On strong domination number of corona related graphs

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Abstract
Let \( G = (V(G), E(G)) \) be a graph and \( uv \in E(G) \) be an edge. A vertex \( u \) strongly dominates \( v \) if \( d_G(u) \geq d_G(v) \). A set \( S \subseteq V(G) \) is a strong dominating set (sd-set) if every vertex \( v \in V(G) - S \) is strongly dominated by some \( u \) in \( S \). The minimum cardinality of a strong dominating set is called the strong domination number of \( G \) which is denoted by \( \gamma_s(G) \). We investigate strong domination number of some corona related graphs.

Keywords
Dominating set, domination number, strong dominating set, strong domination number.

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The concept of strong (weak) domination was introduced by Sampathkumar and Pushpa Latha [13]. Some bounds on \( \gamma_s(G) \) were investigated by Rautenbach [10, 11]. Also some bounds on strong and weak domination numbers were investigated by Sampathkumar and Pushpa Latha [13] while the same authors in [11] have improved these bounds and reported the graphs achieving such bounds. Rautenbach and Zverovich [12] have studied results on the NP-complete problems of strong dominating set and weak dominating set. Gani and Ahamed [4] have introduced the concept of strong and weak domination in fuzzy graphs and provided some examples to explain various notions. Vaidya and Karkar [16, 17] have investigated the strong domination number of some path related graphs and independent strong domination number of the graphs obtained by switching of a vertex in \( P_n \). The strong domination in \( m \)-splitting graph of \( P_n, C_n \) and \( K_{m,n} \) is discussed by Vaidya and Mehta [18] while the same authors in [19] have investigated the strong domination number for the generalized Petersen graph. The relation between the strong domination and weak domination number is given by Boutrig and Chellali [2] while Meena et al. [9] have compared strong efficient domination...
number with strong domination number. The relation between strong domination and maximum degree of the graph as well as weak domination and minimum degree of the graph were revealed by Swaminathan and Thangaraju [15]. To obtain strong domination number of larger graph (super graph) obtained from the given graph is challenging and interesting as well. We have studied such problem in the context of corona of two graphs, edge corona of two graphs and neighbourhood corona of two graphs.

2. Main Results

The concept of corona of two graphs was introduced by Frucht and Harary [3].

Definition 2.1. Let G and H be two graphs on n and m vertices, respectively. The corona of the graphs G and H denoted by $G \circ H$ and is defined as the graph obtained by taking one copy of G and n copies of H, and then joining the $i^{th}$ vertex of G to every vertex in the $i^{th}$ copy of H.

Proposition 2.2. [5] Let G be a connected graph of order n and let H be any graph of order m. Then, $\gamma(G \circ H) = n$.

We prove the following result.

Theorem 2.3. Let G be a connected graph of order n and let H be any simple graph of order m. Then, $\gamma_d(G \circ H) = n$.

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(H) = \{u_1, u_2, \ldots, u_m\}$. In $G \circ H$, denote the $i^{th}$ copy of H by $H_i$ and the vertices of $H_i$ by $u'_1, u'_2, \ldots, u'_m$ for $1 \leq i \leq n$, that is, $V(H_i) = \{u'_1, u'_2, \ldots, u'_m\}$. By the definition of the corona graph, $d_{G \circ H}(v_i) > d_{G \circ H}(u'_i), 1 \leq i \leq n$ and $1 \leq j \leq m$. Therefore, every vertex $v_i$ strongly dominates all the vertices of $H_i$ as well as the vertices of G which are adjacent to $v_i$, and having degree greater than or equal to that of vertex $v_i$. We also observe that any vertex $v_i$ can not strongly dominate any vertex of $H_j$ for $i \neq j$. Therefore, it is enough to include n vertices, namely $v_1, v_2, \ldots, v_n$ in any strong dominating set S. Hence, S becomes the strong dominating set of minimum cardinality implying that $\gamma_d(G \circ H) = n$.

Illustration 2.4. In Figure 1, $S = \{v_1, v_2, v_3, v_4, v_5\}$ is strong dominating set of the graph $C_5 \circ P_2$ and $\gamma_d(C_5 \circ P_2) = 5$. The strong dominating set is shown with solid vertices.

Illustration 2.5. In Figure 2, $S = \{v_1, v_2, v_3, v_4\}$ is a strong dominating set of the graph $P_4 \circ K_1$ and $\gamma_d(P_4 \circ K_1) = 4$. The strong dominating set is shown with solid vertices.

Figure 2. $P_4 \circ K_1$

Definition 2.6. Duplication of a vertex $v_k$ by a new edge $e = v'_k v'_k$ in a graph $G$ produces a new graph $G'$ such that $N(v'_k) = \{v_k, v'_k\}$ and $N(v''_k) = \{v_k, v'_k\}$.

Corollary 2.7. Let G be a graph with $n$ vertices and $G'$ be the graph obtained by duplication of every vertex of a connected graph G by an edge. Then, $\gamma_d(G') = n$.

Proof. In $G \circ H$ let G be any connected graph and $H = P_2$ then $G' \cong G \circ H$. Therefore, by Theorem 2.3 $\gamma_d(G') = \gamma_d(G \circ H) = n$.

The concept of edge corona of two graphs was introduced by Hou and Shiou [8] and defined as follows.

Definition 2.8. Let G and H be two graphs on n and m vertices, k and l edges, respectively. The edge corona $G \bowtie H$ of G and H is defined as the graph obtained by taking one copy of G and k copies of H, and then joining two end vertices of the i-th edge of G to every vertex in the i-th copy of H.

Theorem 2.9. $\gamma_d(P_n \bowtie H) = \left\lceil \frac{n}{2} \right\rceil$, where H is any simple graph.

Proof. Let $V(P_n) = \{v_1, v_2, \ldots, v_n\}, E(P_n) = \{e_1, e_2, \ldots, e_{n-1}\}$ and $V(H) = \{u_1, u_2, \ldots, u_m\}$. In $P_n \bowtie H$, denote the $i^{th}$ copy of H by $H_i$ and the vertices of $H_i$ by $u'_1, u'_2, \ldots, u'_m$ for $1 \leq i \leq n$, that is, $V(H_i) = \{u'_1, u'_2, \ldots, u'_m\}$. By the definition of the edge corona graph, $d_{P_n \bowtie H}(v_i) \geq d_{P_n \bowtie H}(u'_i), 1 \leq i \leq n - 1$ and $1 \leq j \leq m$.

Therefore, the vertex $v_i$ of $P_n$ strongly dominates all vertices of $H_{i-1}$ and $H_i$ as well as the vertices of G which are adjacent to $v_i$ and having degree greater than or equal to that of vertex $v_i$. We also observe that any vertex $v_i$ can not strongly dominate any vertex of $H_j$ for $i \neq j$. Therefore, it is enough to include n vertices, namely $v_1, v_2, v_3, v_4, v_5$ in any strong dominating set S. Hence, S becomes the strong dominating set of minimum cardinality implying that $\gamma_d(G \circ H) = n$.

Illustration 2.5. In Figure 2, $S = \{v_1, v_2, v_3, v_4\}$ is a strong dominating set of the graph $P_4 \circ K_1$ and $\gamma_d(P_4 \circ K_1) = 4$. The strong dominating set is shown with solid vertices.

Figure 2. $P_4 \circ K_1$
Illustration 2.10. In Figure 3, \( S = \{ v_2, v_3 \} \) is a strong dominating set of \( P_3 \circ P_2 \) and \( \gamma_d(P_3 \circ P_2) = 2 \). The strong dominating set is shown with solid vertices.

Definition 2.11. Duplication of an edge \( e = uv \) by a new vertex \( v' \) in a graph \( G \) produces a new graph \( G' \) by adding a vertex \( v' \) such that \( N(v') = \{ u, v \} \).

Corollary 2.12. Let \( G \) be a graph obtained by duplication of each edge of \( P_n \) by a vertex then \( \gamma_d(G) = \left\lceil \frac{n}{2} \right\rceil \).

Proof. Taking \( H = K_1 \) in \( P_n \circ H, G \cong P_n \circ H \). Therefore, by Theorem 2.9 \( \gamma_d(G) = \left\lceil \frac{n}{2} \right\rceil \).

Theorem 2.13. \( \gamma_d(C_n \circ H) = \left\lceil \frac{n}{2} \right\rceil \), where \( H \) is any simple graph.

Proof. Let \( V(C_n) = \{ v_1, v_2, \ldots, v_n \} \), \( E(C_n) = \{ e_1, e_2, \ldots, e_n \} \) and \( V(H) = \{ u_1, u_2, \ldots, u_m \} \). In \( C_n \circ H \), denote the \( i^{th} \) copy of the graph \( H \) by \( H_i \) and the vertices of \( H_i \) by \( u_1, u_2, \ldots, u_m \) for \( 1 \leq i \leq n \). That is, \( V(H_i) = \{ u_1, u_2, \ldots, u_m \} \). By definition of the edge corona graph, \( d_{C_n \circ H}(v_i) = d_{C_n \circ H}(u_i), 1 \leq i \leq n \) and \( 1 \leq j \leq m \). Therefore, the vertex \( v_i \) from \( C_n \) strongly dominates all vertices of \( H_{i-1} \) and \( H_i \) for as well as the vertices of \( G \) which are adjacent to \( v_i \) and having degree greater than or equal to that of vertex \( v_i \) for \( 2 \leq i \leq n \) while the vertex \( v_1 \) strongly dominates all vertices of \( H_0, H_1, v_2 \) and \( v_4 \). If \( C_n \) is a cycle of odd order then, \( v_1, v_3, v_5, \ldots, v_{n-1} \) strongly dominate all the vertices of \( C_n \circ H \). In other words at least \( \left\lceil \frac{n}{2} \right\rceil \) vertices are necessary to strongly dominate all the vertices of \( C_n \circ H \). If \( C_n \) is a cycle of even order then the vertices \( v_2, v_4, v_6, \ldots, v_{n} \) or \( v_1, v_3, v_5, \ldots, v_{n-1} \) strongly dominate all the vertices of \( C_n \circ H \). In other words at least \( \frac{n}{2} \) number of vertices are necessary to strongly dominate all the vertices of the graph \( C_n \circ H \). Therefore, \( \gamma_d(C_n \circ H) = \left\lceil \frac{n}{2} \right\rceil \).

Illustration 2.14. In Figure 4, \( S = \{ v_2, v_4 \} \) is a strong dominating set of graph \( C_4 \circ P_2 \) and \( \gamma_d(C_4 \circ P_2) = 2 \). The strong dominating set is shown with solid vertices.

Definition 2.15. The middle graph \( M(G) \) of a graph \( G \) is the graph whose vertex set is \( V(G) \) \( \cup E(G) \) and in which two vertices are adjacent whenever either they are adjacent edges of \( G \) or one is a vertex of \( G \) and the other is an edge incident with it.

Corollary 2.16. \( \gamma_d(M(C_n)) = \left\lceil \frac{n}{2} \right\rceil \).

Proof. In graph, \( C_n \circ H \) if we consider \( H = K_1 \) then \( C_n \circ H \cong M(C_n) \). Therefore, by Theorem 2.13, \( \gamma_d(M(C_n)) = \left\lceil \frac{n}{2} \right\rceil \).

Theorem 2.17. \( \gamma_d(K_{1,n} \circ H) = 1 \), where \( H \) is any simple graph.

Proof. Let \( V(K_{1,n}) = \{ v_0, v_1, v_2, \ldots, v_n \} \), \( E(K_{1,n}) = \{ e_1, e_2, \ldots, e_n \} \) and \( V(H) = \{ u_1, u_2, \ldots, u_m \} \). By definition of the edge corona graph, \( d_{K_{1,n} \circ H}(v_i) \geq d_{K_{1,n} \circ H}(u_i), 1 \leq i \leq n \) and \( 1 \leq j \leq m \). The vertex \( v_0 \) strongly dominates all vertices of \( K_{1,n} \circ H \). Therefore, \( S = \{ v_0 \} \) is the strong dominating set of minimum cardinality for \( K_{1,n} \circ H \). Hence, \( \gamma_d(K_{1,n} \circ H) = 1 \).

Illustration 2.18. In Figure 5, \( S = \{ v_1 \} \) is a strong dominating set of the graph \( K_{1,n} \circ K_1 \) and \( \gamma_d(K_{1,n} \circ K_1) = 1 \). The strong dominating set is shown with solid vertices.

Definition 2.19. A friendship graph \( F_n \) is a one point union of \( n \) copies of cycle \( C_3 \).

Corollary 2.20. \( \gamma_d(F_n) = 1 \).

Proof. Taking \( H = K_1 \) in \( K_{1,n} \circ H, F_n \cong K_{1,n} \circ K_1 \). Therefore, by Theorem 2.17 \( \gamma_d(K_{1,n} \circ H) = \gamma_d(F_n) = 1 \).

The following concept was introduced by Gopalapillai [6] recently.
Definition 2.21. Let \( G \) and \( H \) be two graphs on \( n \) and \( m \) vertices respectively. Then the neighborhood corona, \( G\bar{\star}H \) is the graph obtained by taking one copy of \( G \) and \( n \) copies of \( H \), and then joining each neighbor of \( i^{th} \) vertex of \( G \) to every vertex in the \( i^{th} \) copy of \( H \).

In neighborhood corona \( G\bar{\star}H \) if we consider \( H = K_1 \) then \( G\bar{\star}H \) becomes a splitting graph. The splitting graph is introduced by Sampathkumar and Walikar [14].

Theorem 2.22. For \( n \geq 4 \),

\[
\gamma_{d}(P_n\bar{\star}G) = \begin{cases} 
\frac{n}{2} & \text{if } n \equiv 0 \pmod{4}, \\
\frac{n+1}{2} & \text{if } n \equiv 1 \text{ or } 3 \pmod{4}, \\
\frac{n+2}{2} & \text{if } n \equiv 2 \pmod{4}.
\end{cases}
\]

Proof. Let \( V(P_n) = \{v_1, v_2, \ldots, v_n\} \) and \( V(G) = \{u_1, u_2, \ldots, u_m\} \). In \( P_n\bar{\star}G \), denote the vertices of \( i^{th} \) copy of the graph \( G \) by \( u_1, u_2, \ldots, u_m \) for \( 1 \leq i \leq n \), and \( H_i = \{u_1, u_2, \ldots, u_m\} \). 

It is very clear that \( d_{P_n\bar{\star}G}(v_i) = (m+1)d_{P_i}(v_i) \), for \( 1 \leq i \leq n \) and \( d_{P_n\bar{\star}G}(u'_j) = d_{P_i}(v_i) + d_{G}(u_j) \), for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \). Hence, \( d_{P_n\bar{\star}G}(v_i) \geq d_{P_i\bar{\star}G}(u'_j) \).

Case: I. \( n \equiv 0 \pmod{4} \)

Let \( V(P_n\bar{\star}G) = \{v_1, v_2, v_3, v_4\} \cup \{v_5, v_6, v_7, v_8\} \cup \ldots \cup \{v_{n-3}, v_{n-2}, v_{n-1}, v_n\} \cup H_1 \cup H_2 \cup \ldots \cup H_n \) be the partition of \( V(P_n\bar{\star}G) \). As per the discussion about the degree of the vertices of graph \( P_n\bar{\star}G \) in beginning of proof, the vertex \( v_1 \) and \( v_n \) strongly dominate only \( m+1 \) vertices while the vertices \( v_2, v_3, v_4, \ldots, v_{n-1} \) strongly dominate \( 2m+3 \) vertices from \( P_n\bar{\star}G \). Therefore, it is very clear that if we consider the vertex \( v_2 \) in a strong dominating set \( S \) then the vertices \( v_1, v_3, v_4 \) and all the vertices of \( H_1 \) and \( H_2 \) are strongly dominated by it. Now, to strongly dominate the vertices of \( H_2 \) the vertex \( v_1 \) or the vertex \( v_3 \) must belong to \( S \). But the vertex \( v_1 \) strongly dominates more vertices than the vertex \( v_1 \). Therefore, to form the strong dominating set \( S \) of minimum cardinality the vertex \( v_3 \) must belong to \( S \). Thus, the vertices \( v_2 \) and \( v_3 \) strongly dominate all the vertices of the sets \( \{v_1, v_2, v_3, v_4\}, H_1, H_2, H_3, \) and \( H_4 \). We can also observe that the vertices \( v_5, v_7, v_8 \) strongly dominate all the vertices of the sets \( \{v_5, v_6, v_7, v_8\}, H_5, H_6, H_7, \) and \( H_8 \). Similarly, the vertices \( v_{n-2} \) and \( v_{n-1} \) strongly dominate all the vertices of the sets \( \{v_{n-3}, v_{n-2}, v_{n-1}, v_n\}, H_{n-3}, H_{n-2}, H_{n-1}, \) and \( H_n \). Therefore, \( v_2, v_3, v_4, v_5, v_6, v_7, v_8, \ldots, v_{n-2}, v_{n-1} \) vertices strongly dominate all the vertices of \( P_n\bar{\star}G \). So, it is enough to consider \( \frac{n}{2} \) vertices from \( V(P_n) \) to strongly dominate all the vertices of the graph \( P_n\bar{\star}G \). Hence, \( \gamma_{d}(P_n\bar{\star}G) = \frac{n}{2} \).

Case: II. \( n \equiv 1 \pmod{4} \)

Let \( V(P_n\bar{\star}G) = \{v_1, v_2, v_3, v_4\} \cup \{v_5, v_6, v_7, v_8\} \cup \ldots \cup \{v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}\} \cup \{v_{n-2}, v_{n-1}, v_n\} \cup H_1 \cup H_2 \cup \ldots \cup H_n \) be the partition of \( V(P_n\bar{\star}G) \). As per the discussion in beginning of the case(I), \( \frac{n-1}{2} \) number of vertices \( \{v_2, v_3, v_6, v_7, \ldots, v_{n-3}, v_{n-2}\} \) from \( V(P_n) \) can strongly dominate the vertices \( v_1, v_2, v_3, \ldots, v_{n-1} \) and all the vertices of sets \( H_1, H_2, \ldots, H_{n-1} \). To strongly dominate the remaining vertices \( v_n \) and all the vertices of the set \( H_n \) the vertex \( v_{n-1} \) must be in a strong dominating set \( S \). Therefore, \( \gamma_{d}(P_n\bar{\star}G) = \frac{n+1}{2} \).

Hence, \( \gamma_{d}(P_n\bar{\star}G) = \frac{n+1}{2} \).

Case: III. \( n \equiv 2 \pmod{4} \)

Let \( V(P_n\bar{\star}G) = \{v_1, v_2, v_3, v_4\} \cup \{v_5, v_6, v_7, v_8\} \cup \ldots \cup \{v_{n-5}, v_{n-4}, v_{n-3}, v_{n-2}\} \cup \{v_{n-4}, v_{n-3}, v_{n-2}, v_{n-1}\} \cup \{v_{n-1}, v_n\} \cup H_1 \cup H_2 \cup \ldots \cup H_n \) be the partition of \( V(P_n\bar{\star}G) \). As per the discussion in beginning of the case(I), \( \frac{n-2}{2} \) number of vertices \( v_2, v_3, v_6, v_7, \ldots, v_{n-4}, v_{n-3} \) from \( V(P_n) \) can strongly dominate the vertices \( v_1, v_2, v_3, \ldots, v_{n-2} \) and all the vertices of the sets \( H_1, H_2, \ldots, H_{n-2} \). To strongly dominate the remaining vertices \( v_{n-1}, v_n \) and all the vertices of the sets \( H_{n-1} \) and the set \( H_n \) the vertex \( v_{n-1} \) and either \( v_{n-2} \) or \( v_n \) must be in a strong dominating set \( S \). Therefore, \( \gamma_{d}(P_n\bar{\star}G) = \frac{n+1}{2} \).

Case: IV. \( n \equiv 3 \pmod{4} \)

Let \( V(P_n\bar{\star}G) = \{v_1, v_2, v_3, v_4\} \cup \{v_5, v_6, v_7, v_8\} \cup \ldots \cup \{v_{n-6}, v_{n-5}, v_{n-4}, v_{n-3}\} \cup \{v_{n-2}, v_{n-1}, v_n\} \cup H_1 \cup H_2 \cup \ldots \cup H_n \) be the partition of \( V(P_n\bar{\star}G) \). As per the discussion in beginning of the case(I), \( \frac{n-3}{2} \) vertices namely \( v_2, v_3, v_6, v_7, \ldots, v_{n-5}, v_{n-4} \) from \( V(P_n) \) can strongly dominate the vertices \( v_1, v_2, v_3, \ldots, v_{n-3} \) and all the vertices of the sets \( H_1, H_2, \ldots, H_{n-3} \). To strongly dominate the remaining vertices \( v_{n-2}, v_{n-1}, v_n \) and all the vertices of the sets \( H_{n-2}, H_{n-1}, H_n \) the vertex \( v_{n-1} \) and either \( v_{n-2} \) or \( v_n \) must be in a strong dominating set \( S \). Therefore, \( \gamma_{d}(P_n\bar{\star}G) = \frac{n+1}{2} \).

Remark 2.23. \( \gamma_{d}(P_2\bar{\star}G) = \gamma_{d}(P_3\bar{\star}G) = 2 \).

Illustration 2.24. In Figure 6, \( S = \{v_2, \ldots, v_4\} \) is a strong dominating set of the graph \( P_3\bar{\star}P_2 \) and \( \gamma_{d}(P_3\bar{\star}P_2) = 3 \). The strong dominating set is shown with solid vertices.
\[
\gamma_{st}(S'(P_n)) = \begin{cases} 
\frac{n}{2} & \text{if } n \equiv 0 \pmod{4}, \\
\frac{n+1}{2} & \text{if } n \equiv 1 \text{ or } 3 \pmod{4}, \\
\frac{n}{2} + 1 & \text{if } n \equiv 2 \pmod{4}.
\end{cases}
\]

**Proof.** In the graph \(P_n \star G\) if we consider \(G = K_1\) then \(P_n \star K_1 \cong S'(P_n)\) becomes splitting graph of path \(P_n\). Thus, \(\gamma_{st}(S'(P_n)) = \gamma_{st}(P_n \star K_1)\). Therefore, by the Theorem 2.22, the result holds.

**Conclusion**

The concept of strong domination is a variant of usual domination in graphs. The strong domination number of some standard graphs are already available in the literature while we have investigated the strong domination number for corona product and corona like products of two graphs. To derive similar results for other graph families as well as in the context of various domination models are potential areas of research.

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**References**


