Malaya J. Mat. 3(2)(2015) 191–201

Malaya Journal of Matematik

www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

On local attractivity of nonlinear functional integral equations via measures of noncompactness

B. C. Dhage[a,*], S. B. Dhage[a], S. K. Ntouyas[b,†] and H. K. Pathak[c,‡]

[a] *Kasubai, Gurukul Colony, Ahmedpur-413 515, Dist: Latur, Maharashtra, India.*

[b] *Department of Mathematics, University of Ioannina, 451 10 Ioannina, Greece.*

[c] *School of Studies in Mathematic, Pt. Ravishankar Shukla University, Raipur (C.G.) 492010, India.*

Abstract

In this paper, we prove the local attractivity of solutions for a certain nonlinear Volterra type functional integral equations. We rely on a measure theoretic fixed point theorem of Dhage (2008) for nonlinear $\mathcal{D}$-set-contraction in Banach spaces. Finally, we furnish an example to validate all the hypotheses of our main result and to ensure the existence and ultimate attractivity of solutions for a numerical nonlinear functional integral equation.

Keywords: Measure of noncompactness, fixed point theorem, functional integral equation, attractivity of solutions.

2010 MSC: 45G10, 45G99.

1 Introduction

The last three decades witnessed the active area of research in the connotation of measure theoretic fixed point theory and its applications to the problems of nonlinear differential and integral equations. The novelty of this approach lies in the advantage that along with existence we obtain some additional information about some characterizations of the solutions automatically. Local and global stability of the solutions of certain functional integral equations is discussed via measures of noncompactness by many researchers (see, for instance, Banas and Goebel [3], Banas and Rzepka [4], Dhage [8, 9], Dhage and Ntouyas [13] and the references therein). Very recently, Dhage [8] derived an abstract fixed point result more general than Darbo [5] fixed point theorem using the notion of measures of noncompactness and applied to stability problem of certain nonlinear functional integral equations. See Dhage and Lakshmikantham [12] and the references therein. Inspired or motivated by the idea of $\mathcal{D}$-functions that given in the examples of Dhage [10, 11], we prove in this paper the local attractivity of solutions for a certain nonlinear Volterra type functional integral equations via Dhage's measure theoretic fixed point theorem.

We now consider the following generalized nonlinear functional integral equation (in short GNFIE)

$$x(t) = u(t, x(t)) + p\Big(\int_0^{\gamma(t)} f(t, s, x(\theta(s)))ds, \int_0^{\sigma(t)} g(t, s, x(\eta(s)))ds\Big) \tag{1.1}$$

*Corresponding author.

†Member of Nonlinear Analysis and Applied Mathematics (NAAM)-Research Group at King Abdulaziz University, Jeddah, Saudi Arabia.

‡This research was supported by the University Grant Commission, New Delhi, India (MRP-MAJOR-MATH-2013-18394).

E-mail address: bcdhage@gmail.com (B. C. Dhage), sbdhage4791@gmail.com(S. B. Dhage), sntouyas@uoi.gr (S. K. Ntouyas), hkpathak05@gmail.com, (H. K. Pathak).

for $t \in \mathbb{R}^+ = [0,\infty) \subset \mathbb{R}$, where $u : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $f, g : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}$, $p : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\gamma, \theta, \sigma, \eta : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions.

Notice that the functional integral equation (1.1) is "general" in the sense that it includes several classes of known integral equations discussed in the literature (see Banas and Rzepka [4], Dhage [8], Dhage and Ntouyas [13], O'Regan and Meehan [15], Krasnoselskii [14], Väth [16], Dhage [7, 8] and the references therein). In this paper, we intend to obtain solution of GNFIE (1.1) in the space $BC(\mathbb{R}^+, \mathbb{R})$ of all bounded and continuous real-valued functions on $\mathbb{R}^+$. We use a fixed point theorem of Dhage [8] involving general measures of noncompactness to prove the existence and ultimate attractivity of solutions of GNFIE (1.1) under certain new conditions. The results of this paper are new to the theory of nonlinear differential and integral equations.

2 Auxiliary Results

This section is devoted to presenting a few auxiliary results needed in the sequel. Assume that E is a Banach space with the norm $\|\cdot\|$ and the zero element θ. Denote by $B[x,r]$ the closed ball centered at x and with radius r. If X, Y are arbitrary subsets of E then the symbols λX and $X + Y$ stand for the usual algebraic operations on those sets. Moreover, we write $\overline{X}$, $\overline{co}\, X$ to denote the closure and the closed convex hull of X, respectively.

Further, let $\mathcal{P}_p(E)$ denote the class of all nonempty subsets of E with a property p. Here p may be p =closed (cl, in short), p =bounded (bd, in short), p =relatively compact (rcp, in short) etc. Thus, $\mathcal{P}_{cl}(E), \mathcal{P}_{bd}(E), \mathcal{P}_{cl,bd}(E)$ and $\mathcal{P}_{rcp}(E)$ denote respectively the classes of closed, bounded, closed and bounded and relatively compact subsets of E.

The axiomatic way of defining the concept of the measure of noncompactness has been adopted in several papers in the literature. See Akhmerov *et al.* [2], Deimling [6], Väth [16] and Zeidler [17]. In this paper, we adopt the following axiomatic definition of the measure of noncompactness in a Banach space given in Banas and Goebel [3] and Dhage [7, 8].

Definition 2.1. *A mapping $\mu : \mathcal{P}_{bd}(E) \to \mathbb{R}^+$ is called the measure of noncompactness in E if it satisfies the following conditions:*

1^o *The family* $\ker\, \mu = \{X \in \mathcal{P}_{bd}(E) : \mu(X) = 0\}$ *is nonempty and* $\ker\, \mu \subset \mathcal{P}_{rcp}(E)$.

2^o $\mu(\overline{X}) = \mu(X)$.

3^o $\mu(\overline{co}\, X) = \mu(X)$.

4^o $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$.

5^o *If $\{X_n\}$ is a decreasing sequence of sets in $\mathcal{P}_{cl,bd}(E)$ such that $\lim_{n\to\infty} \mu(X_n) = 0$, then the intersection set $\overline{X}_\infty = \bigcap_{n=1}^{\infty} X_n$ is nonempty.*

The family $\ker \mu$ described in 1^o is said to be *the kernel of the measure of noncompactness* μ. We refer to [2, 3, 4, 6, 16, 17] for further facts concerning the measures of noncompactness and their properties. Let us only observe that the intersection set X_∞ is a member of the family $\ker\, \mu$. Indeed, since $\mu(X_\infty) \leq \mu(X_n)$ for any n, we infer that $\mu(X_\infty) = 0$. In view of 1^o this yields that $X_\infty \in \ker\, \mu$.

A measure μ of noncompactness is said to be sublinear if

6^o $\mu(X + Y) \leq \mu(A) + \mu(B)$ for all $X, Y \in \mathcal{P}_{bd}(E)$, and

7^o $\mu(\lambda X) \leq |\lambda| \mu(X)$ for all $\lambda \in \mathbb{R}$ and $X \in \mathcal{P}_{bd}(E)$.

Let $E = BC(\mathbb{R}^+, \mathbb{R})$ be the space of all continuous and bounded functions on $\mathbb{R}^+$ and define a norm $\|\cdot\|$ in E by

$$\|x\| = \sup\{|x(t)| : t \geq 0\}.$$

Clearly E is a Banach space with this supremum norm. Let us fix a bounded subset A of E and a positive real number T. For any $x \in A$ and $\epsilon \geq 0$, denote by $\omega^T(x, \epsilon)$, the modulus of continuity of x on the interval $[0, T]$ defined by

$$\omega^T(x, \epsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, T], |t - s| \leq \epsilon\}.$$

Moreover, let

$$\omega^T(A, \epsilon) = \sup\{\omega^T(x, \epsilon) : x \in A\},$$

$$\omega_0^T(A) = \lim_{\epsilon \longrightarrow 0} \omega^T(A, \epsilon),$$

$$\omega_0(A) = \lim_{T \longrightarrow \infty} \omega_0^T(A).$$

By $A(t)$ we mean a set in $\mathbb{R}$ defined by $A(t) = \{x(t) | x \in A\}$ for $t \in \mathbb{R}^+$. We denote diam $(A(t)) = \sup\{|x(t) - y(t)| : x, y \in A\}$. Finally we define a function μ on $\mathcal{P}_{bd}(E)$ by the formula

$$\mu(A) = \omega_0(A) + \limsup_{t \longrightarrow \infty} \operatorname{diam}(A(t)). \tag{2.2}$$

It has been shown in Banas and Goebel [3] that μ is a sublinear measure of noncompactness in E. From the definition of the measure μ, it is clear that the thickness of the bundle of functions $A(t)$ tends to zero as t tends to ∞. This particular characteristic of μ has been utilized in formulating the main existence and attractiivity result of this paper.

Before going to the key tool used in this paper, we recall the following useful definition introduced by Dhage [8].

Definition 2.2. *A mapping $\mathcal{T} : E \to E$ is called $\mathcal{D}$-set-Lipschitz if there exists a upper semi-continuous nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\mu(\mathcal{T}(A)) \leq \varphi(\mu(A)$ for all $A \in \mathcal{P}_{bd}(E)$ with $\mathcal{T}(A) \in P_{bd}(E)$, where $\varphi(0) = 0$. The function φ is sometimes called a $\mathcal{D}$-function of $\mathcal{T}$ on E. Especially when $\varphi(r) = kr, k > 0$, U is called a k-set-Lipschitz mapping and if $k < 1$, then $\mathcal{T}$ is called a k-set-contraction on E. Further, if $\varphi(r) < r$ for $r > 0$, then $\mathcal{T}$ is called a nonlinear $\mathcal{D}$-set-contraction on E.*

Lemma 2.1 (Dhage [8]). *If φ is a $\mathcal{D}$-function with $\varphi(r) < r$ for $r > 0$, then $\lim_{n \to \infty} \varphi^n(t) = 0$ for all $t \in [0, \infty)$ and vice-versa.*

Using Lemma 2.1, Dhage [8] proved the following important result.

Theorem 2.1. *Let C be a closed, convex and bounded subset of a Banach space E and let $\mathcal{T} : C \to C$ be a continuous and nonlinear $\mathcal{D}$-set-contraction. Then $\mathcal{T}$ has a fixed point.*

Remark 2.1. Let us denote by $\operatorname{Fix}(\mathcal{T})$ the set of all fixed points of the operator $\mathcal{T}$ which belong to C. It can be shown that the set $\operatorname{Fix}(\mathcal{T})$ existing in Theorem 2.1 belongs to the family ker μ. Indeed, if $\operatorname{Fix}(\mathcal{T}) \notin$ ker μ, then $\mu(\operatorname{Fix}(\mathcal{T})) > 0$ and $\mathcal{T}(\operatorname{Fix}(\mathcal{T})) = \operatorname{Fix}(\mathcal{T})$. Now from nonlinear set-contractivity it follows that $\mu(\mathcal{T}(\operatorname{Fix}(\mathcal{T}))) \leq \phi(\mu(\operatorname{Fix}(\mathcal{T})))$ which is a contradiction since $\phi(r) < r$ for $r > 0$. Hence Fix $(\mathcal{T}) \in$ ker μ. This particular characteristic has been utilized in our study of local attractivity of the solutions of nonlinear integral equations.

3 Local Attractivity Results

In this section we prove our main existence and attractivity results for the GNFIE (1.1) under some suitable conditions. We need the following definition in what follows. Let us assume that $E = BC(\mathbb{R}^+, \mathbb{R})$ and let Ω be a subset of E. Let $\mathcal{T} : E \to E$ be an operator and consider the operator equation in E,

$$\mathcal{T}x(t) = x(t) \text{ for all } t \in \mathbb{R}^+. \tag{3.3}$$

Below we give an attractivity characterizations of the solutions for the operator equation (3.3) on $\mathbb{R}^+$.

Definition 3.3. *We say that solutions of the equation (3.3) are locally ultimately attractive if there exists a closed ball $B[x_0, r_0]$ in the space $BC(\mathbb{R}^+, \mathbb{R})$ for some $x_0 \in BC(\mathbb{R}^+, \mathbb{R})$ such that, for arbitrary solutions $x = x(t)$ and $y = y(t)$ of equation (3.3) belonging to $B[x_0, r_0] \cap \Omega$, we have*

$$\lim_{t\to\infty} (x(t) - y(t)) = 0. \tag{3.4}$$

In case the limit (3.2) is uniform with respect to the set $B[x_0, r_0] \cap \Omega$, i.e., for each $\epsilon > 0$ there exists $T > 0$ such that

$$|x(t) - y(t)| \leq \epsilon \tag{3.5}$$

for all solutions $x, y \in B[x_0, r_0] \cap \Omega$ of (3.3) and for $t \geq T$, we will then say that solutions of equation (3.3) are uniformly locally ultimately attractive on $\mathbb{R}^+$.

We consider the following set of hypotheses in the sequel.

(H_0) The functions $\gamma, \theta, \sigma, \eta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ are continuous and $\lim_{t\longrightarrow\infty} \gamma(t) = 0$.

(H_1) The function $u : \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exists a continuous and nondecreasing function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$|u(t, x) - u(t, y)| \leq \varphi(|x - y|)$$

for each $t \in \mathbb{R}^+$ and $x, y \in \mathbb{R}$. Moreover, we assume $\varphi(r) < r$ for $r > 0$.

(H_2) The function $U : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ defined by $U(t) = |u(t, 0)|$ is bounded with $c_1 = \sup_{t\geq 0} U(t)$.

(H_3) The function $f : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exist a constant $k > 0$ and a function φ as appears in (H_1) such that

$$|f(t, s, x) - f(t, s, y)| \leq k\, \varphi(|x - y|)$$

for $t, s \in \mathbb{R}^+$ and $x, y \in \mathbb{R}$.

(H_4) The function $F : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ defined by $F(t) = \displaystyle\int_0^{\gamma(t)} |f(t, s, 0)| ds$ is bounded with $c_2 = \sup_{t\geq 0} F(t)$.

(H_5) The function $g : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R} \longrightarrow \mathbb{R}$ is continuous and there exist functions $a, b : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ satisfying

$$|g(t, s, x)| \leq a(t)b(s)$$

for $t, s \in \mathbb{R}^+$ and $x \in \mathbb{R}$. Moreover, $\displaystyle\lim_{t\longrightarrow\infty} a(t) \int_0^{\sigma(t)} b(s)ds = 0$.

(H_6) The function $p : \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfies the following condition

$$|p(t_1, t_2) - p(t_1', t_2')| \leq |t_1 - t_1'| + |t_2 - t_2'|$$

for all $t_1, t_2, t_1', t_2' \in \mathbb{R}$. Moreover, $p(0, 0) = 0$.

Remark 3.2. *Since the hypothesis (H_0) holds, there exists a constant $c_0 > 0$ such that $c_0 = \sup_{t\geq 0} \gamma(t)$. Similarly, since (H_5) holds, the function $v : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $v(t) = a(t) \int_0^{\sigma(t)} b(s)ds$ is continuous and the number $c_3 = \sup_{t\geq 0} v(t)$ exists.*

Theorem 3.2. *Assume that the hypotheses $(H_0) - (H_6)$ hold. Further if there exists a positive solution r_0 of the inequality*

$$(1 + c_0 k)\varphi(r) + q \leq r, \tag{3.6}$$

where q is the constant defined by $q = \sum_{i=1}^{3} c_i$, then the GNFIE (1.1) has a solution and the solutions are uniformly locally ultimately attractive on $\mathbb{R}^+$.

Proof. Now consider the closed ball $B[0, r_0]$ in E centered at origin of radius r_0. Define the mapping $\mathcal{T}$ on E by

$$\mathcal{T}x(t) = u(t, x(t)) + p\Big(\int_0^{\gamma(t)} f(t, s, x(\theta(s)))ds, \int_0^{\sigma(t)} g(t, s, x(\eta(s)))ds\Big) \tag{3.7}$$

for $t \in \mathbb{R}^+$. We shall show that the map $\mathcal{T}$ satisfies all the conditions of Theorem 3.1 on E.

Step I: First we show that $\mathcal{T}$ defines a mapping $\mathcal{T} : E \longrightarrow E$. Since p, q, γ, σ are continuous, $\mathcal{T}x$ is continuous and hence it is measurable on $\mathbb{R}^+$ for each $x \in E$. As $\theta(\mathbb{R}^+) \subseteq \mathbb{R}^+$, we have $\max_{t\geq 0} |x(\theta(t))| \leq \max_{t\geq 0} |x(t)|$. On the other hand, hypotheses $(H_0) - (H_3)$ and (H_5) imply that

$$\begin{aligned}
|\mathcal{T}x(t)| &\leq |u(t, x(t))| + \Big|p\Big(\int_0^{\gamma(t)} f(t, s, x(\theta(s)))ds, \int_0^{\sigma(t)} g(t, s, x(\eta(s)))ds\Big) - p(0, 0)\Big| \\
&\leq |u(t, x(t)) - u(t, 0)| + |u(t, 0)| + \Big|\int_0^{\gamma(t)} f(t, s, x(\theta(s)))ds\Big| + \Big|\int_0^{\sigma(t)} g(t, s, x(\eta(s)))ds\Big| \\
&\leq \varphi(|x(t)|) + |u(t, 0)| + \int_0^{\gamma(t)} |f(t, s, x(\theta(s)))|ds + \int_0^{\sigma(t)} |g(t, s, x(\eta(s)))|ds \\
&\leq \varphi(|x(t)|) + |u(t, 0)| + \int_0^{\gamma(t)} |f(t, s, x(\theta(s))) - f(t, s, 0)|ds + \int_0^{\gamma(t)} |f(t, s, 0)|ds + \int_0^{\sigma(t)} a(t)b(s)ds \\
&\leq \varphi(\|x\|) + U(t) + k\gamma(t)\varphi(\|x\|) + c_2 + v(t) \\
&\leq (1 + c_0 k)\varphi(\|x\|) + q,
\end{aligned}$$

for all $t \in \mathbb{R}^+$. Taking supemum over t, we obtain,

$$\|\mathcal{T}x\| \leq (1 + c_0 k)\varphi(\|x\|) + q \leq r. \tag{3.8}$$

From (3.7), we deduce that $\mathcal{T}x \in E$ and $\mathcal{T}$ defines a mapping $\mathcal{T} : B[0, r_0] \to B[0, r_0]$.

Step II: We show that $\mathcal{T}$ is continuous on $B[0, r_0]$. Let $\epsilon > 0$ be given and let $x, y \in B[0, r_0]$ be such that $\|x - y\| \leq \epsilon$. Then by hypotheses $(H_1) - (H_5)$

$$\begin{aligned}
|\mathcal{T}x(t) - \mathcal{T}y(t)| &\leq |u(t, x(t)) - u(t, y(t))| + \Big|p\Big(\int_0^{\gamma(t)} f(t, s, x(\theta(s)))ds, \int_0^{\sigma(t)} g(t, s, x(\eta(s)))ds\Big) \\
&\quad - p\Big(\int_0^{\gamma(t)} f(t, s, y(\theta(s)))ds, \int_0^{\sigma(t)} g(t, s, y(\eta(s)))ds\Big)\Big| \\
&\leq \varphi(|x(t) - y(t)|) + \Big|\int_0^{\gamma(t)} [f(t, s, x(\theta(s))) - f(t, s, y(\theta(s)))]ds\Big| \\
&\quad + \Big|\int_0^{\sigma(t)} [g(t, s, x(\eta(s))) - g(t, s, y(\eta(s)))]ds\Big| \\
&\leq \varphi(|x(t) - y(t)|) + \int_0^{\gamma(t)} |f(t, s, x(\theta(s))) - f(t, s, y(\theta(s)))|ds \\
&\quad + \int_0^{\sigma(t)} |g(t, s, x(\eta(s))) - g(t, s, y(\eta(s)))|ds \\
&\leq \varphi(\|x - y\|) + k\gamma(t)\varphi(\|x - y\|) + 2\int_0^{\sigma(t)} a(t)b(s)ds \\
&\leq (1 + c_0 k)\varphi(\epsilon) + 2v(t) \\
&\leq (1 + c_0 k)\epsilon + 2v(t).
\end{aligned} \tag{3.9}$$

Since $v(t) \to 0$ as $t \to \infty$, there exists $T > 0$ such that $v(t) \leq \epsilon$, $\forall t > T$. Thus if $t > T$, then from (3.8) we have that

$$|\mathcal{T}x(t) - \mathcal{T}y(t)| \leq (3 + c_0 k)\epsilon.$$

If $t < T$, then define a function $\omega = \omega(\epsilon)$ by the formula

$$\omega(\epsilon) = \sup\{|g(t, s, x) - g(t, s, y)| : t, s \in [0, T], x, y \in [-r_0, r_0], |x - y| \leq \epsilon\}. \tag{3.10}$$

Now $g(t, s, x)$ is continuous and hence uniformly continuous on $[0, T] \times [0, T] \times [-r_0, r_0]$. As a result we have $\omega(\epsilon) \to 0$ as $\epsilon \to 0$. Therefore, from (3.10),

$$|\mathcal{T}x(t) - \mathcal{T}y(t)| \leq (1 + c_0 k)\epsilon + \sigma^* \omega(\epsilon)$$

for all $t \in \mathbb{R}^+$, where $\sigma^* = \max\{\sigma(t) : t \in [0, T]\}$. Hence, it follows that

$$\begin{aligned}\|\mathcal{T}x - \mathcal{T}y\| &\leq \max\{(3 + c_0 k)\epsilon, (1 + c_0 k)\epsilon + \sigma^* \omega(\epsilon)\} \\ &\to 0 \text{ as } \epsilon \to 0.\end{aligned}$$

Hence $\mathcal{T}$ is a continuous mapping from $B[0, r_0]$ into itself.

Step III: Here we show that $\mathcal{T}$ is a nonlinear set-contraction on $B[0, r_0]$. This will be done in the following two cases:

Case I : Let $A \subset B[0, r_0]$ be non-empty. Further fix the number $T > 0$ and $\epsilon > 0$. Since the functions f and g are continuous on compact $[0, T] \times [0, T] \times [-r_0, r_0]$, there are constants $c_4 > 0$ and $c_5 > 0$ such that $|f(t, s, x)| \leq c_4$ and $|g(t, s, x)| \leq c_5$ for all $t, s \in [0, T]$ and $x \in [-r_0, r_0]$. Then choosing $t, \tau \in [0, T]$ such that $|t - \tau| \leq \epsilon$ and taking into account our hypotheses, we obtain

$$\begin{aligned}
|\mathcal{T}x(t) - \mathcal{T}x(\tau)| &\leq |u(t, x(t)) - u(\tau, x(\tau))| + \Big|p\Big(\int_0^{\gamma(t)} f(t, s, x(\theta(s)))ds, \int_0^{\sigma(t)} g(t, s, x(\eta(s)))ds\Big) \\
&\quad - p\Big(\int_0^{\gamma(\tau)} f(\tau, s, x(\theta(s)))ds, \int_0^{\sigma(\tau)} g(\tau, s, x(\eta(s)))ds\Big)\Big| \\
&\leq |u(t, x(t)) - u(t, x(\tau))| + |u(t, x(\tau)) - u(\tau, x(\tau))| \\
&\quad + \left|\int_0^{\gamma(t)} f(t, s, x(\theta(s)))ds - \int_0^{\gamma(\tau)} f(\tau, s, x(\theta(s)))ds\right| \\
&\quad + \left|\int_0^{\sigma(t)} g(t, s, x(\eta(s)))ds - \int_0^{\sigma(\tau)} g(\tau, s, x(\eta(s)))ds\right| \\
&\leq \varphi(|x(t) - x(\tau)|) + |u(t, x(\tau)) - u(\tau, x(\tau))| \\
&\quad + \left|\int_0^{\gamma(t)} f(t, s, x(\theta(s)))ds - \int_0^{\gamma(t)} f(\tau, s, x(\theta(s)))ds\right| \\
&\quad + \left|\int_0^{\gamma(t)} f(\tau, s, x(\theta(s)))ds - \int_0^{\gamma(\tau)} f(\tau, s, x(\theta(s)))ds\right| \\
&\quad + \left|\int_{\sigma(t)}^{\sigma(\tau)} g(t, s, x(\eta(s)))ds\right| \\
&\quad + \left|\int_0^{\sigma(\tau)} [g(t, s, x(\eta(s)))ds - g(\tau, s, x(\eta(s)))]ds\right| \\
&\leq \varphi(|x(t) - x(\tau)|) + |u(t, x(\tau)) - u(\tau, x(\tau))| \\
&\quad + \int_0^{\gamma(t)} |f(t, s, x(\theta(s))) - f(\tau, s, x(\theta(s)))|ds \\
&\quad + \left|\int_{\gamma(\tau)}^{\gamma(t)} |f(\tau, s, x(\theta(s)))|\, ds\right| + \left|\int_{\sigma(t)}^{\sigma(\tau)} |g(t, s, x(\eta(s)))|\, ds\right| \\
&\quad + \int_0^{\sigma(\tau)} |g(t, s, x(\eta(s)))ds - g(\tau, s, x(\eta(s)))|ds \\
&\leq \varphi(|x(t) - x(\tau)|) + \omega^T(u, \epsilon) + k\omega^T(f, \epsilon) + c_4 \omega^T(\gamma, \epsilon) \\
&\quad + T\omega^T(g, \epsilon) + c_5 \omega^T(\sigma, \epsilon),
\end{aligned}$$

where

$$\begin{aligned}
\omega^T(\gamma, \epsilon) &= \sup\{|\gamma(t) - \gamma(\tau)| : t, \tau \in [0, T], |t - \tau| \leq \epsilon\}, \\
\omega^T(\sigma, \epsilon) &= \sup\{|\sigma(t) - \sigma(\tau)| : t, \tau \in [0, T], |t - \tau| \leq \epsilon\},
\end{aligned}$$

$$\begin{aligned}
\omega^T(u,\epsilon) &= \sup\{|u(t,x)-u(\tau,x)| : t,\tau\in[0,T], |t-\tau|\le\epsilon, |x|\le r_0\},\\
\omega^T(f,\epsilon) &= \sup\{|f(t,s,x)-f(\tau,s,x)| : t,\tau\in[0,T], |t-\tau|\le\epsilon, |x|\le r_0\},\\
\omega^T(g,\epsilon) &= \sup\{|g(t,s,x)-g(\tau,s,x)| : t,\tau\in[0,T], |t-\tau|\le\epsilon, |x|\le r_0\}.
\end{aligned}$$

The above inequality further implies that

$$\begin{aligned}
\omega^T(\mathcal{T}x,\epsilon) &\le \varphi(\omega^T(x,\epsilon))+\omega^T(u,\epsilon)+c_0k\omega^T(f,\epsilon)\\
&\quad + c_4\omega^T(\gamma,\epsilon)+T\omega^T(g,\epsilon)+c_5\omega^T(\sigma,\epsilon).
\end{aligned} \tag{3.11}$$

Since by hypotheses, the functions u,φ,γ,σ and f,g are continuous respectively on $[0,T]$ and $[0,T]\times[0,T]\times[-r_0,r_0]$, we infer that they are uniformly continuous there. Hence we deduce that $\varphi(\omega^T(x,\epsilon))\to 0, \omega^T(u,\epsilon)\to 0, \omega^T(\gamma,\epsilon)\to 0, \omega^T(f,\epsilon)\to 0, \omega^T(g,\epsilon)\to 0$ as $\epsilon\to 0$. Hence from the above estimate (3.11), we obtain

$$\omega_0^T(\mathcal{T}(A))=0,$$

and consequently

$$\omega_0(\mathcal{T}(A))=0. \tag{3.12}$$

Case II: Now for any $x,y\in A$ one has:

$$\begin{aligned}
|\mathcal{T}x(t)-\mathcal{T}y(t)| &\le |u(t,x(t))-u(t,y(t))|+\Big|p\Big(\int_0^{\gamma(t)} f(t,s,x(\theta(s)))ds, \int_0^{\sigma(t)} g(t,s,x(\eta(s)))ds\Big)\\
&\quad -p\Big(\int_0^{\gamma(t)} f(t,s,y(\theta(s)))ds, \int_0^{\sigma(t)} g(t,s,y(\eta(s)))ds\Big)\Big|\\
&\le \varphi(|x(t)-y(t)|)+\Big|\int_0^{\gamma(t)} f(t,s,x(\theta(s)))-f(t,s,y(\theta(s)))ds\Big|\\
&\quad +\Big|\int_0^{\sigma(t)} g(t,s,x(\eta(s)))ds-g(t,s,y(\eta(s)))ds\Big|\\
&\le \varphi(\operatorname{diam}(A(t)))+\int_0^{\gamma(t)} |f(t,s,x(\theta(s)))-f(t,s,y(\theta(s)))|ds\\
&\quad +\int_0^{\sigma(t)} |g(t,s,x(\eta(s)))ds-g(t,s,y(\eta(s)))|ds\\
&\le \varphi(\operatorname{diam}(A(t)))+k\int_0^{\gamma(t)} |x(\theta(s))-y(\theta(s))|\,ds+2v(t)\\
&\le \varphi(\operatorname{diam}(A(t)))+k\int_0^{\gamma(t)} \operatorname{diam} A(\theta(s))ds+2v(t)\\
&\le \varphi(\operatorname{diam}(A(t)))+k\int_0^{\gamma(t)} \operatorname{diam}(A)\,ds+2v(t)\\
&\le \varphi(\operatorname{diam}(A(t)))+k\gamma(t)\operatorname{diam}(A)\,ds+2v(t).
\end{aligned}$$

As a result of the above inequality we obtain

$$\operatorname{diam}(\mathcal{T}(A(t)))\le\varphi(\operatorname{diam}(A(t)))+k\gamma(t)\operatorname{diam}(A)+2v(t).$$

Taking the limit superior as $t\to\infty$ in the above inequality yields

$$\begin{aligned}
\limsup_{t\to\infty}\operatorname{diam}(\mathcal{T}(A(t))) &\le \varphi\left(\limsup_{t\to\infty}\operatorname{diam}(A(t))\right)\\
&\quad + k\limsup_{t\to\infty}\gamma(t)\operatorname{diam} A+2\limsup_{t\to\infty} v(t).
\end{aligned}$$

Since both the limits, namely $\lim_{t\to\infty} v(t)$ and $\lim_{t\to\infty}\gamma(t)$ exist and each one is equal to 0, it follows that $\limsup_{t\to\infty} v(t)=0$ and $\limsup_{t\to\infty}\gamma(t)=0$. Hence, from the above inequality, we have

$$\limsup_{t\to\infty}\operatorname{diam}(\mathcal{T}(A(t)))\le\varphi\left(\limsup_{t\to\infty}\operatorname{diam}(A(t))\right). \tag{3.13}$$

Now from the inequalities (3.12), (3.13) and the definition of μ it follows that

$$\begin{aligned}\mu(\mathcal{T}(A)) &= \omega_0(\mathcal{T}(A)) + \limsup_{t\to\infty} \operatorname{diam}(\mathcal{T}(A(t)))\\ &\le \varphi\left(0 + \limsup_{t\to\infty} \operatorname{diam}(A(t))\right)\\ &\le \varphi\left(\omega_0(A) + \limsup_{t\to\infty} \operatorname{diam}(A(t))\right),\end{aligned}$$

or, equivalently,

$$\mu(\mathcal{T}(A)) \le \varphi(\mu(A)), \tag{3.14}$$

where μ is the measure of noncompactness defined in the space $BC(\mathbb{R}^+, \mathbb{R})$. This shows that $\mathcal{T}$ is a nonlinear $\mathcal{D}$-set-contraction on $B[0, r_0]$. Thus, the map $\mathcal{T}$ satisfies all the conditions of Theorem 2.2 with $C = B[0, r_0]$ and an application of it yields that $\mathcal{T}$ has a fixed point in $B[0, r_0]$. This further by definition of $\mathcal{T}$ implies that the GNFIE (1.1) has a solution in $B[0, r_0]$. Moreover, taking into account that the image of $B[0, r_0]$ under the operator $\mathcal{T}$ is again contained in the ball $B[0, r_0]$ we infer that the set $\mathcal{F}(\mathcal{T})$ of all fixed points of $\mathcal{T}$ is contained in $B[0, r_0]$. If the set $\mathcal{F}(\mathcal{T})$ contains all solutions of the equation (1.1), then we conclude from Remark 2.1 that the set $\mathcal{F}(\mathcal{T})$ belongs to the family ker μ. Now, taking into account the description of sets belonging to ker μ (given in Section 2) we deduce that all solutions of the equation (1.1) are uniformly locally ultimately attractive on $\mathbb{R}^+$. This completes the proof. $\square$

4 An Example

As an application, we consider the following nonlinear functional integral equation

$$\begin{aligned}x(t) = \frac{1}{1+t}\ln\left(1 + \frac{1}{2}|x(t)|\right) + \int_0^{\frac{t^2}{t^3+1}} \left(\frac{1+t}{1+t+t^2}\right)\ln\left(1+\frac{1}{2}|x|\right) ds\\ + \int_0^{\frac{t}{1+t}} \exp(-t^2)\frac{s^2\cos x(s)}{1+|\sin x(s)|} ds,\end{aligned} \tag{4.15}$$

for all $t \in \mathbb{R}^+$.

Let

$$p(t,t') = t + t',\ \varphi(t) = \ln\left(1 + \frac{1}{2}t\right),\ \theta(t) = t^2 + 1,\ \sigma(t) = \frac{t}{1+t},\ \eta(t) = t,$$

$$\gamma(t) = \frac{t^2}{t^3+1},\ u(t,x) = \frac{1}{1+t}\ln\left(1 + \frac{1}{2}|x(t)|\right),\ a(t) = (1+t)^3\exp(-t),\ b(s) = s^2,$$

for all $t, t', s \in \mathbb{R}^+$, and

$$f(t,s,x) = \left(\frac{1+t}{1+t+t^2}\right)\ln\left(1+\frac{1}{2}|x|\right),$$

$$g(t,s,x) = (1+t)^3\exp(-t)\frac{s^2\cos x(s)}{1+|\sin x(s)|}$$

for all $t, s \in \mathbb{R}^+$ and $x \in \mathbb{R}$. Notice that:

(i) The functions $\gamma, \theta, \sigma, \eta$ are obviously continuous and $\lim_{t\to\infty}\gamma(t) = \lim_{t\to\infty}\frac{t^2}{1+t^3} = 0$. Also $c_0 = \sup_{t\ge 0}\gamma(t) = \sup_{t\ge 0}\frac{t^2}{1+t^3} \approx 0.441.$

(ii) For arbitrary fixed $x, y \in \mathbb{R}$ we have

$$\begin{aligned} |u(t,x)-u(t,y)| &= \frac{1}{1+t}\left|\ln\left(1+\frac{1}{2}|x|\right)-\ln\left(1+\frac{1}{2}|y|\right)\right| \\ &\leq \ln\frac{1+\frac{1}{2}|x|}{1+\frac{1}{2}|y|} \leq \ln\left(1+\frac{1}{2}\cdot\frac{|x|-|y|}{1+\frac{1}{2}|y|}\right) \\ &< \ln\left(1+\frac{1}{2}|x-y|\right) \\ &= \varphi(|x-y|). \end{aligned}$$

Therefore, hypothesis (H_1) is satisfied with $\varphi(r) = \ln\left(1+\frac{1}{2}r\right) < r$, for $r > 0$.

(iii) (H_2) is satisfied since $U(t) = |u(t,0)| = 0$ and $c_1 = \sup_{t\geq 0}|u(t,0)| = 0$.

(iv) For arbitrary fixed $x, y \in \mathbb{R}$ such that $|x| \geq |y|$ and for $t > 0$ we obtain

$$|f(t,s,x)-f(t,s,y)| = \left(\frac{1+t}{1+t+t^2}\right)\ln\frac{1+\frac{1}{2}|x|}{1+\frac{1}{2}|y|} \leq \varphi(|x-y|),$$

as in (ii). The case is similar when $|y| \geq |x|$. Thus (H_3) is satisfied with $k = 1$ and $\varphi(r) = \ln\left(1+\frac{1}{2}r\right) < r$, for $r > 0$.

(v) Next, hypothesis (H_4) is satisfied, since the function $F : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$F(t) = \int_0^{\gamma(t)} |f(t,s,0)|ds = \int_0^{\frac{t^2}{t^3+1}} 0\, ds = 0$$

is bounded with $c_2 = \sup_{t\geq 0} F(t) = 0$.

(vi) The function g acts continuously from the set $\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}$ into $\mathbb{R}$. Moreover, we have

$$|g(t,s,x)| \leq (1+t)^3 \exp(-t)\, s^2 = a(t)b(s),$$

for all $t, s \in \mathbb{R}^+$ and $x \in \mathbb{R}$, then we can see that hypothesis (H_5) is satisfied. Indeed, we have

$$\begin{aligned} \lim_{t\to\infty} a(t)\int_0^{\frac{t}{1+t}} b(s)ds &= \lim_{t\to\infty}(1+t)^3\exp(-t)\int_0^{\frac{t}{1+t}} s^2 ds \\ &= \frac{1}{3}\lim_{t\to\infty} t^3\exp(-t) = 0. \end{aligned}$$

Also we have $c_3 = \sup_{t\geq 0}\frac{1}{3}t^3\exp(-t) \approx 0.37$.

(vii) Obviously, hypothesis (H_6) is satisfied.

The inequality

$$(1+c_0k)\varphi(r)+q \leq r$$

reduces to the form

$$(1+0.441)\ \ln\left(1+\frac{1}{2}r\right)+0.37 \leq r.$$

It is easily seen that each number $r \geq 0.6$ satisfies the above inequality. Thus, as the number r_0 we can take $r_0 = 0.6$. Note that this estimate of r_0 can be improved.

Keeping in view the above observations, we find that the functions γ, φ, θ, σ, η, u, f, g,a and b satisfy all the conditions of Theorem 3.2 and hence the GNFIE (4.1) has at least one solution in the space $BC(\mathbb{R}^+,\mathbb{R})$ and the solutions of the equation (4.1) are uniformly locally ultimately attractive on $\mathbb{R}^+$ located in the ball $B[0, 0.6]$.

Remark 4.3. We remark that:

(i) Taking $u(t,x(t)) = q(t), p(t,t') = t + t'$ for all $t,t' \in \mathbb{R}^+$ and for any $x \in \mathbb{R}$ the generalized nonlinear functional integral equation (4.1) reduces to the nonlinear functional integral equation considered by Dhage [8] which, in turn, includes several classes of known integral equations discussed in the literature.

(ii) Taking $p(t,t') = t'$, $\gamma(t) = t$ and $\theta(s) = s$ for all $t,s \in \mathbb{R}^+$, we retrieve the functional integral equation studied by Aghajani, Banas and Sabzali [1].

(iii) The authors in [1] generalized Theorem 2.2 under the weaker upper semi-continuity of the $\mathcal{D}$-function ψ and the requirement of the condition that $\lim_{n\to\infty} \psi^n(r) = 0$ for all $t > 0$, however to hold this condition, they needed an additional condition on the function ψ that $\psi(r) < r$ for $r > 0$. But in actual practice, it is very difficult to verify this condition and the authors in [1] did not provide any example of the function ψ illustrating the comparison between two conditions in applications. Moreover, for applications to the existence result, they assumed an additional condition on the function ψ, namely, supperadditivity which automatically yields the upper semi-continuity together with the monotone characterization of the function ψ and so, the existence theorem for the nonlinear integral equation considered in Aghajani *et.al.* [1] follows by a direct application of Theorem 2.2 of Dhage [8].

References

[1] A. Aghajani, J. Banas and N. Sabzali, Some generalizations of Darbo fixed point theorem and applications, *Bull. Belg. Math. Soc. Simon Stevin* **20** (2013), 345-358.

[2] R. R. Akhmerov, M. I. Kamenskii, A. S. Potapov, A. E. Rodhina and B. N. Sadovskii, *Measures of noncompactness and Condensing Operators* , Birkhauser Verlag, 1992.

[3] J. Banas and K. Goebel, *Measures of Noncompactness in Banach Spaces*, LNPAM Vol. 60, Marcel Dekker, New York 1980.

[4] J. Banas and B. Rzepka, An application of measures of noncompactness in the study of asymptotic stability, *Appl. Math. Lett.* **16** (2003), 1-6.

[5] G. Darbo, Punti uniti i transformazion a condominio non compatto, *Rend. Sem. Math. Univ. Padova* **4** (1995), 84-92.

[6] K. Deimling, *Ordinary Differential Equations in Banach Spaces*, LNM Springer Verlag, 1977.

[7] B.C. Dhage, A nonlinear alternative with applications to nonlinear perturbed differential equations,*Nonlinear Studies* **13** (4) (2006), 343-354.

[8] B.C. Dhage, Asymptotic stability of the solution of certain nonlinear functional integral equations via measures of noncompactness, *Comm. Appl. Nonlinear Anal.* **15** (2008), no. 2, 89-101.

[9] B.C. Dhage, Attractivity and positivity results for nonlinear functional integral equations via measures of noncompactness, *Differ. Equ. Appl.* **2** (1) (2010), 299-318.

[10] B.C. Dhage, Partially condensing mappings in ordered normed linear spaces and applications to functional integral equations, *Tamkang J. Math.* **45** (2014), 397-426.

[11] B.C. Dhage, Global attractivity results for comparable solutions of nonlinear hybrid fractional integral equations, *Differ. Equ. Appl.* **6** (2014), 165- 186.

[12] B.C. Dhage, V. Lakshmikatham, On global existence and attrctivity results for nonlinear functional integral equations, *Nonlinear Analysis* **72** (2010), 2219-2227.

[13] B. C. Dhage and S. K.Ntouyas, Existence results for nonlinear functional integral equations via a fixed point theorem of Krasnoselskii-Schaefer type, *Nonlinear Studies* **9** (3) (2002), 307-317.

[14] M.A. Krasnoselskii, *Topological Methods in The Theory of Nonlinear Integral Equations*, Pergamon Press, New York, 1964.

[15] D. O'Regan and M. Meehan, *Existence Theory for Nonlinear Integral and Integro-differential Equations*, Kluwer Academic Dordrechet 1998.

[16] M. Väth, *Voltera and Integral Equations of Vector Functions*, PAM, Marcel Dekker, New York 2000.

[17] E. Zeidler, *Nonlinear Functional Analysis and Its Applications: Part I, Fixed Point Theory*, Springer Verlag, 1985.

Received: December 11, 2014; *Accepted*: March 01, 2015

UNIVERSITY PRESS

Website: http://www.malayajournal.org/