Intersection graph of subgroups of some non-abelian groups

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Abstract

The intersection graph of subgroups of a group $G$ is a graph whose vertex set is the set of all proper subgroups of $G$ and two distinct vertices are adjacent if and only if their intersection is non-trivial. In this paper, we obtain the clique number and degree of vertices of intersection graph of subgroups of dihedral group, quaternion group and quasi-dihedral group.

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1 Introduction

There are several graphs associated with algebraic structures to investigate some specific properties of algebraic structures. Among them the intersection graphs have its own importance, which have been studied in the literature over the past fifty years. In 1964, Bosak [1] initiated the study of the intersection graphs of semigroups. Later, Csákány and Pollák [5] defined the intersection graph of subgroups of finite group. Followed by this, Zelinca investigated the intersection graph of subgroups of a finite abelian group [7]. In the recent years, several interesting properties of the intersection graphs of subgroups groups have been obtained in the literature, see for instance [2], [4], [5], [6] and the references therein.

Let $G$ be a group. The intersection graph of subgroups of $G$, denoted by $\mathcal{I}(G)$, is a graph with all the proper subgroups of $G$ as its vertices and two distinct vertices in $\mathcal{I}(G)$ are adjacent if and only if the corresponding subgroups have a non-trivial intersection in $G$.

Let $G$ be a simple graph. The degree of a vertex $v$ in $G$, denoted by $\deg_G(v)$ is the number of vertices to which $v$ is adjacent. A clique of $G$ is a complete subgraph of $G$. The clique number of $G$ is the is the cardinality of a largest clique in $G$ and it is denoted by $\omega(G)$.

For a positive integer $n$, $\tau(n)$ denotes the number of positive divisor of $n$; $\sigma(n)$ denotes the sum of all the positive divisors of $n$.

The aim of this paper is to find the clique number and degree of vertices of the intersection graph of subgroups of dihedral group, quaternion group and quasi-dihedral group.

We will use the following result of Chakrabarty et al. in the subsequent section.

Theorem 1.1. [3] Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $p_i$’s are distinct primes and $a_i \geq 1$. If $H$ is a proper subgroup of $\mathbb{Z}_n$ with $|H| = p_1^{\beta_1} p_2^{\beta_2} \cdots p_r^{\beta_r}$, then $\deg_{\mathcal{I}(\mathbb{Z}_n)}(H) = \tau(n) - \prod_{j \in \{i_1, i_2, \ldots, i_r\}} (a_j + 1) - 3$.

2 Properties of $\mathcal{I}(D_n), \mathcal{I}(Q_n), \mathcal{I}(QD_{2a})$

First, we start with the dihedral group. The dihedral group of order $2n$ ($n \geq 3$) is defined by

$$D_n = \langle a, b \mid a^n = b^2 = 1, ab = ba^{-1} \rangle.$$
The subgroups of $D_n$ are listed below:

(i) cyclic groups $H_{r0}^i := \langle a^r \rangle$ of order $r$, where $r$ is a divisor of $n$;

(ii) cyclic groups $H_{r1}^i := \langle a^r b \rangle$ of order 2, where $i = 1, 2, \ldots, n$;

(iii) dihedral groups $H_{r}^i := \langle a^r, a^rb \rangle$ of order $2r$, where $r$ is a divisor of $n$, $r \neq 1, n$ and $i = 1, 2, \ldots, \frac{n}{r}$.

The number of subgroups of $D_n$ listed in (i), (ii), (iii) are $\tau(n) - 1$, $n, \sigma(n) - n - 1$ respectively and so the total number of proper subgroups of $D_n$ is $\tau(n) + \sigma(n) - 2$.

**Theorem 2.2.** Let $n \geq 3$ be an integer with $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $p_i$’s are distinct primes and $a_i \geq 1$, and let $r = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k}$ be a divisor of $n$. Then

1. $\deg_{\mathcal{S}(D_n)}(H_{r0}^i) = \tau(n) + \sigma(n) - n - 3 - \sum_{d|n, d \neq 1} \prod_{j \notin \{i, j_2, \ldots, j_r\}} (\alpha_j + 1)$, where $s = \frac{n}{\prod_{j \notin \{i, j_2, \ldots, j_r\}} p_j^{\alpha_j}}$.

2. For each $r \neq 1, n$ and $i = 1, 2, \ldots, \frac{n}{r}$, $\deg_{\mathcal{S}(D_n)}(H_{r1}^i) = \tau(n) + \sigma(n) - n - 2 - \sum_{j \notin \{i, j_2, \ldots, j_r\}} (\alpha_j + 1)$.

3. For each $i = 1, 2, \ldots, n$, $\deg_{\mathcal{S}(D_n)}(H_{r}^i) = \tau(n) - 2$.

**Proof.** (1): First we count the number of subgroups listed in (i) to which $H_{r0}^i$ is adjacent in $\mathcal{S}(D_n)$. Here $\langle a \rangle \cong \mathbb{Z}_n$, so by Theorem 1.1, $H_{r0}^i$ is adjacent with $\tau(n) - 1 - \sum_{j \notin \{i, j_2, \ldots, j_r\}} (\alpha_j + 1)$ subgroups of $\mathbb{Z}_n$ including $\mathbb{Z}_n$. Clearly $H_{r0}^i$ is not adjacent with all the $n$ subgroups of $D_n$ listed in (ii). Finally, we count the number of subgroups listed in (iii) to which $H_{r0}^i$ is adjacent. For every divisor $d \neq 1$ of $s = \frac{n}{\prod_{j \notin \{i, j_2, \ldots, j_r\}} p_j^{\alpha_j}}$, $H_{r0}^i$ is not adjacent with $H_{r1}^{ij}, i = 1, 2, \ldots, \frac{n}{r}$, $H_{r2}^i$ is adjacent with each of the remaining proper subgroups of $D_n$ listed in (iii). The total number of such subgroups is $\sigma(n) - n - 1 - \sum_{d|s, d \neq 1} \frac{n}{d}$. Summing up all these values gives the degree of $H_{r0}^i$.

(2): For each $r \neq 1, n$ and $i = 1, 2, \ldots, \frac{n}{r}$, $H_{r0}^i$ is the maximal cyclic subgroup of $H_{r}^i$ and so the number of subgroups listed in (i) to which $H_{r1}^i$ is adjacent is the same as the number of subgroups listed in (i) to which $H_{r0}^i$ is adjacent including $H_{r0}^i$. The number of such subgroup is $\tau(n) - 1 - \sum_{j \notin \{i, j_2, \ldots, j_r\}} (\alpha_j + 1) - 1$. Among the subgroups of $D_n$ listed in (ii), $H_{r1}^i$ has exactly $r$ subgroups as its subgroups and so $H_{r1}^i$ is adjacent with only these subgroups in the list. Finally, we count the number of subgroups listed in (iii) to which $H_{r1}^i$ is adjacent. For every divisor $d$ of $r$, $H_{r1}^i$ is intersect with $H_{r1}^j$; for every divisor $d$ of $s$, $(d, r) = 1$ and so by chinese remainder theorem there exist an integer, let it be $t$ such that $H_{r1}^i$ is a subgroup of both $H_{r1}^j$ and $H_{r1}^t$. So $H_{r1}^i$ adjacent with all the subgroups of $D_n$ listed in (iii). The total number of such subgroups is $\sigma(n) - n - 1$. The degree of is just the sum of these three values.

(3): For each $i = 1, 2, \ldots, n$, the order of $H_{r1}^i$ is 2. The number of subgroups of $D_n$ contains $H_{r1}^i$ is $\tau(n) - 2$ and $H_{r1}^i$ is not intersect with remaining proper subgroups of $D_n$, since order of $H_{r1}^i$ is prime. This completes the proof.

**Theorem 2.3.** For $n \geq 3$, $\omega(\mathcal{S}(D_n)) = \sigma(n) - n - 1 + \sum_{i=1}^{k} a_i$.

**Proof.** Take $A := C_1 \cup C_2$, where $C_1 := \{H_{r0}^i | r | n, r \neq 1, n, i = 1, 2, \ldots, \frac{n}{r} \}$ and $C_2 := \bigcup \{\langle a^r \rangle | r | n, r \neq 1 \}$ with $r$ has every prime divisor of $n$ as a factor. Clearly $A$ is a maximal clique and $|A| = |C_1| + |C_2| = (\sigma(n) - n - 1) + \sum_{i=1}^{k} a_i$. Let $B$ be another clique different from $A$. Then $B$ should contains either $\langle a^r \rangle$, for some $r | n, r \neq 1$ and $r$ does not contains all the prime divisors of $n$ or $\langle a^r b \rangle$, for some $i = 1, 2, \ldots, n$. If $B$ contains the subgroup $\langle a^r \rangle$, for some $r | n, r \neq 1$ and $r$ does not contains all the prime divisors of $n$, then let $p_j$ be the prime divisor of $n$ which is not a divisor of $r$. Here $G$ has at least two subgroups of order $2p_j$ and so we cannot take the subgroups of order $2p_j$ in $B$. It follows that $|B| < |A|$. If $B$ contains the subgroup $\langle a^r b \rangle$, for some $i = 1, 2, \ldots, n$, then $\langle a^r b \rangle$ adjacent with $\tau(n) - 2$ and so we cannot take $\sigma(n) - \tau(n) + 1$ subgroups in $B$. It follows that $|B| < |A|$. This completes the proof. □
Next, we consider the quaternion group. For any integer \( n > 1 \), the quaternion group of order \( 4n \) is defined by
\[
Q_n = \langle a, b | a^{2n} = b^4 = 1, b^2 = a^n, ab = ba^{-1} \rangle.
\]
The subgroups of \( Q_n \) are listed below:

(i) cyclic groups \( H_{0,r} := \langle a^r \rangle \), of order \( r \), where \( r \) is a divisor of \( 2n \);
(ii) cyclic groups \( H_{i,1} := \langle a^i b \rangle \) of order 4, where \( i = 1, \ldots, n \);
(iii) quaternion groups \( H_{i,r} := \langle a^r, a^ib \rangle \) of order \( 4r \), where \( r \) is a divisor of \( n, i = 1, \ldots, \frac{n}{r} \).

The number of subgroups \( Q_n \) is \( 4n \), since order of \( H_{0,r} \) is 4. Finally we count the number of subgroups listed in (iii) to which \( H_{i,r} \) is adjacent with remaining proper subgroups of \( Q_n \).

**Theorem 2.4.** Let \( n > 1 \) be an integer with \( n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \), where \( p_i \)'s are distinct primes and \( a_i \geq 1 \), and let \( r = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k} \) be a divisor of \( n \).

(1) If \( r \) is even, then \( \deg_{\mathcal{F}(Q_n)}(H_{0,r}) = \tau(2n) + \sigma(n) - 3 - \prod_{j \notin \{i_1, i_2, \ldots, i_r\}} (a_j + 1) \).

(2) If \( r \) is odd, then \( \deg_{\mathcal{F}(Q_n)}(H_{0,r}) = \tau(2n) + \sigma(n) - n - 3 - \prod_{j \notin \{i_1, i_2, \ldots, i_r\}} (a_j + 1) \).\]

(3) For each \( i = 1, \ldots, \frac{n}{r} \), \( \deg_{\mathcal{F}(Q_n)}(H_{i,r}) = \tau(2n) + \sigma(n) - 3 - \prod_{j \notin \{i_1, i_2, \ldots, i_r\}} (a_j + 1) \), where \( a_j \)'s are powers of odd prime factors of \( n \).

**Proof.** (1)-(2): First we count the number of subgroups listed in (i) to which \( H_{0,r} \) is adjacent. Here \( \langle a \rangle \cong \mathbb{Z}_{2n} \), by Theorem 1.1, \( H_{0,r} \) adjacent with \( \tau(2n) - \prod_{j \notin \{i_1, i_2, \ldots, i_r\}} (a_j + 1) - 2 \) subgroups of \( \mathbb{Z}_{2n} \) including \( \mathbb{Z}_{2n} \). Now we consider the following two cases:

**Case a:** \( r \) is even. Here \( Q_n \) has an unique subgroup of order 2 and so every subgroup of even order in \( Q_n \) are adjacent with each other, so \( H_{0,r} \) is adjacent with \( \sigma(n) - 1 \) subgroups of \( Q_n \) excluding \( Q_n \) listed in (ii), (iii). This completes the proof of part (1).

**Case b:** \( r \) is odd. Clearly \( H_{0,r} \) is not adjacent with all the \( n \) subgroups of \( Q_n \) listed in (ii), since order of \( H_{1,r} \) is 4. Finally we count the number of subgroups listed in (iii) to which \( H_{0,r} \) is adjacent. For every divisor \( d \neq 1 \) of \( s = d = p_1^{\beta_1} p_2^{\beta_2} \cdots p_k^{\beta_k} \), \( H_{0,r} \) is not adjacent with \( H_{1,d}, i = 1, 2, \ldots, \frac{n}{d} ; H_{0,r} \) is adjacent with remaining proper subgroups of \( Q_n \) listed in (iii). The total number of such subgroups is \( \sigma(n) - n - 1 - \sum_{d \neq 1} \frac{n}{d} \). This completes the proof of part (2).

(3): For each \( i = 1, \ldots, \frac{n}{r} \), \( H_{0,r} \) is the maximal cyclic subgroup of \( H_{i,r} \) and so the number of subgroups listed in (i) to which \( H_{i,r} \) is adjacent is the same as the number of subgroups listed in (i) to which \( H_{0,r} \) is adjacent including \( H_{0,r} \). The number of such subgroups is \( \tau(n) - \prod_{j \notin \{i_1, i_2, \ldots, i_r\}} (a_j + 1) - 1 \). Also \( Q_n \) has a unique subgroup of order 2 and so \( H_{i,r} \) is adjacent with all the subgroups listed in (i), (ii), since order of subgroups of \( Q_n \) listed in (ii), (iii) is even. The total number of such subgroups is \( \sigma(n) - 1 \). This completes the proof of part (3).

(4): Since \( Q_n \) has an unique subgroup of order 2, so \( H_{1,1} \) is adjacent with all the subgroups listed in (ii), (iii). Also \( H_{1,1} \) is adjacent all the even order subgroups of \( Q_n \) listed in (i). But \( H_{1,1} \) is not adjacent with an odd order subgroups of \( Q_n \) listed in (i). The number of such subgroups is \( \tau(2n) + \sigma(n) - \prod_{j \notin \{i_1, i_2, \ldots, i_r\}} (a_j + 1) - 2 \), where \( a_j \)'s are powers of odd prime factors of \( n \). This completes the proof of part (4).

**Theorem 2.5.** Let \( n > 1 \) be an integer and \( 2n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k} \), where \( p_i \)'s are distinct primes and \( a_i \geq 1 \). Then
\[
\omega(\mathcal{F}(Q_n)) = \sigma(n) + a_1 \prod_{i=2}^{k} (a_i + 1) - 1.
\]
Proof. Let $\mathcal{A}$ be the set of all even order subgroups of $Q_n$. Then $|\mathcal{A}| = \sigma(n) + a_k \prod_{i=2}^{k}(\alpha_i + 1) - 1$ and $\mathcal{A}$ is a maximal clique in $\mathcal{S}(Q_n)$. Let $B$ be another clique different from $\mathcal{A}$. Then $B$ should contains $\langle a_i \rangle$, for some an odd divisor $r$ of $n$, $r \neq 1$. Then $B$ cannot contain the subgroups of order 4. It follows that $|B| < |\mathcal{A}|$. This completes the proof.

Finally, we consider the quasi-dihedral group. For any positive integer $\alpha > 3$, the quasi-dihedral group of order $2^\alpha$, is defined by

$$QD_{2^\alpha} = \langle a, b \mid a^{2^\alpha - 1} = b^2 = 1, bab^{-1} = a^{2^\alpha - 2} - 1 \rangle.$$  

The proper subgroups of $QD_{2^\alpha}$ are listed below:

(i) cyclic groups $H_0 = \langle a^{2^\alpha - 1} \rangle$, where $r$ is a divisor of $2^\alpha - 1$, $r \neq 1$;

(ii) the dihedral group $H_1 = \langle a^2, b \rangle \cong D_{2^{\alpha - 2}}$ and the dihedral subgroups $H_i$ of $H_1^{2^\alpha - 2}$, where $r$ is a divisor of $2^\alpha - 2$, $r \neq 2^\alpha - 2$, $i \in \{1, 2, \ldots, \frac{2^\alpha - 2}{r} \}$;

(iii) the quaternion group $H_2 = \langle a^2, ab \rangle \cong Q_{2^\alpha - 3}$ and the quaternion subgroups $H_i, r \in H_2, 2^\alpha - 3$, where $r$ is a divisor of $2^\alpha - 3$, $r \neq 2^\alpha - 3$, $i \in \{1, 2, \ldots, \frac{2^\alpha - 3}{r} \}$.

The number of subgroups of $QD_{2^\alpha}$ listed in (i), (ii), (iii) are $\tau(2^\alpha - 1) - 1, 2^\alpha - 1, 2^\alpha - 2 - 1$ and so the total number of proper subgroups of $QD_{2^\alpha}$ is $\alpha + 3(2^\alpha - 2) - 1$.

Theorem 2.6. If $\alpha \geq 4$, then

(1) for each divisor $r$ of $2^\alpha - 1$, $r \neq 1$, $\deg_{\mathcal{S}(QD_{2^\alpha})}(H_0) = \alpha + 2^\alpha - 1 - 4$;

(2) for each divisor $r$ of $2^\alpha - 2$, $r \neq 1, i = 1, 2, \ldots, \frac{2^\alpha - 2}{r}$, $\deg_{\mathcal{S}(QD_{2^\alpha})}(H_i) = \alpha + 2^\alpha - 1 - r - 4$;

(3) for each divisor $r$ of $2^\alpha - 3$, $r \neq 1, i = 1, 2, \ldots, \frac{2^\alpha - 3}{r}$, $\deg_{\mathcal{S}(QD_{2^\alpha})}(H_i) = \alpha + 2^\alpha - 1 - 4$;

(4) for $i = 2, 2^2, \ldots, 2^\alpha - 2$, $\deg_{\mathcal{S}(QD_{2^\alpha})}(H_i) = \alpha - 2$;

(5) for $i = 1, 3, \ldots, 2^\alpha - 3$, $\deg_{\mathcal{S}(QD_{2^\alpha})}(H_i) = \alpha + 2^\alpha - 1 - 4$.

Proof. The only maximal subgroups of $QD_{2^\alpha}$ are $H_0 = H_1^{2^\alpha - 1}$, the dihedral subgroup $H_1^{2^\alpha - 2}$ and quaternion subgroup $H_2, 2^\alpha - 3$. Here $H_0$ is a subgroup of all the subgroup of $QD_{2^\alpha}$ other than $H_1^{2^\alpha - 2}$, $i = 2, 2^2, \ldots, 2^\alpha - 2$; also no subgroups listed in (i), (iii) are adjacent with $H_i$, $i = 2, 2^2, \ldots, 2^\alpha - 2$. It follows that $\deg_{\mathcal{S}(QD_{2^\alpha})}(H_0) = \alpha + 3(2^\alpha - 2 - 1) - 2^\alpha - 2 - 1 = \alpha + 2^\alpha - 1 - 4$. Proofs of parts (3) and (5) are similar to the above.

Next, we count the number of subgroups of $QD_{2^\alpha}$ to which $H_i$ is adjacent. By the above argument $H_i$ is adjacent with all the subgroups listed in (i), (iii) and the dihedral subgroups of $H_i^{2^\alpha - 2}$, also $H_i$ has $r$ subgroups of order 2 as its subgroups and so $H_i$ adjacent with these subgroups, so $\deg_{\mathcal{S}(QD_{2^\alpha})}(H_i) = \alpha + 3(2^\alpha - 2 - 1) - 2^\alpha - 2 - r - 1 = \alpha + 2^\alpha - 1 + r - 4$.

Finally, we count the number of subgroups of $QD_{2^\alpha}$ to which $H_i$ is adjacent. Note that $\deg_{\mathcal{S}(QD_{2^\alpha})}(H_1) = \deg_{\mathcal{S}(D_3)}(H_1) + 1$, since order of $H_1$ is 2, and it is not a subgroup of any subgroups of $H_i, 2^\alpha - 3$; $H_i^{2^\alpha - 2}$ is also a vertex of $\mathcal{S}(QD_{2^\alpha})$. So by Theorem 2.2(3), we have $\deg_{\mathcal{S}(QD_{2^\alpha})}(H_1) = \tau(2^\alpha - 1) - 1 = \alpha - 2$. Hence the proof.

Theorem 2.7. For $\alpha \geq 3$, $\omega(\mathcal{S}(QD_{2^\alpha})) = \alpha + 2^\alpha - 1 - 3$.

Proof. Let $\mathcal{A}$ be the set of all subgroups of $QD_{2^\alpha}$ other than $\langle ba^i \rangle$, $i = 2, 2^2, \ldots, 2^\alpha - 2$. Clearly $\mathcal{A}$ is a maximal clique in $\mathcal{S}(QD_{2^\alpha})$ and $|\mathcal{A}| = \alpha + 3(2^\alpha - 2 - 1) - 2^\alpha - 2 = \alpha + 2^\alpha - 1 - 3$. Let $B$ be another clique in $\mathcal{S}(QD_{2^\alpha})$. Then $B$ contains exactly one subgroup of the form $\langle ba^i \rangle$, $i = 2, 2^2, \ldots, 2^\alpha - 2$. It follow that $|B| < |\mathcal{A}|$, since for one cyclic subgroup in $B$, we take more than one quaternion subgroups in $\mathcal{A}$. This completes the proof.
References


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