 4, 5\}$.
2. GR-open sets in $P$ are $P, \emptyset, \{1\}, \{1, 2\}, \{1, 4\}, \{2, 3\}, \{1, 2, 3\}$.
3. $\pi$-open sets in $P$ are $P, \emptyset, \{1\}, \{2, 3\}, \{1, 2, 3\}$.
4. $\delta$-open sets in $P$ are $P, \emptyset, \{1\}, \{2, 3\}, \{1, 2, 3, 4\}$.
5. regular semiopen sets in $P$ are $P, \emptyset, \{1\}, \{2, 3\}, \{1, 4\}, \{1, 4, 5\}, \{2, 3, 5\}$. 

**Remark 3.11.** From the above discussion and known result we have the following implications.

In the following diagram, by $X \rightarrow Y$ means $X$ implies $Y$ but not conversely and $X \Leftrightarrow Y$ means $X$ and $Y$ are independent each other.

![Diagram showing the relationship between different types of open sets](image-url)

**Theorem 3.12.** Intersection of two GR-open sets is a GR-open set in topological spaces.
Proof. Let $M$ and $N$ be two GR-open sets in space $P$. To prove that $M \cap N$ is GR-open set in space $P$, that is to prove that $M \cap N = \text{int}(gcl(M \cap N))$. As $M$ and $N$ are GR-open sets in $P$, $M = \text{int}(gcl(M))$, $N = \text{int}(gcl(N))$. We know that $M \cap N \subseteq M$, $gcl(M \cap N) \subseteq gcl(M)$ also $M \cap N \subseteq N$, $gcl(M \cap N) \subseteq gcl(N)$. Which implies $\text{int}(gcl(M \cap N)) \subseteq \text{int}(gcl(M))$ and $\text{int}(gcl(M \cap N)) \subseteq \text{int}(gcl(N))$. This implies $\text{int}(gcl(M \cap N)) \cap \text{int}(gcl(M)) \cap \text{int}(gcl(N)) = M \cap N$. (i) $M \cap N = \text{int}(M \cap N) \subseteq \text{int}(gcl(M \cap N))$. (ii) $M \cap N = \text{int}(M \cap N)$ because of if $M$ and $N$ are open sets, then every GR-open set is open in $P$. $\text{int}(M \cap N) \subseteq \text{int}(gcl(M \cap N))$. (iii)

From (i) and (ii), $M \cap N = \text{int}(gcl(M \cap N))$. Hence $M \cap N$ is GR-open set in $P$.

Remark 3.13. The union of two GR-open sets is generally not a GR-open set in topological spaces.

Example 3.14. Let $P = \{1, 2, 3, 4\}$ with topology on it $\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$. If $M = \{1, 2\}$ and $N = \{2, 3\}$ are GR-open sets in $P$ but $M \cap N = \{1, 2, 3\}$ is not GR-open set in $P$.

Theorem 3.15. If $M$ is a GR-open then $\text{int}(M) = M$.

Proof. Let $M$ be a GR-open set. To prove $\text{int}(M) = M$. We know that every GR-open set is open, that is $M$ is open set then $\text{int}(M) = M$. The converse of above theorem need not be true.

Example 3.16. Let $P = \{1, 2, 3, 4\}$ with topology on it $\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$, then $\text{GRO}(P) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$. Then the set $M = \{1, 2, 3\}$. Note that $\text{int}(M) = \{1, 2, 3\}$ is not a GR-open set, but it is open set of $P$.

Theorem 3.17. If $M$ is g-closed and open in $P$, then $M$ is GR-open in $P$.

Proof. Let $M$ be a g-closed and open in $P$. To prove that $M$ is GR-open i.e. to prove $M = \text{int}(gcl(M))$. Now $gcl(M) = M$, because $M$ is g-closed set. As $\text{int}(gcl(M)) = \text{int}(M)$ this implies $\text{int}(gcl(M)) = M$, because $M$ is open set. Then $M$ is GR-open in $P$.

Remark 3.18. Complement of a GR-open set need not be a GR-open set.

Example 3.19. Let $P = \{1, 2, 3, 4\}$ with topology on it $\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$. Note that $\{1, 2\}$ is a GR-open set. But $P - \{1, 2\} = \{3\}$ is not a GR-open set in $P$.

4. GR-closed sets and their properties

We introduce GR-closed sets and investigate some of their properties.

Definition 4.1. A subset $M$ of space $P$ is called Generalized Regular closed (briefly, GR-closed) set if $P - M$ is GR-closed in $P$. Then its family is denoted as $\text{GRC}(P)$.

This new class of sets properly lies between the class of regular closed sets and the class of closed sets.

Theorem 4.2. A subset $M$ of $P$ is GR-closed if and only if $M = \text{cl}(gcl(M))$.

Proof. (i) Suppose $M$ is GR-closed. To prove $M = \text{cl}(gcl(M))$. As $M$ is GR-closed, $P - M$ is GR-open in $P$, which implies $P - M = \text{int}(gcl(P - M))$, $P - M = \text{int}(P - gcl(M))$. 

The converse of above theorem need not be true.

Example 4.4. From Example 3.3, the set $\{3, 4\}$ are GR-closed sets but not regular closed in $P$.

Theorem 4.5. Every GR-closed set is closed in $P$.

Proof. Let $M$ be a regular closed set in $P$. Then $M^c$ is a regular open set. By Theorem 3.2, $M^c$ is GR-open set. Therefore $M$ is a GR-closed set in $P$.

Remark 4.7. From Theorem 4.6, we have, every GR-closed set is a closed set but not conversely. Also from Biswas[2], every closed set is semiclosed set but not conversely. Hence every GR-closed set is a semiclosed set but not conversely.

Remark 4.8. From Theorem 4.6, we have, every GR-closed set is a closed set but not conversely. Also from Sheik John[8], every closed set is $\omega$-closed but not conversely. Hence every GR-closed set is a $\omega$-closed set but not conversely.

Remark 4.9. From Theorem 4.6, we know that every GR-closed set is a closed set but not conversely. Also from Levine[6], every closed set is g-closed but not conversely. Hence every GR-closed set is a g-closed set but not conversely.

Remark 4.10. The following example shows that GR-closed sets are independent of $\pi$-closed sets, $\delta$-closed sets and regular semiopen (= regular semiclosed) sets.

Example 4.11. Let $P = \{1, 2, 3, 4, 5\}$ with topology on it $\tau = \{\emptyset, \{1\}, \{1, 4\}, \{2, 3\}, \{1, 2, 3\}, \{1, 2, 3, 4\}\}$. Then
1. closed sets in $P$ are $P, \emptyset, \{5\}, \{4, 5\}, \{1, 4, 5\}, \{2, 3, 5\}, \{2, 3, 4, 5\}$.

2. GR-closed sets in $P$ are $P, \emptyset, \{4, 5\}, \{1, 4, 5\}, \{2, 3, 5\}, \{2, 3, 4, 5\}$.

3. $\pi$-closed sets in $P$ are $P, \emptyset, \{5\}, \{1, 4, 5\}, \{2, 3, 5\}$.

4. $\delta$-closed sets in $P$ are $P, \emptyset, \{5\}, \{1, 4, 5\}, \{2, 3, 5\}$.

5. regular semiopen sets in $P$ are $P, \emptyset, \{1, 4\}, \{2, 3\}, \{1, 4, 5\}, \{2, 3, 5\}$.

Theorem 4.12. Union of two GR-closed sets is a GR-closed set in topological spaces.

Proof. Let $M$ and $N$ be two GR-closed sets in $P$. To prove that $M \cup N = cl(g\text{-int}(M \cup N))$. Assume $M$ and $N$ are GR-closed sets in $P$, $M = cl(g\text{-int}(M))$, $N = cl(g\text{-int}(N))$. We know that $M \subseteq M \cup N$, $g\text{-int}(M) \subseteq g\text{-int}(M \cup N)$ also $N \subseteq M \cup N$, $g\text{-int}(N) \subseteq g\text{-int}(M \cup N)$. Which implies $cl(g\text{-int}(M)) \subseteq cl(g\text{-int}(M \cup N))$ and $cl(g\text{-int}(N)) \subseteq cl(g\text{-int}(M \cup N))$. This implies $cl(g\text{-int}(M)) \cup cl(g\text{-int}(N)) \subseteq cl(g\text{-int}(M \cup N)) \cup cl(g\text{-int}(M \cup N))$. That is $cl(g\text{-int}(M)) \cup cl(g\text{-int}(N)) \subseteq cl(g\text{-int}(M \cup N))$.

Example 4.13. From Example 3.3, then sets $M = \{1, 4\}$ and $N = \{3, 4\}$ are GR-closed sets in $P$ but $M \cap N = \{4\}$ is not GR-closed set in $P$.

Theorem 4.14. If $M$ is a GR-closed if and only if $cl(M) = M$.

Proof. If $M$ is GR-closed. To prove $cl(M) = M$. We know that every GR-closed set is closed set i.e. $M$ is closed then $cl(M) = M$.

Example 4.15. Let $P = \{1, 2, 3, 4\}$ with topology on it $\tau = \{P, \emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}\}$. Then GRC($P$) = $\{P, \emptyset, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$. Then the set $M = \{4\}$. Note that $cl(M) = \{4\}$ is not a GR-closed set, but it is a closed set of $P$.

Theorem 4.16. If $M$ is g-open and closed in $P$, then $M$ is GR-closed set in $P$.

Proof. Let $M$ is g-open and closed set in $P$. To prove that $M$ is GR-closed set i.e. to prove $M = cl(g\text{-int}(M))$. Now $g\text{-int}(M) = M$, because $M$ is g-open set. As $cl(g\text{-int}(M)) = cl(M)$ this implies $cl(g\text{-int}(M)) = M$, because $M$ is closed set. Then $M$ is GR-closed set in $P$. 

5. GR-neighbourhoods and GR-interior

Definition 5.1. (i) Let $P$ be a topological space and $x \in P$, a subset $N$ of $P$ is said to be a GR-neighbourhood (briefly, GR-nhd) of $x$ if and only if there exists a GR-open set $G$ such that $x \in G \subseteq N$.

(ii) The collection of all GR-neighbourhood of $x \in P$ is GR-neighbourhood system at $x$ and is denoted by GR-N(x).

Analogous to interior in a space $P$, we define GR-interior in a space $P$ as follows.

Definition 5.2. Let $M$ be a subset of $P$. A point $x \in M$ is said to be GR-interior point of $M$ if and only if $P$ is a GR-neighbourhood of $x$. The set of all GR-interior points of $M$ is called the GR-interior of $M$ and is denoted as GR-int($M$).

Theorem 5.3. If $M$ is a subset of $P$, then GR-int($M$) = $\bigcup\{G : G$ is GR-open set, $G \subseteq M\}$.

Proof. Let $M$ be a subset of $P, x \in GR\text{-int}(A)$ implies that $x$ is a GR-interior point of $P$ i.e. $M$ is a GR-nhd of point $x$. Then there exists a GR-open set $G$ such that $x \in G \subseteq A$ implies that $x \in \bigcup\{G : G$ is GR-open set, $G \subseteq M\}$. Hence $GR\text{-int}(M) = \bigcup\{G : G$ is GR-open set, $G \subseteq M\}$.

Theorem 5.4. Let $P$ be a topological space and $M \subseteq P$, then show that $M$ is GR-open set if and only if GR-int($M$) = $M$.

Proof. Let $M$ be a GR-open set in $P$. Then clearly the largest GR-open set contained in $M$, is itself $M$. Hence GR-int($M$) = $M$.

Conversely, suppose that $M \subseteq P$ and GR-int($M$) = $M$. Since GR-int($M$) is a GR-open set in $P$, it follows that $M$ is a GR-open set in $P$. 

Theorem 5.5. Let $M$ and $N$ are subset of $P$ Then

1. $GR\text{-int}(P) = P$ and $GR\text{-int}(\emptyset) = \emptyset$.

2. $GR\text{-int}(M) \subseteq M$.

3. If $N$ is any GR-open set contained in $M$, then $N \subseteq GR\text{-int}(M)$.

4. If $M \subseteq N$, then $GR\text{-int}(M) \subseteq GR\text{-int}(N)$.

5. $GR\text{-int}(GR\text{-int}(M)) = GR\text{-int}(M)$.

Proof. (i) Since $P$ and $\emptyset$ are GR-open sets, by Theorem 5.3, $GR\text{-int}(P) = \bigcup\{G : G$ is GR-open set, $G \subseteq P\} = P \cup \{all \ GR-open \ sets\} = P$. That is $GR\text{-int}(P) = P$. Since $\emptyset$ is the only GR-open set contained in $\emptyset$, $GR\text{-int}(\emptyset) = \emptyset$.

(ii) Let $x \in GR\text{-int}(A)$ implies that $x$ is a GR-interior point of $M$. That is $M$ is a GR-nhd of $x$ i.e. $x \in M$. Thus $x \in GR\text{-int}(A)$ implies $x \in A$. Hence $GR\text{-int}(M) \subseteq M$.

(iii) Let $N$ be any GR-open set such that $N \subseteq M$. Let $x \in N$. Since $N$ is a GR-open set contained in $M$, $x$ is a GR-interior point of $M$. That is $x \in GR\text{-int}(M)$. Hence $N \subseteq GR\text{-int}(M)$.

(iv) Let $M$ and $N$ be subsets of $P$ such that $M \subseteq N$. Let $x \in
Let $M$ and $N$ are subset of $P$. Then $GR-$int$(M \cup N) \subseteq GR-$int$(M \cap N)$.
\begin{proof}
We now that $M \subseteq M \cup N$ and $N \subseteq M \cup N$. We have, by Theorem 5.5(iv), $GR-$int$(A) \subseteq GR-$int$(M \cup N)$ and $GR-$int$(N) \subseteq GR-$int$(M \cup N)$. This implies $GR-$int$(M \cup N) \subseteq GR-$int$(M \cap N)$.
\end{proof}

Theorem 5.7. Let $M$ and $N$ are subsets of $P$, then $GR-$int$(M) \cap GR-$int$(N) = GR-$int$(M \cap N)$.
\begin{proof}
We now that $M \cap N \subseteq M$ and $M \cap N \subseteq N$. We have, by Theorem 5.5(iv), $GR-$int$(M \cap N) \subseteq GR-$int$(M)$ and $GR-$int$(M \cap N) \subseteq GR-$int$(N)$. This implies $GR-$int$(M \cap N) \subseteq GR-$int$(M \cap N)$.
\end{proof}

Again, let $x \in GR-$int$(M) \cap GR-$int$(N)$. Then $x \in GR-$int$(M)$ and $x \in GR-$int$(N)$. Hence $x$ is an interior point of each of sets $M$ and $N$. It follows that $M$ and $N$ are GR-nhd of $x$, so that their intersection $M \cap N$ is also a GR-nhd of $x$. Hence $x \in GR-$int$(M \cap N)$. Thus $x \in GR-$int$(M) \cap GR-$int$(N)$ implies that $x \in GR-$int$(M \cap N)$. Therefore $GR-$int$(M) \cap GR-$int$(N) \subseteq GR-$int$(M \cap N)$.

From (i) and (ii), we get $GR-$int$(M) \cap GR-$int$(N) = GR-$int$(M \cap N)$.

6. GR-closure and their properties

Using the Gr-closed sets we can introduce the concept of GR-closure operator in topological spaces.

Definition 6.1. Let $M$ be a subset of a space $P$. We define the GR-closure of $M$ to be the intersection of all GR-closed sets containing $M$. Mathematically, $GR-$cl$(M) = \cap \{ F : M \subseteq F \in GR(P) \}$.

Theorem 6.2. Let $P$ be any topological space and $M \subseteq P$, then show that $M$ is GR-closed set if and only if $GR-$cl$(M) = M$.
\begin{proof}
Let $M$ be a GR-closed set in $P$. Then clearly the smallest GR-closed set contained in $M$, is itself $M$. Hence $GR-$cl$(M) = M$.
Conversely, suppose that $M \subseteq P$ and $GR-$cl$(M) = M$. Since $GR-$cl$(M)$ is a GR-open set in $P$, it follows that $M$ is a GR-closed set in $P$.
\end{proof}

Theorem 6.3. Let $M$ and $N$ are subset of $P$. Then
\begin{enumerate}
\item $GR-$cl$(P) = P$ and $GR-$cl$(\emptyset) = \emptyset$.
\item $M \subseteq GR-$cl$(M)$.
\item If $N$ is any GR-closed set contained in $M$, then $GR-$cl$(M) \subseteq N$.
\item If $M \subseteq N$, then $GR-$cl$(M) \subseteq GR-$cl$(N)$.
\item $GR-$cl$(GR-$cl$(A)) = GR-$cl$(M)$.
\end{enumerate}
\begin{proof}
(i) Obviously.
(ii) By the definition of GR-closure of $M$, it is obvious that $M \subseteq GR-$cl$(M)$.
(iii) Let $N$ be any GR-closed set containing $M$. Since $GR-$cl$(M)$ is the intersection of all GR-closed sets containing $M$, i.e. $GR-$cl$(M)$ is contained in every GR-closed set containing $M$. Hence $GR-$cl$(M) \subseteq N$.
(iv) Let $M$ and $N$ are subsets of $P$ such that $M \subseteq N$. By the definition of GR-closure, $GR-$cl$(N) = \cap \{ F : N \subseteq F \in GR(P) \}$. If $N \subseteq F \in GR(P)$, then $GR-$cl$(N) \subseteq F$. Since $M \subseteq N$, $M \subseteq N \subseteq F \in GR(P)$, we have $GR-$cl$(M) \subseteq F$. Therefore $GR-$cl$(M) \subseteq \cap \{ F : N \subseteq F \in GR(P) \} = GR-$cl$(P)$.
(v) Since $GR-$cl$(M)$ is a GR-closed set in $P$. It follows that $GR-$cl$(GR-$cl$(P)) = P$.
\end{proof}

Theorem 6.4. Let $M$ and $N$ are subsets of $P$, then $GR-$cl$(M \cup N) = GR-$cl$(M) \cup GR-$cl$(N)$.
\begin{proof}
Let $M$ and $N$ are subsets of $P$. Clearly $M \subseteq M \cup N$ and $N \subseteq M \cup N$. We have, by the Theorem 6.3(iv), $GR-$cl$(M \cup N) \subseteq GR-$cl$(M \cup N)$ and $GR-$cl$(N) \subseteq GR-$cl$(M \cup N)$. This implies $GR-$cl$(M \cup N) \subseteq GR-$cl$(M) \cup GR-$cl$(N)$.

Now to prove that $GR-$cl$(M \cup N) \subseteq GR-$cl$(M) \cup GR-$cl$(N)$. Let $x \in GR-$cl$(M \cup N)$ and $x \notin GR-$cl$(M) \cup GR-$cl$(N)$. Then there exists GR-closed sets $M_1$ and $N_1$ with $M \subseteq M_1$, $N \subseteq N_1$, and $x \notin M_1 \cup N_1$. We have $M \cup N \subseteq M_1 \cup N_1$ and $M_1 \cup N_1$ is a GR-closed set by Theorem 6.3, such that $x \notin M_1 \cup N_1$. Thus $x \notin GR-$cl$(M \cup N)$ which is contradiction to $x \in GR-$cl$(M \cup N)$. Hence $GR-$cl$(M \cup N) \subseteq GR-$cl$(M) \cup GR-$cl$(N)$.

From (i) and (ii), we have $GR-$cl$(M \cup N) = GR-$cl$(M) \cup GR-$cl$(N)$.
\end{proof}

Theorem 6.5. Let $M$ and $N$ are subsets of $P$, then $GR-$cl$(M \cap N) \subseteq GR-$cl$(M) \cap GR-$cl$(N)$.
\begin{proof}
Let $M$ and $N$ are subsets of $P$. Clearly $M \cap N \subseteq M$ and $M \cap N \subseteq N$. We have, by Theorem 6.3(iv), $GR-$cl$(M \cap N) \subseteq GR-$cl$(M)$ and $GR-$cl$(M \cap N) \subseteq GR-$cl$(N)$. This implies $GR-$cl$(M \cap N) \subseteq GR-$cl$(M) \cap GR-$cl$(N)$.
\end{proof}

Remark 6.6. In general $GR-$cl$(M) \cap GR-$cl$(N) \not\subseteq GR-$cl$(M \cap N)$, as seen from the following example.

Example 6.7. Consider $P = \{1, 2, 3, 4\}$, topology on it $\tau = \{\emptyset, \{1\}, \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$, $M = \{2, 3\}$, and $N = \{3, 4\}$. Then $GR-$cl$(M) = \{2, 3, 4\}$, $GR-$cl$(N) = \{3, 4\}$, $GR-$cl$(M \cap N) = \{3\}$ and $GR-$cl$(M) \cap GR-$cl$(N) = \{3, 4\}$.

Therefore $GR-$cl$(M) \cap GR-$cl$(N) \not\subseteq GR-$cl$(M \cap N)$.

Theorem 6.8. Let $M$ be a subset of $P$ and $x \in P$. Then $x \in GR-$cl$(M)$ if and only if $F \cap M \neq \emptyset$ for every GR-open set $F$ containing $x$. 

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7. GR-continuous maps and their properties

We introduce a new class of maps called Generalized Regular (brieﬂy, GR-continuous) maps and discuss their characterizations.

Definition 7.1. A map \( f : P \to Q \) is said to be Generalized Regular continuous (brieﬂy, GR-continuous) if the inverse image of every closed set in \( Q \) is GR-closed set in \( P \).

Theorem 7.2. A map \( f : P \to Q \) is GR-continuous if and only if the inverse image of an open set in \( Q \) is GR-open set in \( P \).

Proof. Let \( f : P \to Q \) be GR-continuous and \( M \) be an open set in \( Q \). Then \( M' \) is closed set in \( P \). Since \( f \) is GR-continuous, \( f^{-1}(M') = (f^{-1}(M))' \) and so \( f^{-1}(M) \) is GR-open set in \( P \).

Conversely, assume that \( f^{-1}(M) \) is GR-open set in \( P \) for each open set \( M \) in \( Q \). Let \( F \) be a closed set in \( Q \) and \( f^{-1}(F') \) is GR-open set in \( P \). Since \( f^{-1}(F') = (f^{-1}(F))' \), that is \( f^{-1}(F) \) is GR-closed in \( P \). Therefore \( f \) is GR-continuous.

Theorem 7.3. If a map \( f : P \to Q \) is completely continuous, then it is GR-continuous.

Proof. Let \( f : P \to Q \) be a completely continuous map. To prove that \( f \) is GR-continuous. Let \( F \) be any closed set in \( Q \). Since \( f \) is completely continuous, \( f^{-1}(F) \) is regular closed set in \( P \). By Theorem, every regular closed set is GR-closed set in \( P \). Therefore \( f \) is GR-continuous.

The converse of the above theorem need not be true.

Example 7.4. Let \( P = Q = \{1, 2, 3, 4\} \) be with the topologies \( \tau = \{P, \emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}\} \) and also the \( \sigma = \{Q, \emptyset, \{1\}, \{2\}, \{1, 2, 3\}\} \). Let \( f : P \to Q \) be defined by \( f(1) = 2, f(2) = 3, f(3) = 3 \) and \( f(4) = 4 \). Then \( f \) is GR-continuous but not completely continuous, as inverse image of GR-closed set \( \{2, 3, 4\} \) in \( Q \) is \( \{3, 4\} \) which is not regular closed set in \( P \).

Theorem 7.5. If a map \( f : P \to Q \) is GR-continuous, then \( f \) is continuous.

Proof. Let \( f : P \to Q \) be a GR-continuous map. To prove that \( f \) is continuous map. Let \( f \) be any closed set in \( Q \). Since \( f \) is GR-continuous, \( f^{-1}(F) \) is GR-closed set in \( P \). By Theorem, every GR-closed set is closed in \( P \). Therefore \( f \) is GR-continuous.

The converse of the above theorem need not be true.

Example 7.6. Let \( P = Q = \{1, 2, 3, 4\} \) be with topologies \( \tau = \{P, \emptyset, \{1\}, \{2\}, \{1, 2\}, \{1, 2, 3\}\} \). Let \( f : P \to Q \) be defined by \( f(1) = 4, f(2) = 3, f(3) = 2 \) and \( f(4) = 4 \). Then \( f \) is continuous but not GR-continuous, as inverse image of closed set \( \{3, 4\} \) in \( Q \) is \( \{2, 4\} \) it is not GR-closed set in \( P \).

References
