On coloring of 4-regular graphs

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Abstract
In this paper, we present unique way of coloring of Bi-magic 4-regular graphs with girth 3. It is interesting to note that the relation between bi-magic numbers and chromatic numbers is of importance in the coloring of the vertices.

Keywords
Graph coloring, Chromatic number, Regular graph.

AMS Subject Classification
05C20, 05C90.

1. Introduction
Graph labelings have lately aroused considerable attention. They gave birth to families of graphs with attractive names such as magic, graceful, harmonious, felicitous, sequential and elegant [1]. They exhibited the delicacy of combinatorial constructions and promised interesting applications [2]. Labeled graphs serve as useful models for a broad range of applications such as, coding theory problems, missile guidance codes and convolution codes with optimal auto correlation properties. They facilitate the optimal non-standard encoding of integers. Labeled graphs have also been applied in determining ambiguities in x-ray crystallographic analysis, to design a communication network addressing system, in determining optimal circuit layouts and radio astronomy problems etc., [3, 4]. Most graph labeling methods trace their origin to the one introduced by Rosa [5] or to the one by Graham and Sloane [6].

By $G(p,q)$, we denote a graph having $p$ vertices and $q$ edges, by $V(G)$ and $E(G)$, the vertex-set and the edge set of $G$ respectively. A labeling of graph $G$ is an assignment of labels to either the vertices or the edges of $G$ that induces for each edge $uv$ in the former a label depending on the vertex labels $f(u)$ and in the latter for each vertex $u$ a label depending on the labels of the edges incident with it.

Graph coloring is a special case of graph labeling. It is an assignment of labels traditionally called “colors” to elements of a graph subject to certain constraints. In its simplest form, it is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color. Similarly, an edge coloring assigns a color to each edge so that no two adjacent edges share the same color.

In this paper first section presents basic definitions of labeling and coloring. Second section deals with a 4 regular graphs of girth 3 having bimagic numbers $4n-1$, $5n-1$, but 3 different chromatic numbers. Third section describes with 4 regular bimagic graph of girth 4 having bimagic numbers $4n-2$ and $5n-2$ but 2 different chromatic numbers. In the fourth section, we discuss future work and present the conclusion and in the last section adequate references are given.

2. Preliminaries

Let $G = (V,E)$ be a finite, simple and undirected graph with $p$ vertices and $q$ edges. By a labeling we mean a one-to-one mapping that carries a set of graph elements onto a set of numbers called labels.

Definition 2.1. A regular graph is a graph without loop and without a multiple edges where each vertex has the same degree or valency.

Definition 2.2. A regular graph with vertices of degree $k$ is called $k$-regular graph.

Definition 2.3. A graph $G$ is said to be a vertex magic graph...
if the edges can be labeled with non negative real numbers such that

- different edges have distinct labels and
- the sum of the labels of edges incident to each vertex is constant.

Definition 2.4. A graph $G$ is said to be a bimagic graph if the sum of the labels on the edges incident at the vertices are $k, k_2$ where $k$ and $k_2$ are constants.

Example 2.5.

![Figure 1](image1)

Definition 2.6. Let $G$ be a connected graph. The girth of a graph $G$ is defined as the length of smallest cycle in the graph.

Definition 2.7. A proper vertex coloring of a graph $G$ is an assignment of colors to the vertices of $G$ such that no two adjacent vertices receive the same color.

Definition 2.8. The chromatic number of $G$ is the minimum positive integer $k$ such that there is a proper vertex coloring of $G$ with $k$ colors and it is denoted by $\chi(G)$.

### 3. Theorems

**Theorem 3.1.** For every $n \geq 5$, there exists a 4-regular $(n, 2n)$-bimagic graph of girth 3 with constants $4n - 1$ and $5n - 1$.

**Theorem 3.2.** A 4-regular graph of girth 3, which having bimagic numbers $4n - 1$, $5n - 1$ will have chromatic numbers as follows, where $n$ denotes number of vertices. We present cases depending on the mod values 3, 4 and 5.

**Proof.** Case (i). We consider $n = 0 \pmod 3$. We consider the 4-regular bimagic graph with $n \geq 5$, when $n = 3 + 3k$, $k \geq 1$ its chromatic number is 3. Care should be taken that the coloring is done periodically starting from $v_n$. □

**Example 3.3.** For $n = 9$. (See Fig 2.)

![Figure 2](image2)

**Inference.**

The colors on vertices defined in the map $f : V(G) \to [R, B, G]$. Here $[R, B, G]$ represent 1, 2, 3 colors respectively. Thus we are able to show that $\{v_n, v_1, v_{n-1}\}$ has different colors and constant $5n - 1$, where as the set $\{v_{n-2}, v_{n-3}, \ldots, v_2\}$ has colors repeated. Thus the statement is true.

**Note 3.4.** Here we discusses only $n = 0 \pmod 3$ rest of the vertices 7, 10, 13, 16, ... are explained in terms of mod 4 similarly vertices 8, 11, 14, 17, ... are explained in terms of mod 5 and mod 4. Refer case (ii) case (ii).

**Case (ii).** Consider a 4-regular graphs with the number of vertices $n = 4 + 3k$, $k \geq 1$ has the chromatic number 4, again it is classified into 4 cases.

**Subcase.**

(i) When $n = 7, 10, 31, \ldots$. Since $n = 3 \pmod 4$ coloring is done periodically starting from the vertices $v_n$.

(ii) When $n = 0, 22, 34, \ldots$. Since $n = 2 \pmod 4$ the way of coloring is mentioned subcase (i).

(iii) When $n = 16, 28, 40, \ldots$. Since $n = 0 \pmod 4$ the coloring is mentioned in above 2 cases.

(iv) Exceptional case. When $n = 13, 25, 37, \ldots$. Since $n = 1 \pmod 4$. We know these set of vertices are having chromatic numbers 4, $\{v_n, v_1, v_{n-1}\}$ has different colors and constant $5n - 1$, and coloring is not done periodically.

**Example 3.5.** For $n = 16$. (See Fig 3.)

**Inference.**

The colors on vertices defined in the map $f : V(G) \to [R, B, G, Y]$. Here $[R, B, G, Y]$ represent 1, 2, 3, 4 colors respectively. Thus we based on all the cases 1, 2, 3, 4 we are able to show that $\{v_n, v_1, v_{n-1}\}$ has different colors and constant $5n - 1$ where as the set $\{v_{n-2}, v_{n-3}, \ldots, v_2\}$ has colors repeated. Thus the statement is true.
Consider a 4-regular graphs with the number of vertices
Case (iii).
Subcase.
4. When \( n = 5, 20, 35, 50, \ldots \) Since \( n = 0(\mod 5) \) coloring is done periodically starting from the vertices \( v_n \).
2. When \( n = 8, 23, 38, \ldots \) Since \( n = 3(\mod 5) \) coloring is already mentioned in the subcase (3)
3. When \( n = 11, 26, 39, \ldots \) Since \( n = 1(\mod 5) \) coloring is already mentioned in the subcase (1), (2), (3)
4. When \( n = 14, 29, 44, \ldots \) Since \( n = 4(\mod 5) \) coloring is mentioned in the above subcases (2), (3) and (4)
5. When \( n = 17, 32, \ldots \) Since \( n = 2(\mod 5) \) coloring is mentioned in the above subcases (1), (2), (3), (4).

Example 3.7. For \( n = 16 \). (See Fig 4.)

Inference. The colors on vertices defined in the map \( f : V(G) \rightarrow \{R, B, G, Y, O\} \). Here \( \{R, B, G, Y, O\} \) represent 1,2,3,4,5 colors respectively. Thus we based on all the cases 1, 2, 3, 4, 5, we are able to show that \( \{v_1, v_n, v_{n-1}\} \) has different colors and constant \( 5n - 1 \) where as the set \( \{v_{n-2}, v_{n-3}, \ldots v_2\} \) has colors repeated. Thus the statement is true.

Note 3.8. Here we discussed only \( n = 0(\mod 5), n = 1(\mod 5), n = 2(\mod 5), n = 3(\mod 5) \) and \( n = 4(\mod 5) \). Rest of the vertices are explained in case (i) and case (ii).

4. Conclusion

Thus we classified in a 4-regular graph of girth 3, \( v_n \) the vertices \( \{v_1, v_n, v_{n-1}\} \) has same bimagic number \( 5n - 1 \) but different colors. Coloring also done periodically starting from the vertex \( v_n \).

References