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Classical and partial symmetries of the Benney equation

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Abstract

Lie symmetry group method is applied to study Benney equation. The symmetry group and its optimal system are given,and group invariant solutions associated to the symmetries are obtained. Also the structure of the Lie algebra symmetries is determined. Mainly, we have compared one of the resolved analitical solutions of the Benney equation with one of it's numerical solutions which is obtained via homotopy perturbation method in [4].

Keywords: Lie group analysis, Partial symmetry, Symmetry group, Optimal system, Invariant solution, Benney equation.

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1 Introduction

In the past decades, both mathematicians and physicists have made efforts in the study of exact solutions of partial differential equations. The theory of Lie symmetry groups of differential equations was developed by Sophus Lie [1]. The symmetry group methods provide an ultimate arsenal for analysis of differential equations and is of great importance to understand and to construct solutions of differential equations. Reduction of order of partial differential equations or transformed to ordinary differential equations. According to the standard definition partial symmetries of $\triangle = 0$ as Lie-point invertible transformations T such that there is a non-empty subset $P \subset S_\triangle$such that $T(P) = P$, i.e.such that there is a subset of solutions to $\triangle = 0$ which are transformed into one another. We discuss how to determine both partial symmetries and the invariant set $P \subset S_\triangle$, and show that our procedure is effective by Benney equation.

The homotopy perturbation method (HPM) was established by Ji-Huan He in 1999. In this method, the solution is considered as the summation of an infinite series, which usually converges rapidly to the exact solution. Using the homotopy technique from topology, a homotopy is constructed with an embedding parameter $p \in [0, 1]$, which is considered as a small parameter. The approximations obtained by the homotopy perturbation method are uniformly valid not only for small parameters, but also for very large parameters.

Benney's Equation is

$$u_t + uu_x + u_{xx} + \delta u_{xxx} + u_{xxxx} = 0, \tag{1.1}$$

where u is considered to be periodic in x as an apology for the infinite domain. This equation seems rich in character and has been derived in several physical contexts including the flow of thin liquid films (Benney 1966, Topper and Kawahara 1977).

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2 Lie Symmetry Methods

In this part , we use general for determining symmetries for any system PDE. For this method, use the general case of a nonlinear system of PDE of order n^{th} in p independent and q dependent variables is given as a system of equations:

$$\triangle_\nu(x, u^{(n)}) = 0, \qquad \nu = 1, \cdots, \ell \tag{2.2}$$

via $x = (x^1, \cdots, x^p), u = (u^1, \cdots, u^q)$ and the derivatives of u whit respect to x up to n, and u^n respect all derivatives of u of all orders from 0 to n.

We consider a one-parameter Lie group of infinitesimal transformationsacting on the Jet space of the system (2.2)

$$\tilde{x}^i = x^i + \varepsilon\xi^i(x, u) + O(\varepsilon^2), \qquad \tilde{u}^j = x^j + \varepsilon\eta^j(x, u) + O(\varepsilon^2) \tag{2.3}$$

where ε is the parameter of the transformation and ξ^i, η^j are the infinitesimals of the transformations for the independent and dependent variables, respectively. The infinitesimal generator v associated with the above group of transformations can be written as

$$v = \sum_{i=1}^{p} \xi^i(x, u)\frac{\partial}{\partial x^i} + \sum_{\alpha=1}^{q} \eta_j(x, u)\frac{\partial}{\partial u^j} \tag{2.4}$$

A symmetry of a differential equation is a transformation which maps solutions of the equation to other solutions. (be a vector field on an open subset $M \subset X \times U$). The invariance of the system (2.2) under the infinitesimal transformations leads to the invariance condition

$$\mathrm{pr}^{(n)}v[\triangle_\nu(x, u^{(n)})] = 0, \quad \triangle_\nu(x, u^n) = 0, \qquad \nu = 1, \cdots, \ell, \tag{2.5}$$

where $\mathrm{pr}^{(n)}$ is the n^{th} order prolongation of the infinitesimal generator given by

$$pr^{(n)}v = v + \sum_{\alpha=1}^{q}\sum_{J} \phi_J^\alpha(x, u^{(n)})\frac{\partial}{\partial u_J^\alpha} \tag{2.6}$$

defined on the corresponding jet space $M^{(n)} \subset X \times U^{(n)}$. $(J = (j_1, \cdots, j_k), 1 \leq j_k \leq p, 1 \leq k \leq n)$ with coefficient

$$\phi_\alpha^J(x, u^{(n)}) = D_J(\phi_\alpha - \sum_{i=1}^{p} \xi^i u_i^\alpha) + \sum_{i=1}^{p} \xi^i u_{J,i}^\alpha \tag{2.7}$$

where $u_i^\alpha = \dfrac{\partial u_\alpha}{\partial x_i}$ and $u_{J,i}^\alpha = \dfrac{\partial u_J^\alpha}{\partial x_i}$.

3 Lie Symmetry of the Benney equation

We consider the equation of Benney (1) (with 2 independent variable (x, t) and 1 dependent $u(x, t)$) where x, t are variables, u is a function and δ is a constant.

Let $v = \xi(x, t, u)D_x + \tau(x, t, u)D_t + \phi(x, t, u)D_u$ be a vector field on $X \times U$,$(X = (x, t))$. We wish to determine all possible coefficient function ξ, τ and ϕ so that the corresponding one-parameter group $\exp(\varepsilon v)$ is a symmetry group of the Benney equation, we need to know the fourth prolongation

$$pr^{(4)}v = v + \phi^x\frac{\partial}{\partial u_x} + \phi^t\frac{\partial}{\partial u_t} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \cdots + \phi^{tttt}\frac{\partial}{\partial u_{tttt}}, \tag{3.8}$$

Table 1: Commutation relations satisfied by infinitesimal generators

$[\,,\,]$	v_1	v_2	v_3
v_1	0	v_3	0
v_2	$-v_3$	0	0
v_3	0	0	0

Table 2: Adjoint relations satisfied by infinitesimal generators

$[\,,\,]$	v_1	v_2	v_3
v_1	v_1	$v_2 - \varepsilon v_3$	v_3
v_2	$v_1 + \varepsilon v_3$	v_2	v_3
v_3	v_1	v_2	v_3

of v, with the coefficients :

$$\begin{aligned} \phi^x &= D_x\phi + D_u\phi u_x - u_x D_x\xi - D_u\xi u_x^2 - u_t D_x\tau - u_t D_u\tau u_x, \\ \phi^t &= D_t\phi + D_u\phi u_t - u_x D_t\xi - u_x D_u\xi u_t - u_t D_t\tau - D_u\tau u_t^2, \\ \phi^{xx} &= D_x^2\phi + 2D_{ux}\phi u_x - \cdots - 2\,u_{xt} D_u\tau u_x, \quad \text{and so on.} \end{aligned} \tag{3.9}$$

Applying $pr^{(4)}v$ to Benney equation , we find the infinitesimal criterion. determining equations yields:

$$\begin{array}{lllll} D_x\phi = 0, & D_x\tau = 0, & D_x\xi = 0, & D_t\phi = 0, & D_t\tau = 0, \\ D_u\phi = 0, & D_u\tau = 0, & D_u\xi = 0, & D_t\xi = \phi, & (\delta \neq D_t\tau,\ \ \delta \neq 0) \end{array} \tag{3.10}$$

The solution of the above system gives the following coefficients of the vector field v:

$$\phi\,(x,t,u) = C_2, \quad \tau\,(x,t,u) = C_1, \quad \xi\,(x,t,u) = C_2\,t + C_3, \tag{3.11}$$

Where $\{C_1, C_2, C_3\}$ are arbitrary constants, thus the Lie algebra $\mathbf{g}$ of the Benney equation is spanned by the three vector fields

$$v_1 = D_t, \qquad v_2 = tD_x + D_u, \qquad v_3 = D_x, \tag{3.12}$$

The one-parameter groups G_i generated by the base of $\mathbf{g}$ are given in the following table

$$\begin{aligned} g_1 : (x,t,u) &\longmapsto (x, t+\varepsilon, u), \\ g_2 : (x,t,u) &\longmapsto (\varepsilon t + x, t, \varepsilon + u), \\ g_3 : (x,t,u) &\longmapsto (\varepsilon + x, t, u). \end{aligned} \tag{3.13}$$

Since each group G_i is a symmetry group and if $u = f(x,t)$ is a solution of the Benney equation, so are the functions

$$u^{(1)} = f(x, t-\varepsilon), \quad u^{(2)} = f(x - t\varepsilon, t) - \varepsilon, \quad u^{(3)} = f(x-\varepsilon, t), \tag{3.14}$$

where ε is a real number.

Also their commutator table is and also table Adjoint with (i,j)-th entry indicating $\mathrm{Ad}(\exp(\varepsilon v_i)v_j)$: where ε is a real number. Here we can find the general group of the symmetries by considering a general linear combination "$c_1v_1 + c_2v_2 + c_3v_3$" of the given vector fields. In particular if $\mathbf{g}$ is the action of the symmetry group near the identity, it can be represented in the form $\mathbf{g} = \exp(c_1v_1)\cdots..\exp(c_3v_3)$.

4 Optimal system of the Benney equation

This part using the adjoint representation for classifying group-invariant solutions. let G a Lie group . An *optimal system* of subgroup is a list of conjugate equivalent subgroups with the property that any other subgroup is conjugate to precisely one subgroup in the list. Similarly, a list of subalgebras forms an*optimal system* if every subalgebra of $\mathbf{g}$ is equivalent to a unique member of the list under some element of the adjoint representation:

$$\tilde{h} = \text{Ad}\ \ g(h), g \in G. \tag{4.15}$$

We finding exact solutions and performing symmetry reductions of differential equations. As any transformation in the symmetry group maps a solution to another solution, it is sufficient to find invariant solutions which are not related by transformations in the full symmetry group, this has led to the concept of an optimal system. For one- dimensional subalgebras, this classification problem is essentially the same as the problem of classifying the orbits of the adjoint representation.

The adjoint action is given by the Lie series

$$\text{Ad}(\exp(\varepsilon v_i))v_j = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}(ad\ v_i)^n(v_j), \tag{4.16}$$

where $[v_i, v_j]$ is a commutator for the Lie algebra, ε is a parameter, and $i, j = 1,2,3$.

Let $F_i^\varepsilon : g \longrightarrow g$ defined by $v \longrightarrow \text{Ad}(\exp(\varepsilon v_i)v)$ is a linear map, for $i = 1,2,3$. The matrices M_i^ε of F_i^ε, $i = 1,2,3$ with respect to basis $\{v_1, v_2, v_3\}$

$$M_1^\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\varepsilon \\ 0 & 0 & 1 \end{pmatrix}, \quad M_2^\varepsilon = \begin{pmatrix} 1 & 0 & \varepsilon \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad M_3^\varepsilon = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{4.17}$$

by acting above matrices on a vector field v alternatively we can show that a one-dimensional optimal system of $\mathbf{g}$ is given by

$$Y_1 = v_3, \quad Y_2 = a_1 v_1 + v_2, \quad (a_1 \in \mathbb{R}), \quad Y_3 = v_1, \tag{4.18}$$

5 partial symmetries of Benney equation

Let us consider a general differential problem, given in the form of a system of ℓ differential equations, and briefly denoted, as usual, by

$$\Delta = \Delta(x, u^{(m)}) = 0, \tag{5.19}$$

where $\Delta = (\Delta_1, \cdots, \Delta_\ell)$ are smooth functions involving p independent variables $x = (x_1, \cdots, x_p) \in \mathbb{R}^p$ and q dependent unknown variables $u = (u_1, \cdots, u_q) \in \mathbb{R}^q$ together with the derivatives of the u_α with respect to the $x_i(\alpha = 1, \cdots, q; i = 1, \cdots, p)$ up to some order m. Let

$$X = \xi_i \frac{\partial}{\partial x_i} + \phi_\alpha \frac{\partial}{\partial u_\alpha}, \qquad \xi_i = \xi_i(x,u), \qquad \phi_\alpha = \phi_{\alpha(x,u)}, \tag{5.20}$$

be a given vector field, where ξ_i and ϕ_α are p+q smooth functions. if X is a exact symmetry then $pr^{(m)}X\ \ \Delta = 0$, We now assume that the vector field X is not a symmetry of Δ, hence $pr^{(m)}X\ \ \Delta \neq 0$, let $\Delta^{(1)} = pr^{(m)}X\ \ \Delta$ with this define $\Delta^{(1)}$ of order m' $(m' \leq m)$,

We consider two vector field for this equation.

1) We consider the generic scaling vector fields

$$X_1 = auD_u + bxD_x + ctD_t, \qquad (a, b, c\ \varepsilon\mathbb{R}) \tag{5.21}$$

that is a not a symmetry for Δ.

$$\Delta = u_t + uu_x + u_{xx} + \delta u_{xxx} + u_{xxxx} = 0. \tag{5.22}$$

Applying the fourth prolongation of X_1 on Δ , we obtain

$$\begin{aligned} \Delta^{(1)} &= \mathrm{pr}^{(4)} X_1 \Delta, \\ \Delta^{(1)} &= (2b-a)uu_x + (a-c)u_t + (a-2b)u_{xx} + (a\delta - 3\delta b)u_{xxx} + (a-4b)u_{xxxx}. \end{aligned} \tag{5.23}$$

We rewrite this as

$$\hat{\Delta^{(1)}} = Auu_x + Bu_t - Au_{xx} + Cu_{xxx} + Eu_{xxxx}, \qquad (A, B, C, E \in \mathbb{R}) \tag{5.24}$$

reduction to ODE. If $\langle A = B = E = 0$ and $C \neq 0\rangle$ reduce to $u_{xxx} = 0$, solve this ODE we obtain

$$u(x,t) = \alpha(t) + \beta(t)x + \gamma(t)x^2 \tag{5.25}$$

substituting this into the Δ equation we obtain

$$\beta'(t) + \beta^2(t) = 0, \qquad \alpha'(t) + \alpha(t)\beta(t) = 0, \qquad \gamma(t) = 0, \tag{5.26}$$

from (26) we obtain :

$$\beta(t) = \frac{1}{t + C_1}, \tag{5.27}$$

and from (27)

$$\alpha(t) = \frac{C_2}{t + C_1}, \tag{5.28}$$

in this case we are

$$u(x,t) = \frac{C_2}{t + C_1} + \frac{1}{t + C_1}x. \tag{5.29}$$

2) Consider $X_2 = D_t + D_u$ a arbitrary vector field , Applying the fourth prolongation of X_2 on Δ, we obtain

$$\Delta^{(1)} = \mathrm{pr}^{(4)} X_2 \, \Delta = u_x, \qquad u_x = 0, \tag{5.30}$$

from above equation we obtain :

$$u(x,t) = \alpha(t), \tag{5.31}$$

via substituting $u(x,t) = \alpha(t)$ to Δ we are

$$\alpha'(t) = 0, \tag{5.32}$$

via solving above equation, we obtain :

$$\alpha(t) = C_0, \tag{5.33}$$

In finally we are

$$u(x,t) = C_0. \tag{5.34}$$

6 Compare Numerical Solution and Analytical Solution

In this part we use Homotopy Perturbation Method (HPM), the homotopy perturbation method provides an effective procedure for exact and numerical solutions of partial differential equations. Now we use article's [4] as flows.

Numerical solution :

$$u^*(x,t) = \frac{x}{t} - \frac{1}{4} e^{\sqrt[3]{2}x - 2\delta t}, \tag{6.35}$$

Analytical Solution in generic scaling :

$$u(x,t) = \frac{C_2}{t+C_1} + \frac{x}{t+C_1}, \tag{6.36}$$

With the right choice C_1 and C_2:

$$C_1 = .004801455707 \quad \text{and} \quad C_2 = -.01233867027, \quad (t=1, \delta=1). \tag{6.37}$$

We have :

$$\begin{aligned} u^*(x,1) &= x - \frac{1}{4} 2.718281828^{\sqrt[3]{2}x-2}, & (6.38)\\ u(x,1) &= -.01227970978 + .9952214878\, x. & (6.39) \end{aligned}$$

Now we plot $u(x,1), u^*(x,1)$ to compare

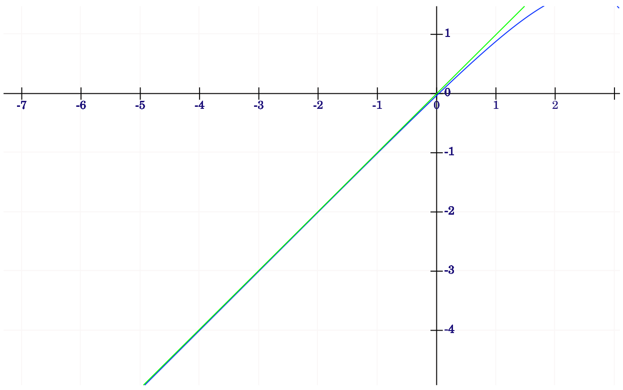

Figure 1: $u(x,1)$(green), $u^*(x,1)$(blue), $\delta = 1$

$$\forall x \in [-2,-1] \Longrightarrow \mid u^*(x,1) - u(x,1) \mid \leq .0131. \tag{6.40}$$

7 Conclusion

In this paper, by applying the criterion of invariance of the equation under the infinitesimal prolonged infinitesimal generators, we find the most general Lie point symmetries group of the Benny equation. Also, we have constructed the optimal system of one-dimensional sub algebras of Benny equation. We find a analytic solution by partial symmetry and compare with numerical solution in order to calculation error.

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