Solution of fractional integro-differential equations by Bernstein polynomials

J.A. Nanware¹*, Parameshwari M. Goud²* and T.L. Holambe³

Abstract
This paper focuses on the study of Bernstein polynomial to approximate the solution of Fractional-integro differential equations (FIDE) with Caputo derivative. This method reduces Fractional-integro differential equations into system of linear equations. Illustrations are given to show the accuracy of the method and results are simulated.

Keywords
Fractional integro-differential equations, Bernstein polynomials, Caputo derivative.

AMS Subject Classification
26A33, 30E25, 34A12, 34A34, 34A37, 37C25, 45J05.

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1. Introduction
The fractional calculus history begins from 30 September 1695 with the derivative of order $\alpha = \frac{1}{2}$. In recent years fractional calculus has attracted many researchers successfully in different disciplines of science and engineering. Compared with integer order differential equation, fractional differential equation has the advantage that it can better describe some natural physical process and dynamic system processes because the fractional order differential operators are non-local operators. The subject of fractional calculus has applications in diverse and wide spread fields of engineering and science such as electro-magnetics, viscoelasticity, fluid mechanics, electro-chemistry and signal processing. In these fields various analytical and numerical methods including their applications to new problems have been proposed in recent years. Fractional integro differential equations plays an significant role in modeling of numerous of physical phenomenon from science and Engineering.


$$D_+^\alpha u(x) = f(x) + \int_a^b k(x,t)u(t)dt, \quad u(a) = u_a, 0 < \alpha \leq 1$$

(1.1)

where $D_+^\alpha u(x)$ indicates the $\alpha^{th}$ Caputo fractional derivative.
of \( u(x) \), \( f(x), k(x, t) \) are given functions, \( x \) and \( t \) are real variables in \([0, 1]\). All the computations done using Scilab software and the results are shown in tabular form.

### 2. Preliminaries

#### 2.1 Basic definitions of fractional calculus:

In this section we present some basic definitions and properties of fractional calculus are given.

**Definition 1** [11] The Abel–Riemann(A-R) fractional integral operator of order \( \alpha > 0 \) of a function \( f(t) \) with \( t \in \mathbb{R}^+ \) is defined as:

\[
J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad t > 0, \alpha > 0 \tag{2.1}
\]

**Definition 2** [11] Caputo fractional derivative of \( f(t) \) is defined as,

\[
D^\alpha f(t) = \begin{cases} 
\frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha+1-m}} d\tau, & m-1 < \alpha < m \\
\frac{d^m}{dt^m} f(t), & \alpha = m
\end{cases} \tag{2.2}
\]

We have the following properties:

(i) \( l^\alpha b^\beta = l^\alpha b + \beta \), for all \( \alpha, \beta \geq 0 \)

(ii) \( l^\alpha t^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma)} t^{\alpha+\gamma} \), \( \alpha \geq 0 \), \( \gamma > -1 \), \( t > 0 \)

(iii) \( D^\alpha r^\gamma = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-\alpha+1)} r^{\gamma-\alpha} \), \( \alpha \geq 0 \), \( \gamma > -1 \), \( t > 0 \)

(iv) \( D^\alpha c = 0 \), \( c \) is constant

(v) \( l^\alpha D^\alpha f(t) = f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(0^+)}{\Gamma(k+1)} t^k \), \( x > 0 \), \( m-1 < \alpha \leq m \)

(vi) \( D^\alpha [\lambda f(t) + \mu g(t)] = \lambda D^\alpha f(t) + \mu D^\alpha g(t) \) where \( \lambda \) and \( \mu \) are constants

#### 2.2 Bernstein Polynomials

[8] The general form of the Bernstein polynomials of \( n^{th} \) degree over the interval is defined by

\[
B_{i,n}(x) = \binom{n}{i} \left( \frac{(x-a)^i (b-x)^{n-i}}{(b-a)^n} \right), a \leq x \leq b, i = 0, 1, 2, \ldots n
\]

where \( \binom{n}{i} = \frac{n!}{i!(n-i)!} \).

The Bernstein polynomials of degree 1, 2 and 3 are:

\[
B_{0,1}(t) = (1-t), \quad B_{1,1}(t) = t \\
B_{0,2}(t) = (1-t)^2, \quad B_{1,2}(t) = 2t(1-t), \quad B_{2,2}(t) = t^2 \\
B_{0,3}(t) = (1-t)^3, \quad B_{1,3}(t) = 3t(1-t)^2, \quad B_{2,3}(t) = 3t^2(1-t), \quad B_{3,3}(t) = t^3
\]

### 3. General method of solution

To obtain the approximate solution of the FIDE by using Bernstein polynomials as follows:

Our method begins by taking the fractional integration to both sides of the equation(1.1) we get:

\[
u(x) = u(0) + I^\alpha f(x) + I^\alpha \left( \int_0^b k(x,t) u(t) dt \right) \tag{3.1}
\]

To determine the approximate solution of (1.1), we use the Bernstein polynomial basis on \([a,b]\) as

\[
u(x) = \sum_{i=0}^{n} a_i B_{i,n}(x) \tag{3.2}
\]

where \( a_i (i = 0, 1, \ldots, n) \) are unknown constants to be determined.

Substituting equation(3.2) into equation(3.1), we obtain:

\[
\sum_{i=0}^{n} a_i B_{i,n}(x) = u(0) + I^\alpha f(x) + I^\alpha \left( \int_0^b k(x,t) \sum_{i=0}^{n} a_i B_{i,n}(t) dt \right)
\]

Hence

\[
\sum_{i=0}^{n} a_i [B_{i,n}(x) - I^\alpha \psi(x)] = u(0) + I^\alpha f(x)
\]

where \( \psi(x) = I^\alpha \left( \sum_{i=0}^{n} a_i B_{i,n}(x) \right) \)

Substituting the values of \( B_{i,n}(x) \) and simplifying the integration.

\[
\sum_{i=0}^{n} a_i [B_{i,n}(x) - I^\alpha \psi(x)] = u(0) + I^\alpha f(x)
\]

Using the Caputo integration and simplifying, now we put \( x = x_m, m = 0, 1, \ldots, n \) into equation(3.3), \( x_m^\alpha \) are being chosen as suitable distinct points in \([a,b]\), putting \( x = x_m \) we obtain the linear system:

\[
\sum_{i=0}^{n} a_i \alpha_i j = \beta_j, j = 0, 1, \ldots, n \tag{3.4}
\]

where \( \alpha_i j = B_{i,n}(x_j) - I^\alpha \psi(x_j) \) and \( \beta_j = u(0) + I^\alpha f(x_j) \).

Solve the linear system of equations by standard methods for the unknown constants \( a_i^\alpha \). Substituting \( a_i (i = 0, 1, 2, \ldots, n) \) in equation(3.2) to obtain the approximate solution of \( u(x) \).

### 4. Illustrations

**Example 1**

Consider the fractional Integro-Differential Equation:

\[
D^\alpha y(x) = \cos x + e^{2x} + \int_0^1 xe^y(t) dt, \quad y(0) = 0, 0 < \alpha \leq 1 \tag{4.1}
\]

By taking the fractional integration for both sides of the above equation, we get:

\[
y(x) = y(0) + I^\alpha (\cos x + e^{2x}) + I^\alpha \left( \int_0^1 xe^y(t) \right) dt \tag{4.2}
\]
To determine the approximate solution of eqn.(4.1), we set
\[ y(x) = \sum_{i=0}^{3} a_i B_{i,1}(x) \]
and after substituting into eqn.(4.2), we get:
\[ \sum_{i=0}^{3} a_i B_{i,1}(x) = I^\alpha (\cos x + e^{2x}) + I^\alpha \left( \int_0^1 x e^x \sum_{i=0}^{3} a_i B_{i,1}(t) dt \right) \]
So,
\[ a_0 (1-x)^3 + a_1 3x (1-x)^2 + a_2 3x^2 (1-x) + a_3 x^3 - \]
\[ I^\alpha \left[ a_0 x \int_0^1 (1-t)^3 e^t dt + a_1 x \int_0^1 (t+1)^3 - 2t^2) e^t dt \right] \]
\[ + 3a_2 x \int_0^1 (t^2 - t^3) e^t dt + a_3 x \int_0^1 t e^t dt \]
\[ = I^\alpha (\cos x + e^{2x}) \]
and in order to avoid the difficulty of evaluating the fractional integration of \( \cos x \) and \( e^{2x} \), we shall use its Maclaurin series
upto five terms we get:
\[ \Rightarrow a_0 (1-x)^3 + a_1 3x (1-x)^2 + a_2 3x^2 (1-x) + a_3 x^3 - \]
\[ I^\alpha \left[ a_0 x \int_0^1 (1-t)^3 e^t dt + a_1 x \int_0^1 (t+1)^3 - 2t^2) e^t dt \right] \]
\[ + 3a_2 x \int_0^1 (t^2 - t^3) e^t dt + a_3 x \int_0^1 t e^t dt \]
\[ = I^\alpha \left[ 2 + 2x + \frac{3}{2} x^2 + \frac{4}{3} x^3 + \frac{17}{24} x^4 + \frac{4}{15} x^5 + \ldots \right] \]
\[ \Rightarrow a_0 (1-x)^3 + a_1 3x (1-x)^2 + a_2 3x^2 (1-x) + a_3 x^3 - \]
\[ I^\alpha a_0 x \left\{ (1-t)^3 e^t - 3(1-t)^2 (t) e^t \right\} \]
\[ + 6(1-t) (e^t - 6e^t) + 3a_1 x \left\{ (t+1)^3 - 2t^2) e^t \right\} \]
\[ - (1 + 3t^2 - 4t) e^t + (6t - 4) e^t - (6e^t) \right\} \]
\[ + 3a_2 x \left\{ (t^2 - t^3) e^t - (2t - 3t^2) e^t \right\} \]
\[ + (2 - 6t) e^t + 6e^t \right\} \]
\[ = I^\alpha \left\{ 2 + 2x + \frac{3}{2} x^2 + \frac{4}{3} x^3 + \frac{17}{24} x^4 + \frac{4}{15} x^5 + \ldots \right\} \]
\[ \Rightarrow a_0 (1-x)^3 + a_1 3x (1-x)^2 + a_2 3x^2 (1-x) + a_3 x^3 - \]
\[ a_0 (6e-16) \Gamma(2) \Gamma(2+\alpha) x^{1+\alpha} - a_0 (11-4e) \Gamma(2) \Gamma(2+\alpha) x^{1+\alpha} \]
\[ - a_3 (6-2e) \Gamma(2) \Gamma(2+\alpha) x^{1+\alpha} - 2 \Gamma(\alpha+3) \]
\[ + \frac{x^\alpha}{\Gamma(\alpha+2)} + \frac{x^{2+\alpha}}{\Gamma(\alpha+3)} + \frac{x^{3+\alpha}}{\Gamma(\alpha+4)} \]
\[ + \frac{x^{4+\alpha}}{\Gamma(\alpha+5)} + \frac{x^{5+\alpha}}{\Gamma(\alpha+6)} \]
\[ \Rightarrow a_0 \left[ (1-x)^3 - (6e-16) \frac{x^{1+\alpha}}{\Gamma(2+\alpha)} \right] + \]
\[ a_1 \left[ 3x (1-x)^2 - 3(11-4e) \frac{x^{1+\alpha}}{\Gamma(2+\alpha)} \right] + \]
\[ a_2 \left[ 3x^2 (1-x) - 3(6-2e) \frac{x^{1+\alpha}}{\Gamma(2+\alpha)} \right] + \]
\[ a_3 \left[ x^3 - (6-2e) \frac{x^{1+\alpha}}{\Gamma(2+\alpha)} \right] \]
\[ + \frac{2x^\alpha}{\Gamma(\alpha+1)} + \frac{x^{2+\alpha}}{\Gamma(\alpha+2)} \]
\[ + \frac{x^{3+\alpha}}{\Gamma(\alpha+3)} + \frac{x^{4+\alpha}}{\Gamma(\alpha+4)} \]
\[ + \frac{x^{5+\alpha}}{\Gamma(\alpha+5)} + \frac{x^{6+\alpha}}{\Gamma(\alpha+6)} \]

\[ (4.3) \]

Case(1): When \( \alpha = 1 \) into equation(4.3)
\[ a_0 [(1-x)^3 - (6e-16) \frac{x^2}{2}] + a_1 [3x (1-x)^2 - 3(11-4e) \frac{x^2}{2}] + \]
\[ a_2 [3x^2 (1-x) - 3(6-2e) \frac{x^2}{2}] + \]
\[ 2x^2 + \frac{x^3}{2} + \frac{x^4}{3} + \frac{17x^5}{24} + \frac{4x^6}{15} + \ldots \]

and after substituting \( x = 0.1, 0.2, 0.3 \) and 0.4 respectively, we get a linear system of equations
\[ a_0 (0.7274515) + a_1 (0.2410969) + a_2 (0.0246773) \]
\[ + a_3 (-0.0018172) = 0.2055348 \]
\[ a_0 (0.5058062) + a_1 (0.3763876) + a_2 (0.0867093) \]
\[ + a_3 (-0.0032687) = 0.4245815 \]
\[ a_0 (0.3290639) + a_1 (0.4238727) + a_2 (0.1680956) \]
\[ + a_3 (0.0016454) = 0.6615767 \]
\[ a_0 (0.1912247) + a_1 (0.4015506) + a_2 (0.2508371) \]
\[ + a_3 (0.0189251) = 0.922166 \]

Solving the above system of equations we get :
\[ a_0 = -0.0012055, a_1 = 0.6736902, a_2 = 2.1856947, \]
\[ a_3 = 5.475391 \]
Thus, the approximate solution of equation(4.1) when \( \alpha = 1 \) becomes;
\[ y(x) = -0.0012055(1-x)^3 + 0.6736902(3x)(1-x)^2 + 2.1856947(3x^2)(1-x) + 5.475391(x^3) \]

Following table(1) and figure(1) represent the approximate solution of example(1) for different values of \( \alpha = 1, 0.5 \) and 0.25

Case(2): When \( \alpha = 0.5 \) into equation(4.3),
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<table>
<thead>
<tr>
<th>x</th>
<th>Approximate solution α = 1</th>
<th>Approximate solution α = 0.5</th>
<th>Approximate solution α = 0.25</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.2273471</td>
<td>1.0945496</td>
<td>2.8216957</td>
</tr>
<tr>
<td>0.2</td>
<td>0.5117096</td>
<td>2.967401</td>
<td>5.2556709</td>
</tr>
<tr>
<td>0.3</td>
<td>0.8576157</td>
<td>3.955534</td>
<td>7.9471628</td>
</tr>
<tr>
<td>0.4</td>
<td>1.2706789</td>
<td>4.5750038</td>
<td>10.867029</td>
</tr>
<tr>
<td>0.5</td>
<td>1.7565425</td>
<td>6.0592217</td>
<td>13.986128</td>
</tr>
<tr>
<td>0.6</td>
<td>2.3208502</td>
<td>7.7122602</td>
<td>17.275317</td>
</tr>
<tr>
<td>0.7</td>
<td>2.9692454</td>
<td>9.538194</td>
<td>20.705454</td>
</tr>
<tr>
<td>0.8</td>
<td>3.7073716</td>
<td>11.541088</td>
<td>24.247397</td>
</tr>
<tr>
<td>0.9</td>
<td>4.5408723</td>
<td>13.725021</td>
<td>27.872003</td>
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</table>

<table>
<thead>
<tr>
<th>Table 1</th>
</tr>
</thead>
<tbody>
<tr>
<td><img src="image" alt="Figure 1" /></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Example(3)</th>
<th>When α = 0.25 into eqn.(4.3),</th>
</tr>
</thead>
<tbody>
<tr>
<td>a0((1−x)³−(6x−16))x¹.²⁵Γ(2.5)</td>
<td>+</td>
</tr>
<tr>
<td>a1[(3)(1−x²)−3(11−4x)]x¹.²⁵Γ(2.5)</td>
<td>+</td>
</tr>
<tr>
<td>a2[3x²(1−x)−3(3−8e)]x¹.²⁵Γ(2.5)</td>
<td>+</td>
</tr>
<tr>
<td>a3[(x³−(6−2e)]x¹.²⁵Γ(2.5)</td>
<td>= 2x³Γ(1.5) + 3x³Γ(3.5) + 8x³Γ(4.5) + 17x³Γ(5.5) + 32x³Γ(6.5) (4.4)</td>
</tr>
</tbody>
</table>

and after substituting x = 0.1, 0.2, 0.3 and 0.4 into eqn.(4.4), respectively, we will get a linear system, that has the following solution,
a0(0.7136292) + a1(0.2241089) + a2(0.0039437) + a3(-0.0269649) = 1.3472748
a0(0.4754418) + a1(0.339069) + a2(0.0411627) + a3(-0.0585122) = 1.748886
a0(0.283125) + a1(0.3664136) + a2(0.0979688) + a3(-0.0834118) = 2.1258189
a0(0.1290494) + a1(0.3251355) + a2(0.1575741) + a3(-0.0941935) = 2.5264427
Solving these systems of equations we get,
a0 = 0.6743795, a1 = 7.3219552, a2 = 19.232883 and a3 = 31.550131
Thus the approximate solution of eqn.(4.1), when α = 0.25 becomes;
y(x) = 0.6743795(1−x)³ + 7.3219552(3x)(1−x)² + 19.232883(3x²)(1−x) + 31.550131(x³)

<table>
<thead>
<tr>
<th>Example(2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Consider the fractional integro differential Equation:</td>
</tr>
<tr>
<td>Dαy(x) = ³x² − 2x² + x + ∫₀¹ xty(t)dt, 0 ≤ x ≤ 1, y(0) = 0</td>
</tr>
<tr>
<td>Using the above mentioned method the table(2) represents the approximate solution of (4.6) when α = 1, 0.5 and 0.25</td>
</tr>
<tr>
<td>Figure(2) represent the approximate solution of example(2) for different values of α = 1, 0.5 and 0.25</td>
</tr>
</tbody>
</table>

Example(3)
Consider the fractional integro differential Equation:

\[ D^\alpha y(x) = 1 - e^x + \int_0^1 xty(t)\,dt, \quad y(0) = 0 \quad (4.7) \]

Using the above mentioned method the table(3) and figure(3) represents the approximate solution of (4.7) when \( \alpha = 1, 0.5 \) and 0.25

### Table 2

<table>
<thead>
<tr>
<th>x</th>
<th>Approximate solution ( \alpha = 1 )</th>
<th>Approximate solution ( \alpha = 0.5 )</th>
<th>Approximate solution ( \alpha = 0.25 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-0.0227339</td>
<td>-0.0899997</td>
<td>-0.173433</td>
</tr>
<tr>
<td>0.2</td>
<td>-0.0599129</td>
<td>-0.1599995</td>
<td>-0.2559222</td>
</tr>
<tr>
<td>0.3</td>
<td>-0.1015801</td>
<td>-0.2099991</td>
<td>-0.2979619</td>
</tr>
<tr>
<td>0.4</td>
<td>-0.1429874</td>
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</tr>
<tr>
<td>0.5</td>
<td>-0.1793867</td>
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</tr>
<tr>
<td>0.6</td>
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<td>0.9</td>
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<td>-0.2145961</td>
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</tbody>
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### Table 3

<table>
<thead>
<tr>
<th>x</th>
<th>Approximate solution ( \alpha = 1 )</th>
<th>Approximate solution ( \alpha = 0.5 )</th>
<th>Approximate solution ( \alpha = 0.25 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>-0.0060008</td>
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<td>0.2</td>
<td>-0.0247223</td>
<td>-0.0920966</td>
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<tr>
<td>0.3</td>
<td>-0.0573279</td>
<td>-0.1749062</td>
<td>-0.3913919</td>
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<td>0.4</td>
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<td>0.9</td>
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</tr>
</tbody>
</table>

### 5. Conclusion

In this study a simple and direct method Bernstein polynomial is used to solve Fractional-integro differential equations, for \( \alpha = 1, 0.5, 0.25 \).

### Acknowledgment

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### References


