Hyers-Ulam-Rassias stability of nth order linear ordinary differential equations with initial conditions

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Abstract

In this paper, we investigate the stability of nth order linear ordinary differential non-homogeneous equation with initial conditions in the Hyers-Ulam-Rassias sense.

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1 Introduction

In 1940, S.M. Ulam while he was giving talk at Wisconsin University, he proposed the following problem: Under what conditions does there exist an additive mapping near an approximately additive mapping? for details see \cite{18}. A year later, D.H. Hyers in \cite{4} gave an answer to the problem of Ulam for additive functions defined on Banach spaces. Let $E_1$ and $E_2$ be two real Banach spaces and $f : X_1 \to X_2$ be a mapping. If there exist an $\epsilon \geq 0$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \epsilon$$

for all $x, y \in X_1$, then there exist a unique additive mapping $g : X_1 \to X_2$ with the property

$$\|f(x) - g(x)\| \leq \epsilon,$$

$\forall x \in X_1$. A generalized solution to Ulam’s problem for approximately linear mappings was proved by Th.M. Rassias in 1978 \cite{13}. He considered a mapping $f : E_1 \to E_2$ such that $t \to f(tx)$ is continuous in $t$ for each fixed $x$. Assume that there exists $\theta \geq 0$ and $0 \geq p < 1$ such that

$$\|f(x+y) - f(x) - f(y)\| \leq \theta (\|x\|^p + \|y\|^p)$$

for any $x, y \in E_1$. After Hyers result, many mathematicians have extended Ulam’s problem to other functional equations and generalized Hyers result in various directions, see \cite{2}, \cite{3}.

Soon-Mo Jung \cite{17}, investigated the Hyers-Ulam stability of a system of first order linear differential equations with constant coefficients. Miura et al \cite{11}, proved the Hyers-Ulam stability of the first-order linear differential equations of the form $y'(t) + g(t)y(t) = 0$, where $g(t)$ is a continuous function, while Jung \cite{14}, proved the Hyers-Ulam stability of differential equations of the form $\varphi(t)y'(t) = y(t)$. Furthermore, the result of Hyers-Ulam stability for first-order linear differential equations has been generalized in \cite{15}, \cite{16}, \cite{19}.

A. Javadian, E. Sorouri, G.H. Kim and M. Eshaghi Gordji \cite{6}, investigated generalized Hyers-Ulam stability

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E-mail address: shckravi@yahoo.co.in (K. Ravi), shcrmurali@yahoo.co.in (R. Murali), antoyellow92@gmail.com (A. Antony Raj).
of the second order linear differential equations of the form $y'' + P(x)y' + q(x)y = f(x)$ with some conditions. Maher Nazmi Qorawani [10], investigated Hyers-Ulam stability of second order linear differential equations of the form $z'' + p(x)z' + (q(x) - α(x))z = 0$ and nonlinear differential equations of the form $z'' + p(x)z' + q(x)z = h(x)\bigl|z\bigl|^{β}\frac{β - 1}{β - 2} f\bigl(p(x)dz\bigl)$ with initial conditions. Li and Yan [8], investigated the Hyers-Ulam stability of nonhomogeneous second order Linear Differential Equations of the form $y'' + r(x)y = 0$ under some special conditions. Pasc Gavruta, Jung, Li [3], investigated the Hyers-Ulam stability for second order linear differential equations with boundary conditions of the form $y'' + β(x)y(x) = 0$. Jinghao Huang, Qusuay H. Alqifiary, and Yongjin Li [7], proved the generalized superstability of nth order linear differential equation $y'' + p(x)y' + q(x)y + r(x) = 0$ and nonlinear differential equations of the form $y'' + β(x)y(x) = 0$. Recently, M.I. Modebei, O.O. Olaiya, I. Otaide [12], investigated generalized Hyers-Ulam stability of second order linear ordinary differential equation $y'' + β(x)y = f(x)$ with initial condition.

In this paper, we investigate the Hyers-Ulam-Rassias Stability of nth order linear ordinary differential equations with initial conditions

$$y^{(n)} + β(x)y(x) = f(x)$$

where $y \in C^n[a, b], β \in C[a, b]$ and $f : [a, b] → \mathbb{R}$ continuous.

Let $(X, \|\cdot\|)$ be a real or complex Banach space with $a, b ∈ \mathbb{R}$ where $-∞ < a < b < ∞$, $ε$ be a positive real number. Let $y : (a, b) → X$ be a continuous function. We consider the following differential equation

$$y^{(n)}(t) = \sum_{k=0}^{n-1} P_ky^{(k)}(t), \quad t \in I$$

and the following differential inequality

$$\left|y^{(n)}(t) - \sum_{k=0}^{n-1} P_ky^{(k)}(t)\right| ≤ ε, \quad t \in I$$

and

$$\left|y^{(n)}(t) - \sum_{k=0}^{n-1} P_ky^{(k)}(t)\right| ≤ φ(t), \quad t \in I$$

**Definition 1.1.** The equation (1.1) is said to have the Hyers-Ulam stability for any $ε > 0$, there exist a real number $K > 0$ such that for each approximate solution $y \in C^n(I, X)$ of (1.2) there exist a solution $y_0 \in C^n(I, X)$ of (1.1) with

$$|y - y_0| ≤ Ke \quad ∀t \in I.$$

**Definition 1.2.** The equation (1.1) is said to have the Hyers-Ulam-Rassias stability if there exist $θ_φ \in C(\mathbb{R}_+, \mathbb{R}_+)$, such that for each approximate solution $y \in C^n(I, X)$ of (1.3) there exist a solution $y_0 \in C^n(I, X)$ of (1.1) with

$$|y - y_0| ≤ θ_φ(t) \quad ∀t \in I.$$

**Definition 1.3.** The equation $y^{(n)}(x) + β(x)y(x) = 0$ has the Hyers-Ulam stability with initial conditions $y(a) = y'(a) = ... = y^{(n-1)}(a) = 0$, if there exists a positive constant $K$ with the following property: For every $ε > 0, y \in C^n[a, b]$, if

$$\left|y^{(n)}(x) + β(x)y(x)\right| ≤ ε,$$

and $y(a) = y'(a) = ... = y^{(n-1)}(a) = 0$, then there exists some $z \in C^n[a, b]$ satisfying $Z^{(n)} + β(x)z(x) = 0$ and $z(a) = z'(a) = ... = z^{(n-1)}(a) = 0$, such that

$$|y(x) - z(x)| ≤ Ke.$$
**Theorem 1.1.** If \( \max |\beta(x)| < \frac{n!}{(b-a)^n} \) Then
\[
y^{(n)}(x) + \beta(x)y(x) = 0
\]
has the Hyers-Ulam stability with initial conditions
\[
y(a) = y'(a) = ... = y^{(n-1)}(a) = 0
\]
where \( y \in C^n[a,b], \beta \in C[a,b] \) and \( f : [a,b] \to \mathbb{R} \) continuous.

**Proof.** For every \( \epsilon > 0 \), By using the Taylor formula, we have
\[
y(x) = y(a) + y'(x-a) + \ldots + \frac{y^{(n)}(\xi)}{n!}(x-a)^n.
\]
Thus
\[
|y(x)| = \left| \frac{y^{(n)}(\xi)}{n!}(x-a)^n \right| \\
\leq \max |y^{(n)}(x)| \frac{(b-a)^n}{n!} \quad \forall x \in [a,b],
\]
then
\[
\max |y(x)| \leq \frac{(b-a)^n}{n!} \left[ \max |y^{(n)}(x) - \beta(x)y(x)| \max |y(x)| \right]
\]
Now using (1.7), we obtain
\[
\max |y(x)| \leq \frac{(b-a)^n}{n!} \left[ \max |y^{(n)}(x) - \beta(x)y(x)| + \max |\beta(x)| \max |y(x)| \right] \\
\leq \frac{(b-a)^n}{n!} \epsilon + \frac{(b-a)^n}{n!} \max |\beta(x)| \max |y(x)|.
\]
Let \( \eta = ((b-a)^n \max |\beta(x)|) / n!, \ K = (b-a)^n / (n!(1-\eta)). \) It is easy to see that \( z_0(\epsilon) = 0 \) is a solution of \( y^{(n)}(x) - \beta(x)y = 0 \) with initial conditions (1.8).

\[
|y - z_0(\epsilon)| \leq Ke.
\]
Hence (1.7) has the Hyers-Ulam-Rassias stability with initial conditions (1.8). \( \square \)

## 2 Main Result

In this section, we shall prove the Generalized Hyers-Ulam-Rassias Stability of the IVP
\[
y^{(n)} + \beta(x)y(x) = f(x)
\]
\[
y(a) = y'(a) = y''(a) = ... = y^{(n-1)}(a) = 0,
\]
where \( y \in C^n[a,b], \beta \in C[a,b] \) and \( f : [a,b] \to \mathbb{R} \) continuous.

**Theorem 2.2.** Suppose \( |\beta(x)| < M \) where \( M = \frac{n!}{(b-a)^n} \), \( \varphi : [a,b] \to [0,\infty) \) in an increasing function. The equation (2.9) has the Hyers-Ulam-Rassias stability if for \( \theta_\varphi \in C(\mathbb{R}_+,\mathbb{R}_+) \) and for each approximate solution \( y \in C^n[a,b] \) of (2.9) satisfying
\[
|y^{(n)} - \beta(x)y(x) - f(x)| \leq \varphi(x)
\]
there exist a solution \( z_0 \in C^n[a,b] \) of (2.9) with condition (2.10) such that
\[
|y(x) - z_0(x)| \leq \theta_\varphi(x).
\]
Proof. From (2.11) we have that

$$-\varphi(x) \leq y^{(n)}(x) - \beta(x)y(x) - f(x) \leq \varphi(x).$$

Integrating from \(a\) to \(x\), and applying condition (2.10) we have

$$-\int_a^x \varphi(s)ds \leq y^{(n-1)}(x) - \int_a^x \beta(s)y(s)ds - \int_a^x f(s)ds \leq \int_a^x \varphi(s)ds.$$

On further integration and also applying condition (2.10) we have

$$-\int_a^{s_1} \int_a^x \varphi(s)dsds_1 \leq y^{(n-2)}(x) - \int_a^{s_1} \int_a^x \beta(s)y(s)dsds_1$$

$$-\int_a^{s_1} \int_a^x f(s)dsds_1 \leq \int_a^{s_1} \int_a^x \varphi(s)dsds_1.$$

Continuing the process finally we can get,

$$-\int_a^{s_n-1} \int_a^{s_{n-2}} \ldots \int_a^{s_2} \int_a^x \varphi(s)dsds_1 \ldots ds_{n-1}$$

$$\leq y(x) - \int_a^{s_n-1} \int_a^{s_{n-2}} \ldots \int_a^{s_2} \int_a^x \beta(s)y(s)dsds_1 \ldots ds_{n-1}$$

$$-\int_a^{s_n-1} \int_a^{s_{n-2}} \ldots \int_a^{s_2} \int_a^x f(s)dsds_1 \ldots ds_{n-1}$$

$$\leq \int_a^{s_n-1} \int_a^{s_{n-2}} \ldots \int_a^{s_2} \int_a^x \varphi(s)dsds_1 \ldots ds_{n-1}.$$

Now applying Lemma (1.1), we obtain

$$-\int_a^x \frac{(x-s)^{n-1}}{(n-1)!} \varphi(s)ds \leq y(x) - \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} \beta(s)y(s)ds - \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} f(s)ds$$

$$\leq \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} \varphi(s)ds.$$

Hence we have

$$\left| y(x) - \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} (\beta(s)y(s)ds + f(s)ds) \right| \leq \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} \varphi(s)ds. \quad (2.13)$$

If we choose \(z_0(x)\) such that it solves equation (2.9) with (2.10) such that

$$z_0(x) = \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} (\beta(s)z_0(s)ds + f(s)ds),$$

thus we estimate

$$|y(x) - z_0(x)| \leq \left| y(x) - \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} (\beta(s)y(s)ds + f(s)ds) \right|$$

$$+ \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} (\beta(s)y(s)ds + f(s)ds) - \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} (\beta(s)z_0(s)ds + f(s)) \right| ds$$

$$\leq \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} \varphi(s)ds + \int_a^x \left| \frac{(x-s)^{n-1}}{(n-1)!} \beta(s)|y(s) - z_0(s)| \right| ds.$$

Now applying (2.13) and Theorem 1.1, we get

$$|y(x) - z_0(x)| \leq \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} \varphi(s)ds + |\beta(s)| \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} |y(s) - z_0(s)| ds$$

$$|y(x) - z_0(x)| \leq \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} \varphi(s)ds + \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} |y(s) - z_0(s)| ds.$$
Applying Gronwall’s inequality, we have
\[
|y(x) - z_0(x)| \leq \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} \varphi(s) ds \exp \left\{ M \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} ds \right\}
\]
\[
|y(x) - z_0(x)| \leq \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} \varphi(s) ds \exp \left\{ M \left[ \frac{(x-a)^n}{n!} \right] \right\}
\]
\[
|y(x) - z_0(x)| \leq c \int_a^x \frac{(x-s)^{n-1}}{(n-1)!} \varphi(s) ds
\]
with
\[
c = \exp \left\{ \left[ \frac{x-a}{b-a} \right]^n \right\}
\]
and the proof is completed.

**Remark:** Note that as \( x \to b \), then the above system considered is Hyers-Ulam stable.

**Conclusion**
We obtained the Hyers-Ulam-Rassias stability of nth order linear ordinary differential nonhomogeneous equation with initial conditions. Hyers-Ulam-Rassias stability guarantees that there is a close exact solution of the system.

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**Competing interests**
The authors declare that they have no competing interests.

**Authors’ contributions**
All authors conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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