Malaya Journal of Matematik, Vol. S, No. 1, 639-647, 2020

https://doi.org/10.26637/MJM0S20/0122

Some new oscillation results for third-order Emden-Fowler type neutral partial differential equations with mixed nonlinearities

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Abstract

In this article, to extend and improve existing oscillatory criteria for third order Emden-Fowler type neutral partial differential equations with mixed nonlinearities subject to the boundary conditions. Several sufficient conditions are obtained for oscillation of solutions of such class of equations by using generalized Riccati and integral average method.

Keywords

Oscillation, Third order, Riccati technique, Emden-Fowler equation.

AMS Subject Classification

34K11, 35B05, 34K40.

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Article History: Received **10** January **2020**; Accepted **01** May **2020**

Contents

1. Introduction

The problem of oscillation and nonoscillation of third order originated by Hanan [14] in this monumental paper published in 1961. Since then a number of researches contributed to the subject investigating various classes of differential equations and applying variety of techniques. A systematic survey of the most significant efforts in this theory can be found in the excellent monographs of Swanson [21], Greguš and the very recent-one one of Padhi and Pati [11,19].

In the middle of the nineteenth century, the Emden-Fowler equations emerged from theories deals with gaseous dynamics in astrophysics. The Emden-Fowler equations are considered to be one of the most important classical objects in the theory of differential equations. This type of equations has variety of interesting physical applications occuring in astrophysics in the form of Emden equation and in atomic physics in the form of Thomas-Fermi equation.

The Emden-Fowler type of equation has significant applications in many fields of scientific and technical world and and this equation has been investigated by many researchers [1-3,5-10,12,13,15-17,23,25,27,30] and the references cited there in.

The Emden-Fowler equations were first considered only for second order equations of the form

$$(p(t)u')' + q(t)u^{\gamma} = 0, t \geq 0. \tag{1.1}$$

By a mixed type Emden-Fowler equation we mean the equation contains a finite sum of powers of x and if there exists in sum exponents of which are both grater than and less than 1. These type of equations arises for instant in the growth of bacteria population with competitive species. As we have known, almost all existing oscillation criteria in the literature, see for example [18, 22, 28] are established for Emden-Fowler type equations with mixed nonlinearities of second order.

In 2007, Xu et al. [29], discussed the Philos-type oscillation criteria for the second order Emden-Fowler neutral delay differential equation of the form

$$\left(|x'(t)|^{\gamma-1}x'(t)\right)' + q_1(t)|y(t-\sigma)|^{\alpha-1}y(t-\sigma) + q_2(t)|y(t-\sigma)|^{\beta-1}y(t-\sigma) = 0,$$

$t \geq 0$, where $x(t) = y(t) + p(t)y(t-\tau)$.

In 2018, Sadhasivam et al. [20], investigated the oscillation of third-order neutral delay mixed type Emden-Fowler differential equations of the form

$$\begin{aligned}&\left(a(t)\left(b(t)|z'(t)|^{\gamma-1}z'(t)\right)'\right)'\\&+p_1(t)|x(\sigma_1(t))|^{\alpha-1}x(\sigma_1(t))\\&+p_2(t)|x(\sigma_2(t))|^{\beta-1}x(\sigma_2(t))=0,\quad t\geq t_0.\end{aligned}$$

Where $z(t)=x(t)+c(t)x(\tau(t))$.

Partial differential equations are used to model a number of real world problems arising in various branches of science and engineering. The oscillation theory for second order partial differential equations have been well developed. See [4,19,24,26,31] and the references cited therein. There are essentially less results on oscillation of third order Emden-Fowler partial differential equations. Motivated by the above observations, we are concerned with the oscillation criteria for third-order Emden-Fowler type neutral partial differential equations with mixed nonlinearities

$$\begin{aligned}&\frac{\partial}{\partial t}\left(r_2(t)\frac{\partial}{\partial t}\left(r_1(t)\left|\frac{\partial z(x,t)}{\partial t}\right|^{\gamma-1}\frac{\partial z}{\partial t}\right)\right)\\&+p_1(x,t)|u(x,t-\delta_1)|^{\alpha-1}u(x,t-\delta_1)\\&+p_2(x,t)|u(x,t-\delta_2)|^{\beta-1}u(x,t-\delta_2)\\&=a(t)\Delta u(x,t)+F(x,t),\ (x,t)\in\Omega\times\mathbb{R}_+.\end{aligned}\tag{1.2}$$

Where $z(x,t)=u(x,t)+q(t)u(x,t-\tau)$. $\mathbb{R}_+=(0,\infty)$, where Ω is bounded domain in $\mathbb{R}^N$ with a piecewise smooth boundary $\partial\Omega$, Δ is the Laplacian operator in the Euclidean N space $\mathbb{R}^N$, i.e, $\Delta u(x,t)=\Sigma_{r=1}^{N}\frac{\partial^2 u(x,t)}{\partial {x_r}^2}$.

Equation (1.1) is supplemented with the boundary conditions with

$$\frac{\partial u(x,t)}{\partial\mu}+g(x,t)u(x,t)=0,\ (x,t)\in\Omega\times\mathbb{R}_+,\tag{1.3}$$

and

$$u(x,t)=0,\ (x,t)\in\Omega\times\mathbb{R}_+.\tag{1.4}$$

Throughout this paper, we will suppose that the following conditions hold:

(A_1) τ, δ_1 and δ_2 are positive constants, α, β and γ are non-negative constants with $0<\alpha<\gamma<\beta$, where γ is ratio of two odd positive integers;

(A_2) $p_1(x,t),p_2(x,t)\in C(\bar{G},\mathbb{R}_+)$, $p_i(t)=min_{x\in\bar{\Omega}}p_i(x,t)$,where i=1,2, $p(t)$ is not identically zero on any ray from $[t_*,\infty)$ for any $t_*\geq 0$ and $q(t)\in C(\mathbb{R}_+,\mathbb{R})$;$0\leq q(t)\leq 1$;

(A_3) $F(x,t)\in C(\bar{G},\mathbb{R})$, such that $\int_\Omega F(x,t)dx\leq 0$.

The following notations will be used for our convenience.

$$U(t)=\frac{1}{|\Omega|}\int_\Omega u(x,t)dx,\tag{1.5}$$

$$Z(t)=U(t)+q(t)U(t-\tau)\tag{1.6}$$

By a solution of (1.1),(1.2) or (1.1),(1.3) we mean a non-trivial function $u(x,t)\in C^3(G)\cap C^2(G)\cap C^1(G)$ that is satisfies the domain G and the boundary condition (1.2), (1.3). A solution $u(x,t)$ of (1.1),(1.2) or (1.1),(1.3) is said to be *oscillatory* in G if it has a zero in $\Omega\times(t,\infty)$ for any $t>0$. Otherwise it is *nonoscillatory*. Equation (1.1) is said to be oscillatory if all its solutions are oscillatory.

This paper is organized as follows: In Section 2, we present some new oscillation criteria for all solutions of (1.1), (1.2) and (1.1), (1.3). In Section 3, examples are provided to illustrate the main results.

2. Main Results

In this section, we state and prove our oscillation results.

Theorem 2.1. *If $u(x,t)$ is a solution of (1.1), (1.2) for which $u(x,t)>0$ in $G_T=\Omega\times[T,\infty),T\geq 0$, then the function $U(t)$ is defined by (1.4) satisfy the differential inequality*

$$\begin{aligned}&\left(r_2(t)\left(r_1(t)\left(z'(t)\right)^\gamma\right)'\right)'\\&+p_1(t)(U(t-\delta_1))^\alpha+p_2(t)(U(t-\delta_2))^\beta\leq 0\end{aligned}\tag{2.1}$$

with $U(t)>0$, $U(t-\tau)>0$ and $U(t-\delta_i)>0$ for $t\geq T$, where $i=1,2$.

Proof. Let $t\geq T$. Integrating (1.1) with respect to x over Ω, we have

$$\begin{aligned}&\int_\Omega\frac{d}{dt}\left(r_2(t)\frac{d}{dt}\left(r_1(t)\left|\frac{dz(x,t)}{dt}\right|^{\gamma-1}\frac{dz}{dt}\right)\right)dx\\&+\int_\Omega p_1(x,t)|u(x,t-\delta_1)|^{\alpha-1}u(x,t-\delta_1)dx\\&+\int_\Omega p_2(x,t)|u(x,t-\delta_2)|^{\beta-1}u(x,t-\delta_2)dx\\&=\int_\Omega a(t)\Delta u(x,t)dx+\int_\Omega F(x,t)dx.\end{aligned}\tag{2.2}$$

Using Green's formula and boundary condition (1.2), we obtain

$$\begin{aligned}\int_\Omega\Delta u(x,t)dx&=\int_{\partial\Omega}\frac{\partial u(x,t)}{\partial\mu}dS\\&=-\int_{\partial\Omega}g(x,t)u(x,t)dS\leq 0,\end{aligned}\tag{2.3}$$

$t \geq T$ .Also from (A_2) and Jensen's inequality, it follows that

$$\begin{aligned}
&\int_\Omega p_1(x,t)|u(x,t-\delta_1)|^{\alpha-1}u(x,t-\delta_1)dx \\
&\geq p_1(t)\int_\Omega (u(x,t-\delta_1))^\alpha dx \\
&\geq p_1(t)\left(\int_\Omega u(x,t-\delta_1)dx\right)^\alpha \\
&\geq p_1(t)\,(U(t-\delta_1))^\alpha,\ t\geq T, \qquad (2.4)\\
&\int_\Omega p_2(x,t)|u(x,t-\delta_2)|^{\beta-1}u(x,t-\delta_2)dx \\
&\geq p_2(t)\int_\Omega (u(x,t-\delta_2))^\beta dx \\
&\geq p_2(t)\left(\int_\Omega u(x,t-\delta_2)dx\right)^\beta \\
&\geq p_2(t)\,(U(t-\delta_2))^\beta,\ t\geq T \qquad (2.5)
\end{aligned}$$

In view of (1.4), (2.3)-(2.5) and (A_4), (2.2) yield

$$\begin{aligned}
&\left(r_2(t)\left(r_1(t)\left(z'(t)\right)^\gamma\right)'\right)' + p_1(t)(U(t-\delta_1))^\alpha \\
&+ p_2(t)(U(t-\delta_2))^\beta \leq 0,\ t\geq T. \qquad (2.6)
\end{aligned}$$

This completes the proof. □

Lemma 2.2. *Assume that $u(x,t)$ is an eventually positive solution of Eq.(1.1). If*

$$\int_0^\infty \frac{1}{r_2(s)}ds = \infty, \qquad (2.7)$$

$$\int_0^\infty \frac{1}{r_1(s)}ds = \infty, \qquad (2.8)$$

$$\int_0^\infty \left(\frac{1}{r_1(\theta)}\int_\theta^\infty \frac{1}{r_2(\xi)}\left(\int_\xi^\infty P(s)ds\right)d\xi\right)^{\frac{1}{\gamma}} d\theta = \infty, \qquad (2.9)$$

where

$$\begin{aligned}
P(t) &= p_1(t)\,(1-q(t-\delta_1))^\alpha L^\alpha \\
&+ p_2(t)\,(1-q(t-\delta_2))^\beta L^\beta \qquad (2.10)
\end{aligned}$$

then there exists a sufficiently large T such that $(r_1(t)(z'(t))^\gamma)' > 0$ on $[T,\infty)$ and either $z'(t) > 0$ on $[T,\infty)$ or $\lim_{t\to\infty} z(t) = 0$.

Proof. Since $u(x,t)$ is an eventually positive solution of (1.1). There exits $t_1 \geq t_0$ such that $u(x,t) > 0$ on $\Omega\times[T,\infty)$, $u(x,t-\tau) > 0, u(x,t-\delta_1) > 0, u(x,t-\delta_2) > 0$ and from (1.5) we have,

$$\begin{aligned}
&\left(r_2(t)\left(r_1(t)(z'(t))^\gamma\right)'\right)' = -p_1(t)(U(t-\delta_1))^\alpha \\
&- p_2(t)(U(t-\delta_2))^\beta < 0,\ \ t\geq t_1. \qquad (2.11)
\end{aligned}$$

Then $\left(r_2(t)\left(r_1(t)(z'(t))^\gamma\right)'\right)'$ is strictly decreasing on $[t_1,\infty)$, and thus $(r_1(t)(z'(t))^\gamma)'$ is eventually of one sign. For $t_2 > t_1$ is sufficiently large on $[t_1,\infty)$, we claim $(r_1(t)(z'(t))^\gamma)' > 0$ on $[t_2,\infty)$. Otherwise, assume that there exists a sufficiently large $t_3 > t_2$ such that $(r_1(t)(z'(t))^\gamma)' < 0$ on $[t_3,\infty)$. Then $r_1(t)(z'(t))^\gamma$ is decreasing on $[t_3,\infty)$, and we have

$$\begin{aligned}
r_1(t)(z'(t))^\gamma - r_1(t_3)(z'(t_3))^\gamma &= \int_{t_3}^t \frac{r_2(s)}{r_2(s)}\left(r_1(s)(z'(s))^\gamma\right)' ds \\
&\leq r_2(t_3)\left(r_1(t_3)(z'(t_3))^\gamma\right)'\int_{t_3}^t \frac{1}{r_2(s)}ds.
\end{aligned}$$

By (2.7), we have $\lim_{t\to\infty} r_1(t)(z'(t))^\gamma = -\infty$. So there exists a sufficiently large t_4 with $t_4 > t_3$ such that $z'(t) < 0, [t_4,\infty)$. Further more

$$\int_{t_4}^t z'(s)ds = z(t) - z(t_4)$$

$$\int_{t_4}^t \frac{r_1(s)}{r_1(s)}z'(s)ds \leq r_1(t_4)z'(t_4)\int_{t_4}^t \frac{1}{r_1(s)}ds. \qquad (2.12)$$

By (2.8), we deduce that $\lim_{t\to\infty} z(t) = -\infty$, which contradicts the fact that $z(t)$ is an eventually positive solution of (2.8). So $(r_1(t)(z'(t))^\gamma)' > 0$ on $[t_2,\infty)$. Thus $z'(t)$ is eventually of one sign. Now, we assume that $z'(t) < 0, t\in[t_5,\infty)$, for sufficiently large $t_5 > t_4$. Since $z'(t) > 0$, further more, we have $\lim_{t\to\infty} z(t) = L \geq 0$. We claim that $L = 0$. Otherwise assume that $L > 0$. Then $z(t) \geq L$ on $[t_5,\infty)$ and for $t\in[t_5,\infty)$ by (2.1),

$$\begin{aligned}
&\left(r_2(t)\left(r_1(t)(z'(t))^\gamma\right)'\right)' = -p_1(t)(U(t-\delta_1))^\alpha \\
&- p_2(t)(U(t-\delta_2))^\beta
\end{aligned}$$

By (1.5), we get

$$U(t) \geq z(t) - q(t)U(t-\tau) \geq z(t)(1-q(t)) \qquad (2.13)$$

Then, for all $t\geq t_5$,

$$U(t-\delta_1) \geq (1-q(t-\delta_1))z(t-\delta_1) \qquad (2.14)$$

$$U(t-\delta_2) \geq (1-q(t-\delta_2))z(t-\delta_2) \qquad (2.15)$$

Then (2.1), implies that,

$$\begin{aligned}
&\left(r_2(t)\left(r_1(t)(z'(t))^\gamma\right)'\right)' + p_1(t)(1-q(t-\delta_1))^\alpha z^\alpha(t-\delta_1) \\
&\qquad + p_2(t)(1-q(t-\delta_2))^\beta z^\beta(t-\delta_2) \leq 0. \qquad (2.16)
\end{aligned}$$

Since $z(t) \geq L$ on $[t_5,\infty)$, we get

$$\begin{aligned}
&\left(r_2(t)\left(r_1(t)(z'(t))^\gamma\right)'\right)' + p_1(t)(1-q(t-\delta_1))^\alpha L^\alpha \\
&\qquad + p_2(t)(1-q(t-\delta_2))^\beta L^\beta \leq 0,\quad t\geq t_5. \qquad (2.17)
\end{aligned}$$

Using (2.10), we have

$$\left(r_2(t)\left(r_1(t)(z'(t))^\gamma\right)'\right)' + P(t) \leq 0,\quad t\geq t_5. \qquad (2.18)$$

Integrating with respect to s from t to ∞ yields,

$$\int_t^{\infty}\left(r_2(s)\left(r_1(s)(z'(s))^{\gamma}\right)'\right)' \leq -\int_t^{\infty}P(s)ds$$
$$\left(r_1(s)(z'(s))^{\gamma}\right)' \geq \frac{1}{r_2(t)}\int_t^{\infty}P(s)ds$$

Integrating with respect to s from t to ∞ yields,

$$\int_t^{\infty}\left(r_1(s)(z'(s))^{\gamma}\right)' ds \geq \int_t^{\infty}\frac{1}{r_2(\xi)}\int_{\xi}^{\infty}P(s)dsd\xi$$
$$r_1(t)(z'(t))^{\gamma} \leq -\int_t^{\infty}\frac{1}{r_2(\xi)}\int_{\xi}^{\infty}P(s)dsd\xi$$
$$z'(t) \leq \left(-\frac{1}{r_1(t)}\int_t^{\infty}\frac{1}{r_2(\xi)}\int_{\xi}^{\infty}P(s)dsd\xi\right)^{\frac{1}{\gamma}}$$

Once again, integrating with respect to s from t_5 to ∞ yields,

$$\int_{t_5}^{t}z'(s)ds \leq -\int_{t_5}^{t}\left[\frac{1}{r_1(\theta)}\int_{\theta}^{\infty}\frac{1}{r_2(\xi)}\int_{\xi}^{\infty}P(s)dsd\xi\right]^{\frac{1}{\gamma}}d\theta$$
$$z(t) \leq z(t_5) - \int_{t_5}^{t}\left[\frac{1}{r_1(\theta)}\int_{\theta}^{\infty}\frac{1}{r_2(\xi)}\int_{\xi}^{\infty}P(s)dsd\xi\right]^{\frac{1}{\gamma}}d\theta.$$

Letting $t \to \infty$, from (2.9) we get $\lim_{t\to\infty} z(t) = -\infty$, which causes a contradiction. So the proof is complete. □

Lemma 2.3. *Assume that $u(x,t)$ is an eventually positive solution of Eq.(1.1) such that $(r_1(t)(z'(t))^{\gamma})' > 0, z'(t) > 0$ on $[t_1,\infty)$, where $t_1 \geq t_0$ is sufficiently large. Then one has*

$$z'(t) \geq \left\{\frac{1}{r_1(t)r_2(t)}\left(r_1(t)(z'(t))^{\gamma}\right)' R_1(t_1,t)\right\}^{\frac{1}{\gamma}} \quad (2.19)$$

or

$$z(t) \geq \left(\left(r_1(t)(z'(t))^{\gamma}\right)'\right)^{\frac{1}{\gamma}} R_2(t_1,t) \quad (2.20)$$

where $R_1(t_1,t) = \int_{t_1}^{t}\frac{1}{r_2(s)}ds, R_2(t_1,t) = \int_{t_1}^{t}\left(\frac{R_1(t_1,s)}{r_1(s)r_2(s)}\right)^{\frac{1}{\gamma}}ds.$

Proof. By (2.11), we obtain that $\left(r_2(t)\left(r_1(t)(z'(t))^{\gamma}\right)'\right)$ is strictly decreasing on $[t_1,\infty)$. So

$$\int_{t_1}^{t}\left(r_1(s)(z'(s))^{\gamma}\right)' ds = r_1(t)(z'(t))^{\gamma} - r_1(t_1)(z'(t_1))^{\gamma}$$

Multiplying and divided by $r_2(t)$,

$$r_1(t)(z'(t))^{\gamma} \geq r_1(t)(z'(t))^{\gamma} - r_1(t_1)(z'(t_1))^{\gamma}$$
$$= \int_{t_1}^{t}\frac{r_2(s)}{r_2(s)}\left(r_1(s)(z'(s))^{\gamma}\right)' ds$$
$$r_1(t)(z'(t))^{\gamma} \geq \frac{1}{r_2(t)}\left(r_1(t)(z'(t))^{\gamma}\right)'\int_{t_1}^{t}\frac{1}{r_2(s)}ds$$
$$(z'(t))^{\gamma} \geq \frac{1}{r_1(t)r_2(t)}\left(r_1(t)(z'(t))^{\gamma}\right)' R_1(t_1,t)$$
$$z'(t) \geq \left\{\frac{1}{r_1(t)r_2(t)}\left(r_1(t)(z'(t))^{\gamma}\right)' R_1(t_1,t)\right\}^{\frac{1}{\gamma}}$$

Integrating with respect to s from t_1 to t we obtain,

$$\int_{t_1}^{t}z'(s)ds \geq \int_{t_1}^{t}\left\{\frac{1}{r_1(s)r_2(s)}\left(r_1(s)(z'(s))^{\gamma}\right)' R_1(t_1,s)\right\}^{\frac{1}{\gamma}}ds$$
$$z(t) \geq \left(\left(r_1(t)(z'(t))^{\gamma}\right)'\right)^{\frac{1}{\gamma}}\int_{t_1}^{t}\left(\frac{R_1(t_1,s)}{r_1(s)r_2(s)}\right)^{\frac{1}{\gamma}}ds$$
$$z(t) \geq \left(\left(r_1(t)(z'(t))^{\gamma}\right)'\right)^{\frac{1}{\gamma}} R_2(t_1,t)$$

This completes the proof. □

In this section, we will obtain Philos - type oscillation criteria for (1.1) under the case when $0 \leq q(t) \leq 1$. The following notations are used in the sequel.

Denote

$$\sigma = min\left\{\frac{\beta-\alpha}{\beta-\gamma}, \frac{\beta-\alpha}{\gamma-\gamma}\right\}, \kappa = \frac{1}{(\gamma+1)^{\gamma+1}}$$
$$P_1(t) = \sigma\left[\left(\frac{p_1(t)}{r_1(t)}\right)^{\beta-\gamma}\left(\frac{p_2(t)}{r_1(t)}\right)^{\gamma-\alpha}\right.$$
$$\left.\times\left(1-q(t-\delta_1)\right)^{\alpha(\beta-\gamma)}\left(1-q(t-\delta_2)\right)^{\beta(\gamma-\alpha)}\right]^{\frac{1}{\beta-\alpha}}$$

Let us define the following Philos functions $\mathbb{J}$.

Let $\mathbb{D}_0 = \left\{(t,s) \in \mathbb{R}^2 : t > s \geq t_0\right\}$ and $\mathbb{D} = \left\{(t,s) \in \mathbb{R}^2 : t \geq s \geq t_0\right\}$. We say that the functions $H \in C(\mathbb{D},\mathbb{R})$ belongs to the class $\mathbb{J}$, denotes by $H \in \mathbb{J}$, if

(H_1) $H(t,t) = 0$ for $t \geq t_0$. $H(t,s) > 0$ on $(t,s) \in \mathbb{D}_0$;

(H_2) H has a continuous and non positive partial derivative on $\mathbb{D}_0$ with respect to the second variable, such that

$$\frac{\partial}{\partial s}H(t,s) = -h(t,s)H(t,s) \quad for \quad (t,s) \in \mathbb{D}_0$$

where $h \in C(\mathbb{D},\mathbb{R})$.

For given functions $h \in C(\mathbb{D},\mathbb{R}), \phi \in C'([\mathbb{R}_+,\infty),\mathbb{R}_+)$ and $\eta \in C'([\mathbb{R}_+,\infty),\mathbb{R})$ we set

$$\rho(t,s) = h(t,s) - \frac{\phi'(s)}{\phi(s)}$$
$$K_1(t,s) = P_1(s) - \eta'(s) + \rho(t,s)\eta(s)$$

Theorem 2.4. *Let $H \in \mathbb{J}$, $\phi \in C'(\mathbb{R}_+,\mathbb{R}_+)$ and $\eta \in C'(\mathbb{R}_+,\mathbb{R})$. Then (1.1),(1.2) is oscillatory provided that the following condition holds*

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\phi(s)\left\{K_1(t,s) - |\lambda(s)||\rho(t,s)|^{\gamma+1}\right\}ds = \infty, \quad (2.21)$$

where

$$\lambda(s) = \frac{(r_2(s)r_1(s))^{1-\gamma}}{(\gamma+1)^{\gamma+1}R_1(t_1,s)}. \quad (2.22)$$

Proof. Let $u(x,t)$ be a nonoscillatory solution of (1.1). Without loss of generality, we may assume that $u(x,t) > 0$ on $\Omega \times [t_0,\infty)$ for some sufficiently large t_0.(The Case $u(x,t) < 0$ can be considered by same method) Let us assume that there exists a $t_1 > t_0$ such that $u(x,t) > 0$, $u(x,t-\tau) > 0, u(x,t-\delta_1) > 0$ and $u(x,t-\delta_2) > 0$ for $t \geq t_1$. Therefore, we get (2.1). Now, define

$$W(t) = \phi(t)\left(\frac{r_2(t)\,(r_1(t)(z'(t))^{\gamma})'}{r_1(t)z^{\gamma}(t-\delta_1)} + \eta(t)\right) \tag{2.23}$$

for $t \geq t_1$. Differentiating (2.23) and (2.16), we have

$$\begin{aligned} W'(t) \leq{}& \frac{\phi'(t)}{\phi(t)}W(t) - \phi(t)\left[\frac{p_1(t)(1-q(t-\delta_1))^{\alpha}z^{\alpha}(t-\delta_1)}{r_1(t)z^{\gamma}(t-\delta_1)}\right] \\ &- \phi(t)\left[\frac{p_2(t)(1-q(t-\delta_2))^{\beta}z^{\beta}(t-\delta_2)}{r_1(t)z^{\gamma}(t-\delta_1)}\right] \\ &- \frac{\phi(t)r_2(t)\,(r_1(t)(z'(t))^{\gamma})'\,(r_1(t)\gamma(z'(t-\delta_1))^{\gamma-1}z'(t-\delta_1))}{(r_1(t)z^{\gamma}(t-\delta_1))^2} \\ &+ \frac{r_1'(t)z^{\gamma}(t-\delta_1)}{(r_1(t)z^{\gamma}(t-\delta_1))^2} + \phi(t)\eta'(t). \end{aligned}$$

For simplicity, let $\delta_1 \geq \delta_2$ (a similar argument holds for $\delta_1 \leq \delta_2$) then
$U(t-\delta_1) \leq U(t-\delta_2)$ and applying Lemma 2.3, We get

$$\begin{aligned} W'(t) \leq{}& \frac{\phi'(t)}{\phi(t)}W(t) \\ &- \phi(t)\left[\frac{p_1(t)}{r_1(t)}(1-q(t-\delta_1))^{\alpha}z^{\alpha-\gamma}(t-\delta_1)\right] \\ &- \phi(t)\left[\frac{p_2(t)}{r_1(t)}(1-q(t-\delta_2))^{\beta}z^{\beta-\gamma}(t-\delta_1)\right] \\ &- \frac{\gamma\phi(t)R_1^{\frac{1}{\gamma}}(t_1,t)}{r_2^{\frac{1-\gamma}{\gamma}}(t)r_1^{\frac{1-\gamma}{\gamma}}(t)}\left(\frac{W(t)}{\phi(t)} - \eta(t)\right)^{1+\frac{1}{\gamma}} + \phi(t)\eta'(t) \end{aligned} \tag{2.24}$$

Using Young's inequality$\left(\frac{l}{p} + \frac{m}{q} \geq l^{\frac{1}{p}}m^{\frac{1}{q}}\right)$, we obtain that

$$\begin{aligned} &\frac{\beta-\gamma}{\beta-\alpha}\left[\frac{p_1(t)}{r_1(t)}(1-q(t-\delta_1))^{\alpha}z^{\alpha-\gamma}(t-\delta_1)\right] \\ &+ \frac{\gamma-\alpha}{\beta-\alpha}\left[\frac{p_2(t)}{r_1(t)}(1-q(t-\delta_2))^{\beta}z^{\beta-\gamma}(t-\delta_1)\right] \\ &\geq \left\{\left(\frac{p_1(t)}{r_1(t)}\right)^{\beta-\gamma}\left(\frac{p_2(t)}{r_1(t)}\right)^{\gamma-\alpha}\right. \\ &\left.\times(1-q(t-\delta_1))^{\alpha(\beta-\gamma)}(1-q(t-\delta_2))^{\beta(\gamma-\alpha)}\right\}^{\frac{1}{\beta-\alpha}} = P_1(t) \end{aligned} \tag{2.25}$$

Combining (2.24) and (2.25) for $t \geq T_0$, we have

$$\begin{aligned} W'(t) \leq{}& \frac{\phi'(t)}{\phi(t)}W(t) - \phi(t)\left[P_1(t) - \eta(t)\right] \\ &- \frac{\gamma\phi(t)R_1^{\frac{1}{\gamma}}(t_1,t)}{r_2^{\frac{1-\gamma}{\gamma}}(t)r_1^{\frac{1-\gamma}{\gamma}}(t)}\left|\frac{W(t)}{\phi(t)} - \eta(t)\right|^{1+\frac{1}{\gamma}} \end{aligned} \tag{2.26}$$

Replacing t in (2.26) by s, then multiplying (2.26) by $H(t,s)$ and integrating on $[T,t]$, it follows from (H_2) that for all $t \geq T \geq T_0$,

$$\begin{aligned} &\int_T^t H(t,s)\phi(s)\left[P_1(s) - \eta(s)\right]ds \\ &\leq -\int_T^t H(t,s)W'(s)ds + \int_T^t H(t,s)\frac{\phi'(s)}{\phi(s)}W(s)ds \\ &\quad - \gamma\int_T^t \frac{\phi(s)R_1^{\frac{1}{\gamma}}(t_1,s)}{r_2^{\frac{1-\gamma}{\gamma}}(s)r_1^{\frac{1-\gamma}{\gamma}}(s)}H(t,s)\left|\frac{W(s)}{\phi(s)} - \eta(s)\right|^{\frac{\gamma+1}{\gamma}}ds \\ &= H(t,T)W(t) + \int_T^t H(t,s)\rho(t,s)W(s)ds \\ &\quad - \gamma\int_T^t \frac{\phi(s)R_1^{\frac{1}{\gamma}}(t_1,s)}{r_2^{\frac{1-\gamma}{\gamma}}(s)r_1^{\frac{1-\gamma}{\gamma}}(s)}H(t,s)\left|\frac{W(s)}{\phi(s)} - \eta(s)\right|^{\frac{\gamma+1}{\gamma}}ds \end{aligned}$$

$$\begin{aligned} &\int_T^t H(t,s)\phi(s)K_1(t,s)ds \leq H(t,T)W(t) \\ &+ \int_T^t H(t,s)\phi(t,s)|\rho(s)|\left|\frac{W(s)}{\phi(s)} - \eta(s)\right|ds \\ &- \gamma\int_T^t H(t,s)\phi(s)\left|\frac{R_1^{\frac{1}{\gamma}}(t_1,s)}{r_2^{\frac{1-\gamma}{\gamma}}(s)r_1^{\frac{1-\gamma}{\gamma}}(s)}\right|\left|\frac{W(s)}{\phi(s)} - \eta(s)\right|^{\frac{\gamma+1}{\gamma}}ds \end{aligned} \tag{2.27}$$

For given t and s, $t \neq s$, set

$$F(\omega) := |\rho||\omega| - \gamma\left|\frac{R_1^{\frac{1}{\gamma}}(t_1,t)}{r_2^{\frac{1-\gamma}{\gamma}}(t)r_1^{\frac{1-\gamma}{\gamma}}(t)}\right||\omega|^{\frac{\gamma+1}{\gamma}}, \quad \omega > 0.$$

$F(\omega)$ attains its maximum at $(\gamma+1)^{-\gamma}\left|\frac{r_2^{1-\gamma}(t)r_1^{1-\gamma}(t)}{R_1(t_1,t)}\right||\rho|^{\gamma}$ and

$$F(\omega) \leq F_{max} \leq |\lambda(t)||\rho|^{\gamma+1} \tag{2.28}$$

Substituting (2.28) into (2.27), we have

$$\begin{aligned} &\int_T^t H(t,s)\phi(s)K_1(t,s)ds \leq H(t,T)W(t) \\ &+ \int_T^t H(t,s)\phi(s)|\lambda(s)||\rho(t,s)|^{\gamma+1}ds \end{aligned} \tag{2.29}$$

Set $T = T_0$, so

$$\int_{T_0}^t H(t,s)\phi(s)\left[K_1(t,s) - |\lambda(s)||\rho(t,s)|^{\gamma+1}\right]ds \leq H(t,T_0)W(T_0)$$

Thus by (H_2), we obtain

$$\begin{aligned} &\int_{T_0}^t H(t,s)\phi(s)\left[K_1(t,s) - |\lambda(s)||\rho(t,s)|^{\gamma+1}\right]ds \\ &\quad = \left(\int_{t_0}^{T_0} + \int_{T_0}^t\right)H(t,s)\phi(s) \\ &\left[K_1(t,s) - |\lambda(s)||\rho(t,s)|^{\gamma+1}\right]ds \\ &\leq H(t,t_0)\left(\int_{t_0}^{T_0}\phi(s)\left[K_1(t,s) - |\lambda(s)||\rho(t,s)|^{\gamma+1}\right]ds\right. \\ &\qquad\left. + |W(T_0)|\right), \end{aligned} \tag{2.30}$$

we divide (2.30) through by $H(t,t_0)$ and take lim sup in it as $t\to\infty$. Eq. (2.21) gives a desired contradiction in (2.13). This proves the theorem. □

Theorem 2.5. *Let H,ϕ,η be as in Theorem 2.1. Suppose that*

$$0<\inf_{s\geq t_0}\left\{\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right\}\leq\infty \tag{2.31}$$

and

$$\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}H(t,s)\phi(s)|\rho(t,s)|^{\gamma+1}ds<\infty. \tag{2.32}$$

Then (1.1),(1.2) is oscillatory provided the following condition holds. There exists $\Theta\in C([(t_0),\infty),\mathbb{R})$

$$\int^{\infty}\phi(s)\left(\frac{\Theta(s)}{\phi(s)}-\eta(s)\right)_{+}^{\frac{\gamma+1}{\gamma}}ds=\infty, \tag{2.33}$$

and for any $T\geq t_0$,

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}H(t,s)\phi(s)\times\left[K_1(t,s)-|\lambda(s)||\rho(t,s)|^{\gamma+1}\right]ds\geq\Theta(t) \tag{2.34}$$

where $\Theta_+(s)=max\{\Theta(s),0\}$.

Proof. Proceeding as in the proof of Theorem 2.2, we have that (2.27) and (2.29) hold. Therefore, from (2.29), for all $t>T\geq T_0$,

$$\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_{T}^{t}H(t,s)\phi(s)\times\left[K_1(t,s)-|\lambda(s)||\rho(t,s)|^{\gamma+1}\right]ds\leq W(T)$$

Also by (2.34), we have

$$\Theta(t)\leq W(T),\quad T\geq T_0. \tag{2.35}$$

Define

$$Q_1(t)=\frac{1}{H(t,T_0)}\int_{T_0}^{t}H(t,s)\phi(t,s)|\rho(s)|\left|\frac{W(s)}{\phi(s)}-\eta(s)\right|ds$$

and

$$Q_2(t)=\frac{\gamma}{H(t,T_0)}\int_{T_0}^{t}H(t,s)\phi(s)\left|\frac{R_1^{\frac{1}{\gamma}}(t_1,s)}{r_2^{\frac{1-\gamma}{\gamma}}(s)r_1^{\frac{1-\gamma}{\gamma}}(s)}\right|\times\left|\frac{W(s)}{\phi(s)}-\eta(s)\right|^{\frac{\gamma+1}{\gamma}}ds$$

Then by (2.27) and (2.34), we see that

$$\begin{aligned}\liminf_{t\to\infty}[Q_2(t)-Q_1(t)]&\leq W(T_0)-\limsup_{t\to\infty}\frac{1}{H(t,t_0)}\int_{T_0}^{t}H(t,s)\phi(s)K_1(t,s)ds\\&\leq W(T_0)-\Theta(T_0)<\infty.\end{aligned} \tag{2.36}$$

Now, we claim that

$$\int_{T_0}^{\infty}\phi(s)\left|\frac{R_1^{\frac{1}{\gamma}}(t_1,s)}{r_2^{\frac{1-\gamma}{\gamma}}(s)r_1^{\frac{1-\gamma}{\gamma}}(s)}\right|\left|\frac{W(s)}{\phi(s)}-\eta(s)\right|^{\frac{\gamma+1}{\gamma}}ds<\infty \tag{2.37}$$

Suppose to the contrary that

$$\int_{T_0}^{\infty}\phi(s)\left|\frac{R_1^{\frac{1}{\gamma}}(t_1,s)}{r_2^{\frac{1-\gamma}{\gamma}}(s)r_1^{\frac{1-\gamma}{\gamma}}(s)}\right|\left|\frac{W(s)}{\phi(s)}-\eta(s)\right|^{\frac{\gamma+1}{\gamma}}ds=\infty \tag{2.38}$$

By (2.31), there exists a positive constant l_1 such that

$$\inf_{s\geq t_0}\left\{\liminf_{t\to\infty}\frac{H(t,s)}{H(t,t_0)}\right\}\geq l_1 \tag{2.39}$$

Let l_2 be an arbitrary positive number, then it follows from (2.38) that there exists a $T_1\geq T_0$ such that

$$\int_{T_0}^{t}\phi(s)\left|\frac{R_1^{\frac{1}{\gamma}}(t_1,s)}{r_2^{\frac{1-\gamma}{\gamma}}(s)r_1^{\frac{1-\gamma}{\gamma}}(s)}\right|\left|\frac{W(s)}{\phi(s)}-\eta(s)\right|^{\frac{\gamma+1}{\gamma}}ds\geq\frac{l_2}{l_1},\quad t\geq T_1 \tag{2.40}$$

Therefore,

$$\begin{aligned}Q_2(t)&=\frac{\gamma}{H(t,T_0)}\int_{T_0}^{t}H(t,s)\times d\left(\int_{T_0}^{s}\phi(\tau)\left|\frac{R_1^{\frac{1}{\gamma}}(t_1,\tau)}{r_2^{\frac{1-\gamma}{\gamma}}(\tau)r_1^{\frac{1-\gamma}{\gamma}}(\tau)}\right|\left|\frac{W(\tau)}{\phi(\tau)}-\eta(\tau)\right|^{\frac{\gamma+1}{\gamma}}d\tau\right)\\&\geq\frac{\gamma}{H(t,T_0)}\int_{T_1}^{t}\left(\frac{-\partial H(t,s)}{\partial s}\right)\int_{T_0}^{s}\phi(\tau)\left|\frac{R_1^{\frac{1}{\gamma}}(t_1,\tau)}{r_2^{\frac{1-\gamma}{\gamma}}(\tau)r_1^{\frac{1-\gamma}{\gamma}}(\tau)}\right|\times\left|\frac{W(\tau)}{\phi(\tau)}-\eta(\tau)\right|^{\frac{\gamma+1}{\gamma}}d\tau ds\\&\geq\frac{l_2}{l_1}\frac{\gamma}{H(t,T_0)}\int_{T_1}^{t}\left(\frac{-\partial H(t,s)}{\partial s}\right)ds\\&\geq\frac{l_2}{l_1}\frac{\gamma}{H(t,T_0)}H(t,T_1)\end{aligned}$$

By (2.39), there exists a $T_2\geq T_1$ such that $\frac{H(t,T_1)}{H(t,T_0)}\geq l_1$ for all $t\geq T_2$, which implies that $Q_2(t)\geq l_2\gamma$. Since l_2 is arbitrary, then

$$\lim_{t\to\infty}Q_2(t)=\infty. \tag{2.41}$$

Next, in view of (2.36), we may consider a sequence $\{T_n\}_{n=1}^{\infty}$ in $[t_0,\infty)$ satisfying

$$\lim_{n\to\infty}[Q_2(T_n)-Q_1(T_n)]=\liminf_{t\to\infty}[Q_2(t)-Q_1(t)]<\infty.$$

Then there exists a constants M such that

$$Q_2(T_n)-Q_1(T_n)\leq M \tag{2.42}$$

for all sufficiently large $n \in \mathbb{N}$. Since (2.41) ensure that

$$\lim_{n\to\infty} Q_2(T_n) = \infty \tag{2.43}$$

and we have (2.42) implies that

$$\lim_{n\to\infty} Q_1(T_n) = \infty \tag{2.44}$$

Further, (2.42) and (2.44) yield the inequalities

$$\frac{Q_1(T_n)}{Q_2(T_n)} - 1 \geq -\frac{M}{Q_2(T_n)} > -\frac{1}{2} \quad or \quad \frac{Q_1(T_n)}{Q_2(T_n)} \geq \frac{1}{2}$$

hold for all sufficiently large $n \in \mathbb{N}$. In view of this and (2.44) we have

$$\lim_{n\to\infty} \frac{Q_1^{\gamma+1}(T_n)}{Q_2^{\gamma}(T_n)} = \infty \tag{2.45}$$

On the other hand, from the definition of Q_1 we can obtain by H*ö*lder's inequality

$$Q_1(T_n) \leq \left(\frac{\gamma}{H(T_n,T_0)} \int_{T_0}^{T_n} H(T_n,s)\phi(s) \left| \frac{R_1^{\frac{1}{\gamma}}(t_1,s)}{r_2^{\frac{1-\gamma}{\gamma}}(s) r_1^{\frac{1-\gamma}{\gamma}}(s)} \right| \right.$$
$$\left. \times \left| \frac{W(s)}{\phi(s)} - \eta(s) \right|^{\frac{\gamma+1}{\gamma}} ds \right)^{\frac{\gamma}{\gamma+1}}$$
$$\times \left(\frac{1}{\gamma^{\gamma} H(T_n,T_0)} \int_{T_0}^{T_n} H(T_n,s)\phi(s)|\rho(t,s)|^{\gamma+1} ds \right)^{\frac{1}{\gamma+1}}$$

and accordingly,

$$\frac{Q_1^{\gamma+1}(T_n)}{Q_2^{\gamma}(T_n)} \leq \frac{1}{\gamma^{\gamma} H(T_n,T_0)} \int_{T_0}^{T_n} H(T_n,s)\phi(s)|\rho(t,s)|^{\gamma+1} ds$$

So, because of (2.45), we have

$$\lim_{n\to\infty} \frac{1}{H(T_n,T_0)} \int_{T_0}^{T_n} H(T_n,s)\phi(s)|\rho(t,s)|^{\gamma+1} ds = \infty$$

which gives that

$$\limsup_{n\to\infty} \frac{1}{H(t,T_0)} \int_{T_0}^{t} H(t,s)\phi(s)|\rho(t,s)|^{\gamma+1} ds = \infty$$

Contradicting (2.32). Therefore (2.37) holds. Now, in view of (2.35) and (2.37) we obtain

$$\int_{T_0}^{\infty} \phi(s) \left(\frac{\Theta(s)}{\phi(s)} - \eta(s) \right)_+^{\frac{\gamma+1}{\gamma}} ds$$
$$\leq \int_{T_0}^{\infty} \phi(s) \left| \frac{W(s)}{\phi(s)} - \eta(s) \right|^{\frac{\gamma+1}{\gamma}} ds < \infty$$

which contradicts (2.33). This completes the proof. □

Theorem 2.6. *Let H, ϕ and η be as in Theorem 2.2, suppose that (2.31) holds and*

$$\liminf_{t\to\infty} \int_{t_0}^{t} H(t,s)\phi(s)|\rho(t,s)|^{\gamma+1} ds < \infty. \tag{2.46}$$

then (1.1) is oscillatory provided the following condition holds.

There exists $\Theta \in C([t_0,\infty),\mathbb{R})$ such that (2.33) holds, and for any $T \geq t_0$,

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} H(t,s)\phi(s) \left[K_1(t,s) - |\lambda(s)||\rho(t,s)|^{\gamma+1}\right] ds \leq \Theta(t). \tag{2.47}$$

Theorem 2.7. *Let H, ϕ and η be as in Theorem 2.2, suppose that (2.31) holds.Then (1.1) is oscillatory provided the following condition holds.*

$$\liminf_{t\to\infty} \frac{1}{H(t,T)} \int_{T}^{t} H(t,s)\phi(s)K_1(t,s) ds < \infty. \tag{2.48}$$

Further, suppose that there exists $\phi \in C([t_0,\infty),\mathbb{R})$ such that (2.33), (2.47) hold.

In this section we establish sufficient condition for the oscillation of all solutions of equations (1.1), (1.3). For this we need the following.

The smallest eigen value β_0 of the Dirichlet problem

$$\Delta\omega(x) + \Lambda\omega(x) = 0 \ \ in \ \ \Omega$$
$$\omega(x) = 0 \ \ on \ \ \partial\Omega$$

is positive and the corresponding eigen function $\varphi(x)$ is positive in Ω.

Theorem 2.8. *Let all conditions of Theorem 2.2 be hold. Assume that $|\varphi(x)| \leq M$ for $x \in \bar{\Omega}$. Then every solution of equations (1.1), (1.3) oscillatory in G.*

Proof. Suppose that $u(x,t)$ is a nonoscillatory solution of (1.1),(1.3). Without loss of generality, we may assume that $u(x,t) > 0, u(x,t-\tau) > 0, u(x,t-\delta_1) > 0$ and $u(x,t-\delta_2) > 0$ in $\Omega \times [t_0,\infty)$ for some s $t_0 > 0$. Multiplying both sides of (1.1) by $\varphi > 0$ and integrating with respect to x over Ω, we obtain for $t \geq t_1$,

$$\begin{aligned}
&\int_{\Omega} \frac{d}{dt}\left(r_2(t)\frac{d}{dt}\left(r_1(t)|\frac{dz(x,t)}{dt}|^{\gamma-1}\frac{dz}{dt}\right)\right)\varphi(x)dx \\
&+ \int_{\Omega} p_1(x,t)|u(x,t-\delta_1)|^{\alpha-1}u(x,t-\delta_1)\varphi(x)dx \\
&+ \int_{\Omega} p_2(x,t)|u(x,t-\delta_2)|^{\beta-1}u(x,t-\delta_2)\varphi(x)dx \\
&= \int_{\Omega} a(t)\Delta u(x,t)\varphi(x)dx \\
&+ \int_{\Omega} F(x,t)\varphi(x)dx.
\end{aligned} \tag{2.49}$$

Using Green's formula and boundary condition (1.3), it follows that

$$\begin{aligned}
\int_{\Omega} \Delta u(x,t)\varphi(x)dx &= \int_{\Omega} u(x,t)\Delta\varphi(x)dx \\
&= -\Lambda_0 \int_{\Omega} u(x,t)\varphi(x)dx \leq 0, \ \ t \geq t_1.
\end{aligned} \tag{2.50}$$

Also from (A_2) and Jensen's inequality, it follows that

$$\begin{aligned}
&\int_\Omega p_1(x,t)|u(x,t-\delta_1)|^{\alpha-1}u(x,t-\delta_1)\varphi(x)dx \\
&\geq p_1(t)\int_\Omega (u(x,t-\delta_1))^\alpha \varphi(x)dx \\
&\geq p_1(t)\left(\int_\Omega u(x,t-\delta_1)^\alpha(\varphi(x))^{1-\alpha+\alpha}dx\right) \\
&\geq M^{1-\alpha}p_1(t)\left(U(t-\delta_1)\right)^\alpha,\ t\geq T, \qquad (2.51)\\
&\int_\Omega p_2(x,t)|u(x,t-\delta_2)|^{\beta-1}u(x,t-\delta_2)\varphi(x)dx \\
&\geq M^{1-\beta}p_2(t)\left(U(t-\delta_2)\right)^\beta,\ t\geq T \qquad (2.52)
\end{aligned}$$

Set

$$V(t)=\int_\Omega u(x,t)\varphi(x)dx, \quad t\geq t_1 \qquad (2.53)$$

In view of (2.50)-(2.52) and (A_3), (2.49) yield

$$\begin{aligned}
&\left(r_2(t)\left(r_1(t)\left(z'(t)\right)^\gamma\right)'\right)'+M^{1-\alpha}p_1(t)(V(t-\delta_1))^\alpha \\
&+M^{1-\beta}p_2(t)(V(t-\delta_2))^\beta\leq 0,\ t\geq T. \qquad (2.54)
\end{aligned}$$

Rest of the proof is similar to that of Theorem 2.2 and hence the details are omitted. □

Theorem 2.9. *Let the conditions of Theorem 2.3 hold, then every solution $u(x,t)$ of (1.1), (1.3) is oscillatory in G.*

Theorem 2.10. *Let the conditions of Theorem 2.4 hold, then every solution $u(x,t)$ of (1.1), (1.3) is oscillatory in G.*

Theorem 2.11. *Let the conditions of Theorem 2.5 hold, then every solution $u(x,t)$ of (1.1), (1.3) is oscillatory in G.*

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ISSN(P):2319 − 3786
Malaya Journal of Matematik
ISSN(O):2321 − 5666
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