New exact solutions for general Boussinesq equation

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Abstract
In this paper, four different families of exact solutions for Boussinesq equation are derived. Boussinesq equation is a fourth order nonlinear partial differential equation and this equation represents the dynamics of several physical phenomena. Since this equation is nonlinear, finding exact solutions are very difficult. The required solutions are derived after assuming ansatz forms for the solutions. These exact solutions can be used to derive exact solutions for different variants of the Boussinesq equation.

Keywords
Boussinesq equation, Exact solutions, Periodic solutions, Hyperbolic solutions.

AMS Subject Classification
35Q51, 35Q53, 37K40.

1 Introduction
Different physical phenomena appearing in nature are usually represented by nonlinear partial differential equations. But, obtaining exact solutions for such equations are highly complicated. In most of the cases only a few exact solutions are available. There are different methods to find out exact solutions for nonlinear equations. Backlund transformation, Darboux transformation and inverse scattering method are some of the available methods[1–3, 9, 14, 17, 20]. Travelling wave solutions are widely used to derive exact solutions for most of the nonlinear partial differential equations.

Boussinesq equation is a fourth order non-linear partial differential equation. Applications of this equation appear in the fields such as the dynamics of shallow water waves in coastal and ocean regions, the dynamics of thin inviscid layers and non-linear lattice waves[5, 6, 11, 12, 19]. Two different types of Boussinesq equation widely discussed in the literature are given by

\[ f_{tt} − f_{xx} − (ff_x)_x + f_{xxxx} = 0 \] (1.1)

and

\[ f_{tt} − f_{xx} − (ff_x)_x − f_{xxxx} = 0 \] (1.2)

where \( f \) is a function of the variables \( x \) and \( t \). There are other variants of Boussinesq equation that are discussed by some authors in the literature. The Boussinesq equation that is considered in this paper is given by

\[ f_{tt} + c_1 f_{xx} + c_2 (ff_x)_x + c_3 f_{xxxx} = 0 \] (1.3)

where \( c_1, c_2 \) and \( c_3 \) are arbitrary real parameters, which generalizes both the above equations.

The different ansatz methods such as tanh-method, sech-method, Exp-function method \((G'/G)\)-Expansion method and Jacobi elliptic function method are used to derive exact solutions for differential equations[8, 10, 13, 15, 16, 21]. Using such methods different solutions for Boussinesq equations are obtained in [4, 7, 18, 19].

The general Boussinesq equation (1.3) is solved to obtain new exact solutions using some ansatz methods, since solving this equation using direct methods are very difficult. The required computations are done with the help of computer algebra system. From these solutions new exact solutions for the equations (1.1) and (1.2) can also be derived.
2. The ansatz method

The solutions of the general Boussinesq equation (1.3) are derived using the traveling wave ansatz of the form

\[ f(t, x) = g(at + bx) \]  \hspace{1cm} (2.1)

where \( a \) and \( c \) are arbitrary real parameters. Then the equation (1.3) becomes the ordinary differential equation

\[ b^4 c_3 g^{(4)}(u) + g''(u) \left( a^2 + b^2 c_2 g(u) + b^2 c_1 \right) + b^2 c_2 g'(u)^2 = 0 \]  \hspace{1cm} (2.2)

where \( u = ax + ct \). To derive the required solutions several forms of the ansatz function \( g(u) \) will be considered. Consider the ansatz form

\[ g = A_0 + \sum_{j=-N,j\neq 0, 1}^N F_j (C_j g_1 + D_j g_2 + E_j) \]  \hspace{1cm} (2.3)

where \( g_1 \) and \( g_2 \) are functions of the variable \( u = at + bx \), and \( A_0, C_j, D_j, E_j \)’s are parameters to be determined. In this paper the functions \( g_1 \) and \( g_2 \) are taken to be trigonometric functions and hyperbolic functions. These ansatz are substituted in the corresponding equation and simplified and derive the relations among the parameters to obtain the exact solutions. The computations are done using any of the computational algebra system. These solutions will lead to new exact solutions for different variants of Boussinesq equation.

3. Trigonometric solutions

To obtain the first exact solutions we assume that only the parameters \( A_0, C_{-1}, D_{-1}, E_{-1} \) and \( F_{-1} \) are non zero in equation (2.3). Then taking \( g_1 \) and \( g_2 \) as sine and cosine functions the ansatz form becomes

\[ g(u) = A_0 + \frac{F_{-1}}{(C_{-1} \sin u + D_{-1} \cos u + E_{-1})} \]  \hspace{1cm} (3.1)

Now, we substitute this in the equation (1.3) and simplify. Then this ansatz function will be a solution if the following algebraic equations are satisfied.

\[ -a^2 - b^2 (A_0 c_2 + b^2 (-c_3) + c_1) = 0, \]
\[ 24b^4 c_3 \left( C_{-1} + D_{-1}^2 - E_{-1}^2 \right)^2 = 0, \]
\[ 3b^2 \left( C_{-1}^2 + D_{-1}^2 - E_{-1}^2 \right) \left( 20E_{-1} b^2 c_3 + c_2 F_{-1} \right) = 0, \]
\[ 3E_{-1} a^2 + b^2 \left( 3E_{-1} (A_0 c_2 - 5b^2 c_3 + c_1) - 2c_2 F_{-1} \right) = 0 \]
\[ 2a^2 \left( C_{-1}^2 + D_{-1}^2 - E_{-1}^2 \right) + b^2 \left( 2C_{-1} (A_0 c_2 - 10b^2 c_3 + c_1) + 2D_{-1} (A_0 c_2 - 10b^2 c_3 + c_1) + E_{-1} \right) = 0 \]

Solving these equations simultaneously we get

\[ A_0 = -\frac{a^2 + b^4 c_3 - b^2 c_1}{b^2 c_2} \]  \hspace{1cm} (3.2)
\[ D_{-1} = \pm \sqrt{E_{-1}^2 - C_{-1}^2} \]

and

\[ F_{-1} = -\frac{6E_{-1} b^2 c_3}{c_2} \]  \hspace{1cm} (3.3)

Substituting these values in the ansatz function given by (3.1) we get the following family of exact solutions for the general Boussinesq equation

\[ f(t, x) = -\frac{a^2 + b^4 c_3 - b^2 c_1}{b^2 c_2} - \frac{6E_{-1} b^2 c_3}{c_2} \left( D_{-1} + C_{-1} \sin u \pm \sqrt{E_{-1}^2 - C_{-1}^2} \cos u \right) \]  \hspace{1cm} (3.4)

where \( u = at + bx \).

To obtain another family of exact solutions, we consider the following ansatz form given by equation (2.3) where \( A_0, C_{-1}, D_{-1}, E_{-1} \) and \( F_{-1} \) are the only non zero parameters. Explicitly this ansatz form is given by

\[ g(u) = A_0 + \frac{F_{-2}}{(C_{-2} \sin(at + bx) + D_{-2} \cos(at + bx))^2 + E_{-2}} \]  \hspace{1cm} (3.5)

Substitute this in the general Boussinesq equation and simplify it. Then this ansatz is a solution to the equation (1.3) if the following algebraic equations are satisfied.

\[ a^2 + b^2 (A_0 c_2 - 4b^2 c_3 + c_1) = 0, \]
\[ 1536E_{-2} b^4 c_3 \left( C_{-2}^2 + D_{-2}^2 + E_{-2} \right)^2 = 0, \]
\[ 24E_{-2} b^2 \left( (C_{-2} + D_{-2} + E_{-2}) (40b^2 c_3 (C_{-2}^2 + D_{-2}^2 + E_{-2}) + 2E_{-2}) + 4c_2 F_{-2} \right) = 0, \]
\[ b^2 \left( 3 (C_{-2}^2 + D_{-2}^2 + E_{-2}) (A_0 c_2 - 20b^2 c_3 + c_1) - 4c_2 F_{-2} \right) + 3a^2 \left( C_{-2}^2 + D_{-2}^2 + E_{-2} \right) = 0, \]
\[ b^2 \left( 2c_2 \left( 5F_{-2} \left( C_{-2}^2 + D_{-2}^2 + E_{-2} \right) - 4E_{-2} A_0 \right) (C_{-2}^2 + D_{-2}^2 + E_{-2}) \right) + 20b^2 c_3 \left( 3 (C_{-2}^2 + D_{-2}^2 + E_{-2})^2 + 20E_{-2} \left( C_{-2}^2 + D_{-2}^2 + E_{-2} \right) - 4E_{-2} c_1 \right) (C_{-2}^2 + D_{-2}^2 + E_{-2}) = 0 \]

Solving this system of equations we get the following relationship among the parameters

\[ A_0 = -\frac{a^2 + b^4 c_3 - b^2 c_1}{b^2 c_2}, \]
\[ E_{-2} = \left( C_{-2}^2 + D_{-2}^2 \right), \]
\[ F_{-2} = \frac{12b^2 c_3 (C_{-2}^2 + D_{-2}^2)}{c_2} \]  \hspace{1cm} (3.7)

So the new family of exact solution for the general Boussinesq equation is obtained by substituting these values in
we get the following family of exact solutions for the general Boussinesq equation, which gives

\[ f(t, x) = \frac{-a^2 + 4b^4c_3 - b^2c_1}{b^2c_2} + \frac{12b^2c_3 (C_{-2} + D_{-2})}{c_2 (- (C_{-2} + D_{-2}) + (C_{-2} \sin u + D_{-2} \cos u)^2)} \]  

(3.8)

where \( u = at + bx \)

### 4. Hyperbolic solutions

In the previous section we have obtained families of exact solutions for the general Boussinesq equation by ansatz method involving sine and cosine functions. Now we derive exact solutions for the non-linear partial differential equation (1.3) using hyperbolic functions. To obtain such exact solutions we assume that only the parameters \( A_0, C_{-1}, D_{-1}, E_{-1} \) and \( F_{-1} \) are non zero in equation (2.3). Then taking \( g_1 = \sinh u \) and \( g_2 = \cosh u \) the ansatz form becomes

\[ g(u) = A_0 + \frac{F_{-1}}{(C_{-1} \sinh u + D_{-1} \cosh u + E_{-1})} \]  

(4.1)

Now we substitute this in the equation (1.3) and simplify. Then this ansatz function will be a solution if the following algebraic equations are satisfied.

\[ 24b^4c_3 (C_{-1}^2 - D_{-1}^2 + E_{-1}^2) = 0, \]
\[ 3b^2 (-C_{-1}^2 + D_{-1}^2 - E_{-1}^2) (20E_{-1}b^2c_3 - c_2F_{-1}) = 0, \]
\[ 3E_{-1}a^2 + b^2 (3E_{-1} (A_0c_2 + 5b^2c_3 + c_1) - 2c_2F_{-1}) = 0, \]
\[ a^2 + b^2 (A_0c_2 + 2b^2c_3 + c_1) = 0 \]

and

\[ 2a^2 (-C_{-1}^2 + D_{-1}^2 - E_{-1}^2) + b^2 (-2C_{-1}^2 (A_0c_2 + 10b^2c_3 + c_1) + 2D_{-1}^2 (A_0c_2 + 10b^2c_3 + c_1) - E_{-1} (2E_{-1} (A_0c_2 + 25b^2c_3 + c_1) - 5c_2F_{-1}) = 0 \]  

(4.2)

Solving these equations simultaneously we get

\[ A_0 = \frac{-a^2 + b^4 (-c_3) - b^2c_1}{b^2c_2}, \]  

(4.3)

\[ D_{-1} = \pm \sqrt{c_1^2 + e_{-1}^2} \]

and

\[ F_{-1} = \frac{6e_{-1}b^2c_3}{c_2} \]  

(4.4)

Substituting these values in the ansatz function given by (4.1) we get the following family of exact solutions for the general Boussinesq equation.

\[ f(t, x) = -\frac{a^2 + b^4c_3 + b^2c_1}{b^2c_2} - \frac{6E_{-1}b^2c_3}{c_2 \left( E_{-1} - \sqrt{D_{-1}^2 - E_{-1}^2} \sinh u + D_{-1} \cosh u \right)} \]  

(4.5)

where \( u = at + bx \)

To obtain another family of exact solutions, we consider the following ansatz form given by equation (2.3) where \( A_0, C_{-2}, D_{-2}, E_{-2} \) and \( F_{-2} \) are the only non zero parameters. Explicitly this ansatz form is given by

\[ g(u) = A_0 + \frac{F_{-2}}{(C_{-2} \sinh(at + bx) + D_{-2} \cosh(at + bx))^2 + E_{-2}} \]  

(4.6)

Substitute this in the general Boussinesq equation and simplify it. Then this ansatz is a solution to the equation (1.3) if the following algebraic equations are satisfied.

\[ 5b^2 (-C_{-2}^2 + D_{-2}^2 - E_{-2}^2) (12b^2c_3 (-C_{-2}^2 + D_{-2}^2 - E_{-2}^2) - c_2F_{-2}) = 0, \]
\[ 3e_{-2}b^2 (56b^2c_3 (C_{-2}^2 - D_{-2}^2 + E_{-2}^2) + 3c_2F_{-2}) = 0, \]
\[ 5e_{-2} (a^2 + b^2 (A_0c_2 + 13b^2c_3 + c_1)) = 0, \]
\[ 2 (a^2 + b^2 (A_0c_2 + 4b^2c_3 + c_1)) = 0, \]
\[ 3a^2 (C_{-2}^2 - D_{-2}^2 + E_{-2}^2) + b^2 (3C_{-2}^2 (A_0c_2 + 20b^2c_3 + c_1) + 3e_{-2}c_1 - 3D_{-2}^2 (A_0c_2 + 20b^2c_3 + c_1) + 3e_{-2}A_0c_2 + 165e_{-2}b^2c_3 + 4c_2F_{-2}) = 0 \]  

(4.7)

Solving this system of equations we get the following relationship among the parameters

\[ A_0 = \frac{-a^2 - 4b^4c_3 - b^2c_1}{b^2c_2}, \]
\[ E_{-2} = 0, \]
\[ F_{-2} = \frac{12b^2c_3 (D_{-2}^2 - C_{-2}^2)}{c_2} \]  

(4.8)

So the new family of exact solution for the general Boussinesq equation is obtained by substituting these values in (4.6), which gives

\[ f(t, x) = -\frac{a^2 - 4b^4c_3 - b^2c_1}{b^2c_2} + \frac{12b^2c_3 (D_{-2}^2 - C_{-2}^2)}{c_2 \left( C_{-2} \sinh(at + bx) + D_{-2} \cosh(at + bx) \right)^2} \]  

(4.9)

### 5. Conclusion

In this paper four families of new exact solutions for general Boussinesq equation (1.3) have been derived. These solutions are derived using ansatz method and computer algebra system. The solutions are obtained in terms of trigonometric functions. The new exact solutions are given by equations (4.1), (4.4), and (4.9).
and hyperbolic sine and cosine functions. These solutions can be used to generate exact solutions for the different variants of Boussinesq equation such as given by (1.1) and (1.2) by a suitable choice for the values of the parameters. It is to be noted that the solutions obtained here are valid for arbitrary values of $a$ and $b$ where $u = at + bx$. These solutions can be used for analyzing the different physical phenomena suitably represented by these equations such as the dynamics of shallow water waves in coastal and ocean regions, the dynamics of thin inviscid layers and non-linear lattice waves. The accuracy of several approximate methods used to derive numerical solutions for Boussinesq equation can also be tested using these solutions.

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**References**


