ORTHOGONAL STABILITY OF THE NEW GENERALIZED QUADRATIC FUNCTIONAL EQUATION

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Abstract

In this paper, the authors investigate the Hyers - Ulam - Rassias stability and J. M. Rassias mixed type product-sum of powers of norms stability of a orthogonally generalized quadratic functional equation of the form

$$f(nx+y) + f(nx-y) = n[f(x+y) + f(x-y)] + 2n(n-1)f(x) - 2(n-1)f(y).$$

Where $f : A \rightarrow B$ be a mapping from a orthogonality normed space $A$ into a Banach Space $B$, $\perp$ is orthogonality in the sense of Ratz with $x \perp y$ for all $x, y \in A$.

Keywords: Hyers - Ulam - Rassias stability, J. M. Rassias mixed type product-sum of powers of norms stability, Example, Orthogonally quadratic functional equation, Orthogonality space, Quadratic mapping.

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1 Introduction


The idea of generalized Hyers-Ulam stability is extended to various functional equations like additive equations, Jensen’s equations, Hosszu’s equations, homogeneous equations, logarithmic equations, exponential equations, multiplicative equations, trigonometric and gamma functional equations.

It is easy to see that the quadratic function $f(x) = kx^2$ is a solution of each of the following functional equations

$$f(x+y) + f(x-y) = 2f(x) + 2f(y),$$

$$f(x+y+z) + f(x) + f(y) + f(z) = f(x+y) + f(y+z) + f(z+x),$$

$$f(x-y-z) + f(x) + f(y) + f(z) = f(x-y) + f(y+z) + f(z-x),$$

$$f(x+y+z) + f(x-y+z) + f(x+y-z) + f(x-y-z) = 4f(x) + 4f(y) + 4f(z).$$

So it is natural that each equation is called a quadratic functional equation. In particular, every solution of the quadratic equation (1.7) is said to be quadratic function. It is well known that a function $f$ between real vector spaces is quadratic if and only if there exists a unique symmetric biadditive function $B$ such that $f(x) = B(x, x)$ for all $x$(see[1,9]). The bi-additive function $B$ is given by

$$B(x, y) = \frac{1}{4}[f(x+y) - f(x-y)].$$
Hyers-Ulam-Rassias stability problem for the quadratic functional equation \([1.1]\) was first treated by F. Skof for functions \(f : A \to B\) where \(A\) is a normed space and \(B\) is a Banach space (see \([17]\)). Cholewa \([2]\) noticed that the theorem of Skof is still true if relevant domain \(A\) is replaced by abelian group. Czerwik \([3]\) proved the Hyers-Ulam-Rassias stability of the equation \([1.1]\).

In 1982-1984, J.M. Rassias \([12,13]\) proved the following theorem in which he generalized the Hyers stability result by presenting a weaker condition controlled by a product of different powers of norms.

**Theorem 1.1.** \([12,13]\) Let \(f : E \to E'\) be a mapping from a normed vector space \(E\) into a Banach space \(E'\) subject to the inequality
\[
\|f(x + y) - f(x) - f(y)\| \leq \epsilon \|x\|^p \|y\|^q
\]
for all \(x, y \in E\), where \(\epsilon\) and \(p, q\) are constants with \(\epsilon > 0\) and \(r = p + q \neq 1\). Then the limit
\[
L(x) = \lim_{n \to \infty} \frac{f(2^nx)}{2^n}
\]
exists for all \(x \in E\) and \(L : E \to E'\) is the unique additive mapping which satisfies
\[
\|f(x) - L(x)\| \leq \frac{\epsilon}{2 - 2^r} \|x\|^r
\]
for all \(x \in E\). If, in addition, for every \(x \in E\), \(f(tx)\) is continuous in real \(t\) for each fixed \(x\), then \(L\) is linear.

The above-mentioned stability involving a product of different powers of norms is called Ulam-Gavruta-Rassias stability. Later, J.M.Rassias \([15]\) discussed the stability of quadratic functional equation
\[
f(mx + y) + f(mx - y) = 2f(x + y) + 2f(x - y) + 2(m^2 - 2)f(x) - 2f(y)
\]
for any arbitrary but fixed real constant \(m\) with \(m \neq 0; m \neq \pm 1; m \neq \pm \sqrt{2}\) using the mixed powers of norms.

Now we present the results connected with functional equation in orthogonal space. The orthogonal Cauchy functional equation
\[
f(x + y) = f(x) + f(y), x \perp y
\]
in which \(\perp\) is an abstract orthogonally was first investigated by S. Gudder and D. Strawther. R. Ger and J. Sikorska discussed the orthogonal stability of the equation \([1.5]\) in \([7]\). The orthogonally quadratic functional equation \([1.1]\) was first investigated by F. Vajzovic \([20]\) when \(X\) is a Hilbert space, \(Y\) is the scalar field, \(f\) is continuous and \(\perp\) means the Hilbert space orthogonality. This result was then generalized by H. Drljevic \([4]\), M. Fochi \([5]\), M. Moslehian \([10,11]\) and G. Szabo \([18]\).

**Definition 1.1.** A vector space \(X\) is called an orthogonality vector space if there is a relation \(x \perp y\) on \(X\) such that
(i) totality of \(\perp\) for zero: \(x \perp 0, 0 \perp x\) for all \(x \in X\);
(ii) independence: if \(x \perp y\) and \(x, y \neq 0\), then \(x, y\) are linearly independent;
(iii) homogeneity: if \(x \perp y\), then \(ax \perp by\) for all \(a, b \in \mathbb{R}\);
(iv) the Thalesian property: if \(P\) is a two-dimensional subspace of \(X\); then
(a) for every \(x \in P\) there exists \(0 \neq y \in P\) such that \(x \perp y\);
(b) there exists vectors \(x, y \neq 0\) such that \(x \perp y\) and \(x + y \perp x - y\).

Any vector space can be made into an orthogonality vector space if we define \(x \perp 0, 0 \perp x\) for all \(x\) and for non zero vector \(x, y\) define \(x \perp y\) if \(f\) \(x, y\) are linearly independent. The relation \(\perp\) is called symmetric if \(x \perp y\) implies that \(y \perp x\) for all \(x, y \in X\).

**Definition 1.2.** The pair \((x, \perp)\) is called an orthogonality space. It becomes orthogonality normed space when the orthogonality space is equipped with a norm.

**Definition 1.3.** Let \(X\) be an orthogonality space and \(Y\) be a real Banach space. A mapping \(f : X \to Y\) is called orthogonally quadratic if it satisfies the so-called orthogonally Euler-Lagrange (or Jordan-von Neumann) quadratic functional equation
\[
f(x + y) + f(x - y) = 2f(x) + 2f(y)
\]
for all \(x, y \in X\) with \(x \perp y).
In this paper, we obtain the general solution of new quadratic functional equation

\[ f(nx + y) + f(nx - y) = n[f(x + y) + f(x - y)] + 2n(n - 1)f(x) - 2(n - 1)f(y) \] (1.8)

and study the Hyers - Ulam - Rassias stability and J. M. Rassias mixed type product-sum of powers of norms stability in the concept of orthogonality.

**Definition 1.4.** A mapping \( f : A \rightarrow B \) is called orthogonally quadratic if it satisfies the quadratic functional equation (1.8) for all \( x, y \in A \) with \( x \perp y \) where \( A \) be an orthogonality space and \( B \) be a real Banach space.

Throughout this paper, let \( (A, \perp) \) denote an orthogonality normed space with norm \( \| \cdot \|_A \) and \( (B, \| \cdot \|_B) \) is a Banach space. We define

\[ Df(x, y) = f(nx + y) + f(nx - y) \]

\[ - n[f(x + y) + f(x - y)] - 2n(n - 1)f(x) + 2(n - 1)f(y). \] (1.9)

for all \( x, y \in A \) with \( x \perp y \).

Now we proceed to find the general solution of the functional equation (1.8).

### 2 The General Solution of the Functional Equation (1.8)

In this section, we obtain the general solution of the functional equation (1.8). Throughout this section, let \( X \) and \( Y \) be real vector spaces.

**Theorem 2.2.** Let \( X \) and \( Y \) be real vector spaces. A function \( f : X \rightarrow Y \) satisfies the functional equation

\[ f(nx + y) + f(nx - y) = n[f(x + y) + f(x - y)] + 2n(n - 1)f(x) - 2(n - 1)f(y) \] (2.1)

for all \( x, y \in X \) if and only if it satisfies the quadratic functional equation

\[ f(x + y) + f(x - y) = 2f(x) + 2f(y) \] (2.2)

for all \( x, y \in X \).

**Proof.** Suppose a function \( f : X \rightarrow Y \) satisfies (2.1). Putting \( x = y = 0 \) in (2.1), we get \( f(0) = 0 \). Let \( x = 0 \) and \( y = 0 \) in (2.1), we obtain \( f(-y) = f(y) \) and \( f(nx) = n^2f(x) \), respectively. Setting \( (x, y) = (x, x + y) \) in (2.1), we obtain

\[ f((n + 1)x + y) + f((n + 1)x - y) = n[f(2x + y) + f(-y)] + 2f(nx) - 2nf(x) \] (2.3)

for all \( x, y \in X \). Replacing \( y \) by \(-y\) in (2.3) and adding the resultant with (2.3), we obtain

\[ f((n + 1)x + y) + f((n + 1)x - y) + f((n - 1)x + y) + f((n - 1)x - y) \]

\[ = n[f(2x + y) + f(2x - y)] + 2n[f(x + y) + f(x - y)] + 2[f(x + y) + f(x - y)] \]

\[ + 2nf(y) + 4f(nx) - 4nf(x) \] (2.4)

for all \( x, y \in X \). Setting \( n = n + 1, \) \( n = n - 1 \) and \( n = 2 \) respectively in (2.1), we obtain the following equations

\[ f((n + 1)x + y) + f((n + 1)x - y) \]

\[ = (n + 1)[f(x + y) + f(x - y)] + 2n^2f(x) + 2nf(x) - 2nf(y) \] (2.5)

\[ f((n - 1)x + y) + f((n - 1)x - y) = (n - 1)[f(x + y) + f(x - y)] \]

\[ + 2n^2f(x) - 6nf(x) + 4f(x) - 2nf(y) + 4f(y) \] (2.6)

\[ f(2x + y) + f(2x - y) = 2[f(x + y) + f(x - y)] + 4f(x) - 2f(y) \] (2.7)

for all \( x, y \in X \). Substitute (2.5), (2.6) and (2.7) in (2.4), we arrive (2.2).
Conversely, assume \( f \) satisfies the functional equation (2.2). Letting \((x, y)\) by \((0, 0)\) in (2.2), we get \( f(0) = 0 \). Putting \( x = 0 \) in (2.2), we obtain \( f(-y) = f(y) \) for all \( y \in X \). Thus \( f \) is an even function. Substituting \((x, y)\) by \((x, x)\) and \((x, 2x)\) in (2.2), we get
\[
f(2x) = 4f(x), f(3x) = 9f(x)
\]
(2.8)
respectively for all \( x \in X \). Setting \((x, y) = (nx + y, nx - y)\) in (2.2), we obtain
\[
f(nx + y) + f(nx - y) = 2n^2f(x) + 2f(y)
\]
(2.9)
for all \( x, y \in X \). Multiplying (2.2) by \( n \) and subtracting the resultant from (2.9), we arrive (2.1). \( \square \)

3 \: Hyers - Ulam - Rassias Stability of (1.8)

In this section, we present the Hyers - Ulam - Rassias stability of the functional equation (1.8) involving sum of powers of norms.

**Theorem 3.3.** Let \( \mu \) and \( s (s < 2) \) be non-negative real numbers. Let \( f : A \to B \) be a mapping fulfilling
\[
\|D f(x, y)\|_B \leq \mu \{ \|x\|_A^s + \|y\|_A^s \}
\]
(3.1)
for all \( x, y \in A \) with \( x \perp y \). Then there exists a unique orthogonally quadratic mapping \( Q : A \to B \) such that
\[
\|f(x) - Q(x)\|_B \leq \frac{\mu}{2(n^2 - n^s)} \|x\|_A^s
\]
(3.2)
for all \( x \in A \). The function \( Q(x) \) is defined by
\[
Q(x) = \lim_{k \to \infty} \frac{f(n^kx)}{(n^2)^k}
\]
(3.3)
for all \( x \in A \).

**Proof.** Replacing \((x, y)\) by \((0, 0)\) in (3.1) we get \( f(0) = 0 \). Setting \((x, y)\) by \((x, 0)\) in (3.1), we obtain
\[
\|f(nx) - n^2f(x)\|_B \leq \frac{\mu}{2} (\|x\|_A^s)
\]
(3.4)
for all \( x \in A \). Since \( x \perp 0 \), we have
\[
\left\| \frac{f(nx)}{n^2} - f(x) \right\|_B \leq \frac{\mu}{2n^2} \|x\|_A^s
\]
(3.5)
for all \( x \in A \). Now replacing \( x \) by \( nx \) and dividing by \( n^2 \) in (3.5) and summing resulting inequality with (3.5), we arrive
\[
\left\| \frac{f(n^kx)}{(n^2)^k} - f(x) \right\|_B \leq \frac{\mu}{2n^2} \left( 1 + \frac{n^s}{n^2} \right) \|x\|_A^s
\]
(3.6)
for all \( x \in A \). In general, using induction on a positive integer \( n \) we obtain that
\[
\left\| \frac{f(n^kx)}{(n^2)^k} - f(x) \right\|_B \leq \frac{\mu}{2n^2} \sum_{t=0}^{k-1} \frac{n^s^t}{(n^2)^t} \|x\|_A^s
\]
(3.7)
for all \( x \in A \). In order to prove the convergence of the sequence \( \{ f(n^kx) / (n^2)^k \} \) replace \( x \) by \( n^m x \) and divide by \( (n^2)^m \) in (3.7), for any \( k, m > 0 \), we obtain
\[
\left\| \frac{f(n^k n^m x)}{(n^2)^{k+m}} - f(n^m x) \right\|_B = \frac{1}{(n^2)^m} \left\| \frac{f(n^k n^m x)}{(n^2)^k} - f(n^m x) \right\|_B
\]
\[
\leq \frac{1}{(n^2)^m} \frac{\mu}{2n^2} \sum_{t=0}^{k-1} \frac{n^s^t}{(n^2)^t} \|n^m x\|_A^s
\]
\[
\leq \frac{\mu}{2n^2} \sum_{t=0}^{\infty} \frac{1}{n^{2s(t+m)}} \|x\|_A^s.
\]
(3.8)
As \( s < 2 \), the right hand side of (3.8) tends to 0 as \( m \to \infty \) for all \( x \in A \). Thus \( \{ f(n^k x) / (n^2)^k \} \) is a Cauchy sequence. Since \( B \) is complete, there exists a mapping \( Q : A \to B \) such that

\[
Q(x) = \lim_{k \to \infty} \frac{f(n^k x)}{(n^2)^k} \quad \forall x \in A.
\]

Letting \( k \to \infty \) in (3.7), we arrive the formula (3.2) for all \( x \in A \). To prove \( Q \) satisfies (1.8), replace \((x,y)\) by \((n^k x, n^k y)\) in (3.1) and divide by \((n^2)^k\) then it follows that

\[
\frac{1}{(n^2)^k} \| f(n^k (nx + y)) + f(n^k (ny - x)) - n[f(n^k (x + y)) - f(n^k (x - y))] \\
- 2n(n-1)f(n^k x) - 2(n-1)f(n^k y) \|_B \leq \frac{\mu}{(n^2)^k} \left\{ \left\| n^k x \right\|_A^s + \left\| n^k y \right\|_A^s \right\}.
\]

Taking limit as \( n \to \infty \) in the above inequality, we get

\[
\| Q(nx + y) + Q(nx - y) - n[Q(x + y) - Q(x - y)] \\
- 2n(n-1)Q(x) + 2(n-1)Q(y) \|_B \leq 0,
\]

which gives

\[
Q(nx + y) + Q(nx - y) = n[Q(x + y) - Q(x - y)] + 2n(n-1)Q(x) - 2(n-1)Q(y)
\]

by taking limit as \( k \to \infty \) in (3.7), we obtain

\[
\| f(x) - Q(x) \|_B \leq \frac{\mu}{2(n^2 - n^s)} \| x \|_A^s \quad (3.9)
\]

for all \( x, y \in A \) with \( x \perp y \). Therefore \( Q : A \to B \) is an orthogonally quadratic mapping which satisfies (1.8). To prove the uniqueness: Let \( Q' \) be another orthogonally quadratic mapping satisfying (1.8) and the inequality (3.2). Then

\[
\| Q(x) - Q'(x) \|_B = \frac{1}{(n^2)^k} \| Q \left( n^k x \right) - Q' \left( n^k x \right) \|_B \\
\leq \frac{1}{(n^2)^k} \left\{ \| Q(k^l x) - f(k^l x) \|_B + \| f \left( n^k x \right) - Q' \left( n^k x \right) \|_B \right\} \\
\leq \frac{\mu}{n^2 - n^s} \frac{1}{n^{(2-s)}} \| x \|_A^s \\
\to 0 \quad \text{as} \quad k \to \infty
\]

for all \( x \in A \). Therefore \( Q \) is unique. This completes the proof of the theorem.


**Theorem 3.4.** Let \( \mu \) and \( s (s > 2) \) be nonnegative real numbers. Let \( f : A \to B \) be a mapping satisfying (3.1) for all \( x, y \in A \) with \( x \perp y \). Then there exists a unique orthogonally quadratic mapping \( Q : A \to B \) such that

\[
\| f(x) - Q(x) \|_B \leq \frac{\mu}{2(n^2 - n^s)} \| x \|_A^s \quad (3.10)
\]

for all \( x \in A \). The function \( Q(x) \) is defined by

\[
Q(x) = \lim_{k \to \infty} (n^2)^k f \left( \frac{x}{n^k} \right) \quad (3.11)
\]

for all \( x \in A \).

**Proof.** Replacing \( x \) by \( \frac{x}{n^k} \) in (3.4), the rest of the proof is similar to that of Theorem 3.1.
4 J.M. Rassias Mixed Type Product - Sum of Powers of Norms Stability of (1.8)

In this section, we discuss the J.M. Rassias mixed type product - sum of powers of norms stability of the functional equation (1.8).

**Theorem 4.5.** Let \( f: A \to B \) be a mapping satisfying the inequality

\[
\| D f(x,y) \|_B \leq \mu \left\{ \| x \|_A^{2s} + \| y \|_A^{2s} + \| x \|_A \| y \|_A \right\}
\]

(4.1)

for all \( x, y \in A \) where \( \mu \) and \( s \) are constants with, \( \mu, s > 0 \) and \( s < 1 \). Then the limit

\[
Q(x) = \lim_{k \to \infty} \frac{f \left( \frac{n^k x}{(n^2)^k} \right)}{n^k}
\]

(4.2)

exists for all \( x \in A \) and \( Q: A \to B \) is the unique quadratic mapping such that

\[
\| f(x) - Q(x) \|_B \leq \frac{\mu}{2(n^2 - n^{2s})} \| x \|_A^{ns}
\]

(4.3)

for all \( x \in A \).

**Proof.** Letting \((x, y)\) by \((0, 0)\) in (4.1), we get \( f(0) = 0 \). Again substituting \((x, y)\) by \((x, 0)\) in (4.1), we obtain

\[
\left\| \frac{f(nx)}{n^2} - f(x) \right\|_B \leq \frac{\mu}{2n^2} \| x \|_A^{ns}
\]

(4.4)

for all \( x \in A \). Now replacing \( x \) by \( nx \) and dividing by \( n^2 \) in (4.4) and summing resulting inequality with (4.4), we arrive

\[
\left\| \frac{f \left( \frac{n^2 x}{(n^2)^2} \right)}{(n^2)^2} - f(x) \right\|_B \leq \frac{\mu}{2n^2} \left\{ 1 + \frac{n^{2s}}{n^2} \right\} \| x \|_A^{2s}
\]

(4.5)

for all \( x \in A \). Using induction on a positive integer \( k \), we obtain that

\[
\left\| \frac{f \left( \frac{n^k x}{(n^2)^k} \right)}{(n^2)^k} - f(x) \right\|_B \leq \frac{\mu k}{2n^2} \sum_{t=0}^{k-1} \left( \frac{n^{2s}}{n^2} \right)^t \| x \|_A^{2s}
\]

(4.6)

for all \( x \in A \). In order to prove the convergence of the sequence \( \{ f(n^k x) / (n^2)^k \} \) replace \( x \) by \( m^m x \) and divide by \((n^2)^m \) in (4.6), for any \( k, m > 0 \), we obtain

\[
\left\| \frac{f \left( \frac{n^k m^m x}{(n^2)^k} \right)}{(n^2)^k m^m} - f \left( \frac{n^m x}{(n^2)^m} \right) \right\|_B \leq \frac{1}{(n^2)^m} \left\| \frac{f \left( \frac{n^k m^m x}{(n^2)^k} \right)}{(n^2)^k} - f \left( \frac{n^m x}{(n^2)^m} \right) \right\|_B
\]

\[
\leq \frac{\mu k}{2n^2} \sum_{t=0}^{k-1} \left( \frac{n^{2s}}{n^2} \right)^t \| m^m x \|_A^{2s}
\]

\[
\leq \frac{\mu}{n^2} \sum_{t=0}^{m} \frac{1}{n(n-2s)(t+1-m)} \| x \|_A^{2s}
\]

(4.7)

As \( s < 1 \), the right hand side of (4.7) tends to 0 as \( m \to \infty \) for all \( x \in A \). Thus \( \{ f(n^k x) / (n^2)^k \} \) is a Cauchy sequence. Since \( B \) is complete, there exists a mapping \( Q: A \to B \) such that

\[
Q(x) = \lim_{k \to \infty} \frac{f(n^k x)}{(n^2)^k} \quad \forall x \in A.
\]

Letting \( n \to \infty \) in (4.6), we arrive the formula (4.2) for all \( x \in A \). To show that \( Q \) is unique and it satisfies (1.8), the rest of the proof is similar to that of theorem 3.1.
Theorem 4.6. Let \( f : A \to B \) be a mapping satisfying the inequality (4.7) for all \( x, y \in A \) where \( \mu \) and \( s \) are constants with, \( \mu, s > 0 \) and \( s > 2 \). Then the limit

\[
Q(x) = \lim_{k \to \infty} (n^2)^k f \left( \frac{x}{n^k} \right)
\]

(4.8)

exists for all \( x \in A \) and \( Q : A \to B \) is the unique quadratic mapping such that

\[
\| f(x) - Q(x) \|_B \leq \frac{\mu}{2(n^{2s} - n^{2})} \| x \|^{2s}
\]

(4.9)

for all \( x \in A \).

Proof. Replacing \( x \) by \( \frac{x}{2} \) in (4.4), the proof is similar to that of Theorem 4.5.

Now we will provide an example to illustrate that the functional equation (1.8) is not stable for \( s = 2 \).

Example 4.1. Let \( \varphi : X \to X \) be a function defined by

\[
\phi(x) = \begin{cases} 
\mu \| x \|^2, & \| x \| < 1 \\
\mu & \text{otherwise}
\end{cases}
\]

(4.10)

where \( \mu > 0 \) is a constant and we define a function \( f : X \to Y \) by

\[
f(x) = \sum_{m=0}^{\infty} \frac{\varphi(n^m x)}{(n^2)^m}
\]

(4.11)

for all \( x \in X \). Then \( f \) satisfies the functional inequality

\[
\| D(f(x), y) \| \leq \frac{2n^2}{(n-1)} \mu (\| x \|^2 + \| y \|^2)
\]

(4.12)

for all \( x, y \in X \). Then there exist any quadratic mapping \( Q : X \to Y \) satisfying

\[
\| f(x) - Q(x) \| \leq \eta \| x \|^2
\]

(4.13)

for \( x \in X \).

Proof. From the equation (4.10) and (4.11), we obtain

\[
f(x) \leq \sum_{m=0}^{\infty} \frac{\varphi(n^m x)}{n^{2m}} = \sum_{k=0}^{\infty} \frac{\mu}{n^{k}} \leq \mu \left( \frac{n^2}{n^{2} - 1} \right)
\]

(4.14)

for all \( x \in X \). Therefore we see that \( f \) is bounded. We are going to prove that \( f \) satisfies (4.12).

If \( (\| x \|^2 + \| y \|^2) \geq 1 \) then the left hand side of (4.12) is less than

\[
\frac{2n^2}{(n-1)}
\]

. Now we suppose that \( 0 \leq \| x \|^2 + \| y \|^2 \leq 1 \). Then there exist a positive integer \( k \) such that

\[
\frac{1}{n^{2k-1}} \leq \| x \|^2 + \| y \|^2 < \frac{1}{n^{2k}}
\]

(4.15)

for all \( x \in X \), so that

\[
n^{2k} \| x \|^2 < 1, n^{2k} \| y \|^2 < 1
\]

and consequently, \( n^{k-1} \| x \| < 1, n^{k-1} \| y \| < 1, n^{k-1} \| x + y \| < 1, n^{k-1} \| x - y \| < 1, n^{k-1} \| nx + y \| < 1, n^{k} \| nx - y \| < 1 \) for all \( m \in 0, 1, 2, ..., k-1 \)

\[
n^{k-1} \| x \| < 1, n^{k-1} \| y \| < 1, n^{k-1} \| x + y \| < 1, n^{k-1} \| x - y \| < 1, n^{k-1} \| nx + y \| < 1, n^{k} \| nx - y \| < 1.
\]
for all \( x \in \{0, 1, 2, \ldots, k - 1\} \).

\[
\|D(f(x,y))\| \leq \sum_{m=k}^{\infty} \frac{2n(n+1)}{n^{2m}} \mu
\leq \frac{2n(n+1)}{n^{2m}} \left( \frac{n^2}{n^2 - 1} \right) \mu
\leq \frac{2n^2}{n-1} \mu \left( \|x\|^2 + \|y\|^2 \right)
\]

Thus \( f \) satisfies the inequality (4.12) Let us consider the an orthogonally quadratic mapping satisfying \( Q : X \rightarrow Y \) and a constant \( \eta > 0 \) such that

\[
\|f(x) - Q(x)\| \leq \eta \|x\|^2
\]

for all \( x \in X \). Since \( f \) is bounded, \( Q \) is also bounded on any open interval containing the origin zero. we have

\[
Q(x) = c \|x\|^2
\]

for all \( x \in X \) and \( c \) is constant. Thus we obtain

\[
\|f(x) - c \|x\|^2\| \leq \eta \|x\|^2
\]

\[
\|f(x)\| \leq (|c| + \eta) \|x\|^2
\]

(4.16)

for all \( x \in X \). But we can choose a positive integer

\[
p, p\mu > \eta + |c|
\]

. If \( x \in \left(0, \frac{1}{n^p}\right)\), then \( n^m x \in (0, 1) \) for all \( m = 0, 1, \ldots, p - 1\). For this \( x \), we get

\[
f(x) = \sum_{m=0}^{\infty} \frac{\phi(n^m x)}{n^{2m}} \geq \sum_{m=0}^{\infty} \frac{\mu(n^m)}{n^{2m}} \|x\|^2 = p\mu \|x\|^2 > (\eta + |c|) \|x\|^2
\]

which contradicts (4.16). Therefore the functional equation (1.8) is not stable in sense of Ulam, Hyers and Rassias if \( s = 2 \), assumed in the inequality (4.16).

\[
\square
\]

References


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