Solution of two-dimensional non-linear Burgers’ equations with nonlocal boundary condition

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Abstract
In this article, we tried to find the solution in Burgers’ equations by iteration method. Also we have proposed a numerical method by using finite difference method.

Keywords
Two Dimensional Parabolic Equation, Periodic Boundary Condition, Finite Difference, Boundary Value.

AMS Subject Classification

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1. Introduction

Burgers’ equations are a special case of the Navier-Stokes equation. Burgers’ equations are an important differential equation from fluid dynamics, and are used to describe various natural phenomena such as mathematically shown that the turbulence and modeling of gas dynamics, shock waves, etc. Numerical methods for Burgers’ problems: finite-element, finite-difference methods, spectral methods. Superpositions for nonlinear differential operators constructed in 1893 by Vessiot [10]. Key references can be found in [10, 11]. In this study, we use the superposition principle for nonlinear Burgers’ equations.

Consider 2-dimensional nonlinear Burgers’ problem taken from [2],

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{R} \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) \quad (1.1)
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \frac{1}{R} \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right) \quad (1.2)
\]

Here

\[
\Omega = \{(x,y) : a \leq x \leq b, a \leq y \leq b, 0 < t < T \}
\]

\[\partial \Omega\] denotes the boundary of \(\Omega\), \(u(x,y,t)\) and \(v(x,y,t)\) are the velocity components to be determinant \(f_1, g_1\) and \(f_2, g_2\) are known functions and \(R\) is Reynolds number.

The analytical solution of equation (1.1) and (1.2) were given by Fletcher. Fletcher used to the transformation Hopf-Cole [3]. Fletcher made a comparison of the different numerical methods [4]. Wubs and Geode have made an explicit-implicit method [5], Goyon used the many multilevel methods used by ADI [6]. Bahadır used the fully implicit finite-difference method[3]. Srivastava et al used The Crank- Nicholson scheme [7].

In this paper, an iterative method is presented to find numerical solutions problem. We use the superposition principle for nonlinear partial equations [10, 11].

\[\Omega\] denotes the domain

\[\Omega := \{(x,y,t) : 0 < x < 1, 0 < y < 1, 0 < t < T \}\]
Partial differential equation (1.1) can be solved by splitting it into two one-dimensional equation from [10] rather than discretising the complete two-dimensional Burgers' equation to give an approximating equation based on two-dimensional computational moleculke, as seen [8]. Let consider

\[ u_x \approx f(x,t,u) \]
\[ u_y \approx f(y,t,u). \]

We can write equation (1.1) as equations (1.5) and (1.6)

\[ \frac{1}{2} u_t - \frac{1}{2R} u_{xx} = uf(x,t,u), (x,t) \in \Omega \] (1.5)
\[ \frac{1}{2} u_t - \frac{1}{2R} u_{yy} = vf(y,t,u), (y,t) \in \Omega \] (1.6)

Applying the same estimations for equation (1.2) we can write as equations (1.7) and (1.8)

\[ \frac{1}{2} v_t - \frac{1}{2R} v_{xx} = ug(x,t,v), (x,t) \in \Omega \] (1.7)
\[ \frac{1}{2} v_t - \frac{1}{2R} v_{yy} = vg(y,t,v), (y,t) \in \Omega \] (1.8)

with the periodic and initial boundary condition

\[ u(x,0) = \varphi(x), u(y,0) = \varphi(y) x,y \in [0,1] \]
\[ u(0,t) = u(1,t), u_x(1,t) = 0 \text{ for } x \]
\[ u(0,t) = u(1,t), u_y(1,t) = 0 \text{ for } y \] (1.9)
\[ v(x,0) = \varphi(x), v(y,0) = \varphi(y) x,y \in [0,1], \]
\[ v(0,t) = v(1,t), v_x(1,t) = 0 \text{ for } x \]
\[ v(0,t) = v(1,t), v_y(1,t) = 0 \text{ for } y \]

The functions \( \varphi(x), \varphi(y) \) and \( f(x,t,u), f(y,t,u), g(x,t,v) \) and \( g(y,t,v) \) are defined functions on \([0,1]\) and \( \partial \Omega \times (\infty,\infty) \), respectively.

The functions \( u(x,t), v(x,t), u(y,t), v(y,t) \) are solutions. The periodic boundary conditions are encountered very often [1].

**Definition 1.1.** \( \{u(x,t), v(x,t), u(y,t), v(y,t)\} \) from the class \( C^{,1}(\Omega) \cap C^{1,0}(\partial \Omega) \) is defined the classical solution of system (1.5)-(1.9).

**2. The Fourier Method for Burgers' Problem**

Let these conditions are valid:

(a1) \( \varphi(x), \varphi(y) \in C[0,1], \)
\( \varphi(0) = \varphi(1), \varphi_x(1) = 0, \varphi(0) = \varphi(1), \varphi_y(1) = 0, \)

(a2) Let \( f(x,t,u), g(x,t,v), f(y,t,u) \) and \( g(y,t,v) \) is continuous

\[ ||f(x,t,u) - f(x,t,u)|| \leq L(x,t)||u - \tilde{u}|| \] (2.1)
\[ ||g(x,t,v) - g(x,t,v)|| \leq L(x,t)||v - \tilde{v}||, \]

where \( L(x,t) \in L_2(\Omega) \), \( b(x,t) \geq 0, \)

\[ ||g(x,t,v) - g(x,t,v)|| \leq L(x,t)||u - \tilde{u}|| \] (2.2)
\[ ||g(y,t,v) - g(y,t,v)|| \leq L(x,t)||v - \tilde{v}|| \]

By Fourier method, the solution of (1.5)-(1.6) and (1.7)-(1.8):

\[ u(x,t) = \sum_{k=1}^\infty \sum_{\lambda=1}^{\infty} \left[ \alpha_k \left( \frac{\sin \pi k x}{\sin \frac{\pi k}{2} \lambda} \right) \right] e^{-\frac{\pi^2}{4R} \frac{\lambda^2 t}{2}} + \sum_{k=1}^\infty \sum_{\lambda=1}^{\infty} \left[ \beta_k \left( \frac{\cos \pi k x}{\cos \frac{\pi k}{2} \lambda} \right) \right] e^{-\frac{\pi^2}{4R} \frac{\lambda^2 t}{2}} \] (2.3)

\[ v(x,t) = \sum_{k=1}^\infty \sum_{\lambda=1}^{\infty} \left[ \gamma_k \left( \frac{\cos \pi k x}{\cos \frac{\pi k}{2} \lambda} \right) \right] e^{-\frac{\pi^2}{4R} \frac{\lambda^2 t}{2}} \] (2.4)

**For equation (1.5) Fourier coefficient is:**

\[ v_{0}(t) = \mathcal{A} - \int_0^1 \alpha_k(t)x(x,\alpha,\beta)e dx, \]
\[ v_{2k-1}(t) = \mathcal{A} \int_0^1 \beta_k(t)x(x,\alpha,\beta)e dx, \] (2.5)

\[ v_{2k}(t) = \mathcal{A} \int_0^1 \beta_k(t)x(x,\alpha,\beta)e dx, \]

**For the (1.6) equation Fourier coefficient is:**

\[ v_{0}(t) = \mathcal{A} - \int_0^1 \beta_k(t)x(x,\alpha,\beta)e dx, \] (2.6)
\[ v_{2k-1}(t) = \mathcal{A} \int_0^1 \beta_k(t)x(x,\alpha,\beta)e dx, \]
\[ v_{2k}(t) = \mathcal{A} \int_0^1 \beta_k(t)x(x,\alpha,\beta)e dx, \] (2.7)

Same estimation for Fourier coefficient to (1.7)-(1.8) equation.
Definition 2.1. Let
\[ \{ u(\theta) \} = \left\{ u_{0}(\theta), u_{1}(\theta), u_{2}(\theta), \ldots, u_{k}(\theta), \ldots \right\} , \]
is satisfied that
\[ \max_{0 \leq \theta \leq T} \left| u_{k}(\theta) \right| \leq \frac{4}{N} \sum_{k=1}^{\infty} \left| u_{k-1}(\theta) \right| + \frac{4}{N} \sum_{k=1}^{\infty} \left| u_{k-2}^{(1)}(\theta) \right| \]
for the norm in \( B_{1} \).

\( B_{1} \) is called the Banach spaces.

Denote the set
\[ \{ u(\theta) \} = \left\{ u_{0}(\theta), u_{1}(\theta), u_{2}(\theta), \ldots, u_{k}(\theta), \ldots \right\} \]
is satisfied that \( \max_{0 \leq \theta \leq T} \left| u_{k}(\theta) \right| \leq \frac{4}{N} \sum_{k=1}^{\infty} \left| u_{k-1}(\theta) \right| \)
for the norm in \( B_{2} \).

\( B_{2} \) is called the Banach spaces.

Theorem 2.2. If (a1)-(a2) are satisfied then the equation (1.5)-(1.9) has a unique solution .

Proof.

\[ \left( N(1) \right)_{0} = u_{0}(\theta), \quad \left( N(2) \right)_{0} = u_{0}(\theta) + \int_{0}^{\theta} u_{0}(\theta) \phi_{0}(\theta) \, d\theta, \]
(2.8)

\[ \left( N(3) \right)_{0} = u_{0}(\theta), \quad \left( N(4) \right)_{0} = u_{0}(\theta) + \int_{0}^{\theta} u_{0}(\theta) \phi_{0}(\theta) \, d\theta, \]
(2.9)

\[ u_{0}(\theta) = u_{0}(\theta) + \int_{0}^{\theta} u_{0}(\theta) \phi_{0}(\theta) \, d\theta, \]
(2.10)

From the theorem \( u(0) \in B_{1} \), \( t \in [0, T] \).

For \( N=0 \) in the above equation and using Cauchy inequality, Lipschitz condition and taking the maximum, we have

\[ \max_{0 \leq \theta \leq T} \left| u_{k}(\theta) \right| \leq 2 \left| u_{0}(\theta) \right| + 2 \left| \int_{0}^{\theta} u_{0}(\theta) \phi_{0}(\theta) \, d\theta \right| \]
(2.11)

Applying Cauchy, Bessel, Hölder, Lipschitz conditions and taking maximum:

\[ \max_{0 \leq \theta \leq T} \left| u_{k}(\theta) \right| \leq 2 \left| u_{0}(\theta) \right| + \sqrt{N} \left| \int_{0}^{\theta} u_{0}(\theta) \phi_{0}(\theta) \, d\theta \right| \]
(2.12)

According to the same estimates ,

\[ \max_{0 \leq \theta \leq T} \left| u_{k}(\theta) \right| \leq 2 \left| u_{0}(\theta) \right| + \sqrt{N} \left| \int_{0}^{\theta} u_{0}(\theta) \phi_{0}(\theta) \, d\theta \right| \]
(2.13)

\[ \left| u(\theta) \right|_{B_{1}} \leq \left| u_{0}(\theta) \right| + \left| \int_{0}^{\theta} u_{0}(\theta) \phi_{0}(\theta) \, d\theta \right| \]
(2.14)

\[ \left| u(\theta) \right|_{B_{2}} \leq \left| u_{0}(\theta) \right| + \left| \int_{0}^{\theta} u_{0}(\theta) \phi_{0}(\theta) \, d\theta \right| \]
(2.15)

From the theorem \( u(0) \in B_{1} \),
Same estimations and using Gronwall’s inequality
\[ ||u(t) - u^{(n)}||_{B_1} \leq \sqrt{2(\tau + \frac{3\tau^2}{2} + 2\tau^3)} \left( \frac{\tau^{n+1}}{||u(x, t)||_{L^2(\Omega)}} \right)^2 \]
\[ + \frac{1}{2} \sqrt{2(\tau + \frac{3\tau^2}{2} + 2\tau^3)} \left( \frac{\tau^{(n+1)}}{||u(x, t)||_{L^2(\Omega)}} \right)^2 \]
\[ \leq \sqrt{2(\tau + \frac{3\tau^2}{2} + 2\tau^3)} \left( \frac{\tau^{N+1}}{||u(x, t)||_{L^2(\Omega)}} \right)^2 \]
\[ + \frac{1}{2} \sqrt{2(\tau + \frac{3\tau^2}{2} + 2\tau^3)} \left( \frac{\tau^{(N+1)}}{||u(x, t)||_{L^2(\Omega)}} \right)^2 \]

\[ ||v(t) - v^{(n)}||_{B_2} \leq \sqrt{2(\tau + \frac{3\tau^2}{2} + 2\tau^3)} \left( \frac{\tau^{(n+1)}}{||v(x, t)||_{L^2(\Omega)}} \right)^2 \]
\[ + \frac{1}{2} \sqrt{2(\tau + \frac{3\tau^2}{2} + 2\tau^3)} \left( \frac{\tau^{(N+1)}}{||v(x, t)||_{L^2(\Omega)}} \right)^2 \]

\[ u^{(N+1)}(t) \text{ and } v^{(N+1)}(t) \quad (N \to \infty) \text{ converge in } B_1 \text{ and } B_2, \]
respectively.

For the uniqueness, \((u, v)\) and \((\pi, \tau)\) are two solution of problem. If the same estimation is used, \( |u(t) - \pi(t)| \) and \( |v(t) - \tau(t)| \).

\[ ||u(t) - \pi(t)||_{B_1} \leq \sqrt{2(\tau + \frac{3\tau^2}{2} + 2\tau^3)} \left( \frac{\tau^{(n+1)}}{||u(x, t)||_{L^2(\Omega)}} \right)^2 \]
\[ + \frac{1}{2} \sqrt{2(\tau + \frac{3\tau^2}{2} + 2\tau^3)} \left( \frac{\tau^{(N+1)}}{||u(x, t)||_{L^2(\Omega)}} \right)^2 \]
\[ ||v(t) - \tau(t)||_{B_2} \leq \sqrt{2(\tau + \frac{3\tau^2}{2} + 2\tau^3)} \left( \frac{\tau^{(n+1)}}{||v(x, t)||_{L^2(\Omega)}} \right)^2 \]
\[ + \frac{1}{2} \sqrt{2(\tau + \frac{3\tau^2}{2} + 2\tau^3)} \left( \frac{\tau^{(N+1)}}{||v(x, t)||_{L^2(\Omega)}} \right)^2 \]

From Gronwall’s inequality \( u(t) = \pi(t) \).
Same estimation for \( ||v(t) - \tau(t)||_{B_2} \), we have \( v(t) = \tau(t) \).

**Theorem 2.3.** Solution \((u, v)\) of the problem (1.5)-(1.9) depends continuously on the data \( \phi \) under assumptions (a1)-(a2).

**Proof.** By using same operations, we obtain:

\[ ||(u - \pi)||_{B_1} \leq ||(v - \tau)||_{B_2} \leq \sqrt{2(\tau + \frac{3\tau^2}{2} + 2\tau^3)} \left( \frac{\tau^{(n+1)}}{||u(x, t)||_{L^2(\Omega)}} \right)^2 \]
\[ + \frac{1}{2} \sqrt{2(\tau + \frac{3\tau^2}{2} + 2\tau^3)} \left( \frac{\tau^{(N+1)}}{||u(x, t)||_{L^2(\Omega)}} \right)^2 \]
\[ ||(v - \tau)||_{B_2} \leq \sqrt{2(\tau + \frac{3\tau^2}{2} + 2\tau^3)} \left( \frac{\tau^{(n+1)}}{||v(x, t)||_{L^2(\Omega)}} \right)^2 \]
\[ + \frac{1}{2} \sqrt{2(\tau + \frac{3\tau^2}{2} + 2\tau^3)} \left( \frac{\tau^{(N+1)}}{||v(x, t)||_{L^2(\Omega)}} \right)^2 \]

For \( \phi \to \phi \) then \( u \to \pi \) and \( v \to \tau \).

3. **Numerical Method**

In this Section, we construct the numerical solutions of problems (1.5)-(1.9). In order to solve these problems numerically, we have to use linearization of the nonlinear terms:

\[ \frac{1}{2} u_t^{(n)} - \frac{1}{2R} u_{xx}^{(n)} = u^{(n)} f(x, t, u^{(n-1)}), (x, t) \in \Omega \quad (3.1) \]
\[ \frac{1}{2} u_t^{(n)} - \frac{1}{2R} u_{yy}^{(n)} = v^{(n)} f(y, t, u^{(n-1)}), (y, t) \in \Omega \quad (3.2) \]
\[ \frac{1}{2} v_t^{(n)} - \frac{1}{2R} v_{xx}^{(n)} = u^{(n)} g(x, t, v^{(n-1)}), (x, t) \in \Omega \quad (3.3) \]
\[ \frac{1}{2} v_t^{(n)} - \frac{1}{2R} v_{yy}^{(n)} = v^{(n)} g(y, t, v^{(n-1)}), (y, t) \in \Omega \quad (3.4) \]

Let \( u^{(n)}(x, t) = u(x, t), u^{(n)}(y, t) = v(y, t), \) \( w^{(n)}(x, t) = v(x, t), \)
\( w^{(n)}(y, t) = g(y, t) \) and \( f(x, t, u^{(n-1)}) = f(x, t), f(y, t, u^{(n-1)}) =

\[ \tilde{f}(x, t), g(x, t, v^{(n-1)}) = \tilde{g}(x, t), g(y, t, v^{(n-1)}) = \tilde{g}(y, t) \].

After this transformation, we obtain a linear problem:

\[ \frac{1}{2} u_t - \frac{1}{2R} u_{xx} = u \tilde{f}(x, t), (x, t) \in \Omega \quad (3.5) \]
\[ \frac{1}{2} u_t - \frac{1}{2R} u_{yy} = v \tilde{f}(y, t), (y, t) \in \Omega \quad (3.6) \]
\[ \frac{1}{2} v_t - \frac{1}{2R} v_{xx} = u \tilde{g}(x, t), (x, t) \in \Omega \quad (3.7) \]
\[ \frac{1}{2} v_t - \frac{1}{2R} v_{yy} = v \tilde{g}(y, t), (y, t) \in \Omega \quad (3.8) \]

with the conditions (1.9).

We divide the intervals \([0, 1]\) and \([0, T]\) into subintervals \(N_x\) and \(N_y\). We obtain equal lengths \( h = \frac{1}{N_x}\) and \( \tau = \frac{T}{N_y}\). We use implicit finite-difference approximation for discretizing equation (3.5):

\[ \frac{1}{2\tau} (u_{i+1}^{k+1} - u_i^k) - \frac{1}{2h^2} (u_{i-1}^{k+1} - 2u_i^{k+1} + u_{i+1}^{k+1}) = u_i^{k+1}f_i^k \quad (3.9) \]

where \( 1 \leq i \leq N_x \) and \( 0 \leq k \leq N_t \) are the indices, \( u_i^k = u(x_i, t_k), f_i^k = \tilde{f}(x_i, t_k), x_i = ih, t_k = k\tau \).

We obtain

\[ -Pu_{i+1}^{k+1} + (1 + 2P)u_i^{k+1} - Pu_{i-1}^{k+1} = u_i^{k+1}(1 + 2\tau)f_i^k \]

where \( P = \frac{1}{h^2}\). By using Gauss Elimination Method, we can solve this equation. Equations (3.6), (3.7) and (3.8) can be solved in a similar way.

**References**

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