Existence of solutions of $q$-functional integral equations with deviated argument


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Abstract

In this paper, we study the existence of solutions for $q$-functional integral equations in Banach space $C[0, T]$. The existence and uniqueness of solutions for the problems are proved by means of the Banach contraction principle.

Keywords: $q$-functional integral equations; Banach contraction principle; Deviated argument; existence.

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1 Introduction

The quantum calculus or $q$-difference calculus is an old subject that was first developed by Jackson ([12], [13]), while basic definitions and properties can be found in [15]. Studies on $q$-difference equations appeared already at the beginning of the last century in intensive works especially by F H Jackson [14], R D Carmichael [6], T E Mason [19], C R Adams [1], W J Trjitzinsky [21] and other authors [5]. Recently, $q$-calculus has served as an bridge between mathematics and physics. It has a lot of applications in mathematics and physics([7], [8], [17], [22]).

In this paper, we are concerned with the $q$-functional integral equations

\[ x(t) = g(t) + \int_0^t f_1(t, s, x(\phi(s))) \, dq_s, \quad t \in [0, T] \]  

(1.1)

and

\[ x(t) = g(t) + f_2(t, \int_0^t g(s, x(\phi(s))) \, dq_s), \quad t \in [0, T] \]  

(1.2)

where $\phi$ is deviated function. The existence of continuous solutions of the $q$-functional integral equation (1.1) in the Banach space $C[0, T]$ will be proved. The monotonicity of the solution of the equation (1.1) will be studied. The existence of continuous solutions of the $q$-functional integral equation (1.2) in Banach space $C[0, T]$ will be proved.

2 preliminaries

Here, we give the definition of $q$-derivative and $q$-integral and some of their properties which is referred to ([2], [15]).

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Let \( q \in (0, 1) \) and define
\[
[n]_q = \frac{q^n - 1}{q - 1} = 1 + q + q^2 + \cdots + q^{n-1}, \quad n \in \mathbb{R}
\]
which is called the \( q \)-analogue of \( n \).

**Definition 2.1.** The \( q \)-derivative of a real valued function \( f \) is defined by
\[
D_q f(t) = \frac{d_q f(t)}{d_q t} = \frac{f(qt) - f(t)}{qt - t}, \quad D_q f(0) = \lim_{t \to 0} D_q f(t)
\]
Note that \( \lim_{q \to 1} D_q f(t) = f'(t) \) if \( f(t) \) is differentiable.

The higher order \( q \)-derivative are defined as
\[
D_q^n f(t) = f(t), \quad D_q^n f(t) = D_q D_q^{n-1} f(t), \quad n \in \mathbb{N}.
\]

**Definition 2.2.** Suppose \( 0 < a < b \). The definite \( q \)-integral is defined as
\[
I_q f(x) = \int_0^b f(x) \, d_q x = (1 - q) b \sum_{j=0}^{\infty} q^j f(q^j b).
\]
and
\[
\int_a^b f(x) \, d_q x = \int_a^b f(x) \, d_q x - \int_a^0 f(x) \, d_q x.
\]
Similarly, we have
\[
I_q^n f(t) = f(t), \quad I_q^n f(t) = I_q I_q^{n-1} f(t), \quad n \in \mathbb{N}.
\]

**Theorem 2.1** (see [15]). *(Fundamental Theorem of \( q \)-Calculus)*
If \( F(x) \) is an antiderivative of \( f(x) \), and \( F(x) \) is continuous at \( x = 0 \), then
\[
\int_a^b f(x) \, d_q x = F(b) - F(a), \quad \quad 0 \leq a < b \leq \infty.
\]

**Theorem 2.2.** (see [4], [15]) For any function \( f \) one has
\[
D_q I_q f(x) = f(x). \quad \quad (2.3)
\]

**Theorem 2.3.** (see [2]) Let \( f \) be a function defined on \([a, b]\), \( 0 \leq a \leq b \), and \( c \) is a fixed point in \([a, b]\). Assume that there exists, \( 0 \leq \gamma < 1 \) such that \( x^\gamma f(x) \) is continuous on \([a, b]\). Let
\[
F(x) = \int_c^x f(t) \, d_q t, \quad \quad x \in [a, b].
\]
Then \( F(x) \) is a continuous function on \([a, b]\).

**Lemma 2.1.** If
\[
F(t) = \int_0^t f(s) \, d_q s, \quad \quad \text{for } t \in [a, b],
\]
is continuous, then for every \( \epsilon > 0 \) \( \exists \delta > 0 \), such that \( t_2, t_2 \in [0, T], \ |t_2 - t_1| < \delta \), then
\[
|F(t_2) - F(t_1)| < \epsilon
\]
i.e.,
\[
|\int_0^{t_2} f(s) \, d_q s - \int_0^{t_1} f(s) \, d_q s| < \epsilon.
\]

**Lemma 2.2.** (see [18])
(1) If \( f \) and \( g \) are \( q \)-integrable on \([a, b]\), \( a, c \in [a, b] \), then
\[
(i) \int_a^b [f(x) + g(x)] \, d_q x = \int_a^b f(x) \, d_q x + \int_a^b g(x) \, d_q x,
(ii) \int_a^b a f(x) \, d_q x = a \int_a^b f(x) \, d_q x,
\]
Main results

First, we study the existence and uniqueness of the solution of the functional integral equation (1.1) by

\[ Fx(t) = g(t) + \int_0^t f_1(t, s, x(\phi(s))) \, dq_s. \]

To show that \( F : C[0, T] \rightarrow C[0, T] \), let \( x \in C[0, T] \), \( t_1, t_2 \in [0, T] \), then

\[
|Fx(t_2) - Fx(t_1)| = |g(t_2) - g(t_1)| + \int_0^{t_2} f_1(t_2, s, x(\phi(s))) \, dq_s - \int_0^{t_1} f_1(t_1, s, x(\phi(s))) \, dq_s |
\]

\[
\leq |g(t_2) - g(t_1)| + |\int_0^{t_2} f_1(t_2, s, x(\phi(s))) \, dq_s - \int_0^{t_1} f_1(t_1, s, x(\phi(s))) \, dq_s |
\]

\[
\leq |g(t_2) - g(t_1)| + |\int_0^{t_2} f_1(t_2, s, x(\phi(s))) \, dq_s - \int_0^{t_1} f_1(t_1, s, x(\phi(s))) \, dq_s |
\]

\[
+ |\int_0^{t_2} f_1(t_1, s, x(\phi(s))) \, dq_s - \int_0^{t_1} f_1(t_1, s, x(\phi(s))) \, dq_s |
\]

\[
\leq |g(t_2) - g(t_1)| + \int_0^{t_2} |f_1(t_2, s, x(\phi(s))) - f_1(t_1, s, x(\phi(s)))| \, dq_s 
\]

\[
+ |\int_0^{t_2} f_1(t_1, s, x(\phi(s))) \, dq_s - \int_0^{t_1} f_1(t_1, s, x(\phi(s))) \, dq_s |
\]

\[
+ |\int_0^{t_2} f_1(t_1, s, x(\phi(s))) \, dq_s - \int_0^{t_1} f_1(t_1, s, x(\phi(s))) \, dq_s |
\]

\[
+ |\int_0^{t_2} f_1(t_1, s, x(\phi(s))) \, dq_s - \int_0^{t_1} f_1(t_1, s, x(\phi(s))) \, dq_s |
\]

\[
+ |\int_0^{t_2} f_1(t_1, s, x(\phi(s))) \, dq_s - \int_0^{t_1} f_1(t_1, s, x(\phi(s))) \, dq_s |
\]

3 Main results

Let \( X \) be the class of all continuous functions, \( x \in C[0, T] \) with the norm

\[ \|x\| = \sup_{t \in [0,T]} |x(t)|. \]

First, we study the existence and uniqueness of the solution of the \( q \)-functional integral equation (1.1) and then we proved the monotonicity for the solution.

Consider the \( q \)-functional integral equation (1.1) under the following assumptions

(i) \( g : [0, T] \rightarrow R \) is continuous.
(ii) \( f_1 : [0, T] \times [0, T] \times R \rightarrow R \) is continuous.
(iii) \( f_1 \) satisfies the Lipschitz condition

\[ |f_1(t, s, x) - f_1(t, s, y)| \leq k(t, s) |x - y|. \]

(iv)

\[ \sup_t \int_0^t k(t, s) \, dq_s \leq K \]

Now for the existence of a unique continuous solution of the \( q \)-functional integral equation (1.1) we have the following theorem.

**Theorem 3.4.** Let the assumptions (i)-(iv) be satisfied. If \( K < 1 \), then the \( q \)-functional integral equation (1.1) has a unique solution \( x \in C[0, T] \).

**Proof.** Define the operator \( F \) associated with the \( q \)-functional integral equation (1.1) by

\[ Fx(t) = g(t) + \int_0^t f_1(t, s, x(\phi(s))) \, dq_s. \]
applying Theorem 2.3 and Lemma 2.1, then we deduce that

$$ F : C[0, T] \to C[0, T]. $$

Let \( x, y \in C[0, T] \), we have

\[
|Fx(t) - Fy(t)| = |g(t) + \int_0^t f_1(t, s, x(\phi(s))) \, dq s - g(t) - \int_0^t f_1(t, s, y(\phi(s))) \, dq s|
\]

\[
= |\int_0^t f_1(t, s, x(\phi(s))) \, dq s - \int_0^t f_1(t, s, y(\phi(s))) \, dq s|
\]

\[
\leq \int_0^t |f_1(t, s, x(\phi(s))) - f_1(t, s, y(\phi(s)))| \, dq s
\]

\[
\leq \int_0^t k(t, s) |x(\phi(s)) - y(\phi(s))| \, dq s
\]

\[
\leq \|x - y\| \int_0^t k(t, s) \, dq s
\]

\[
\leq K \|x - y\|.
\]

This means that \( F \) is contraction.

Applying Banach contraction principle ([10], [16]), then we deduce that there exists a unique solution \( x \in C[0, T] \) of the \( q \)-functional integral equation (1.1).

The following theorem prove the monotonicity for the solution of the \( q \)-functional integral equation (1.1).

**Theorem 3.5.** Let the assumptions (i)-(iv) of Theorem (3.1) be satisfied. If \( f_1(t, s, x(\phi(s))) \) and \( g(t) \) are monotonic nonincreasing (nondecreasing) in \( t \) for each \( t \in [0, T] \), then the \( q \)-integral equation (1.1) has a unique monotonic nonincreasing (nondecreasing) solution \( x \in C[0, T] \).

**Proof.** Let \( f, g \) be monotonic nonincreasing functions in \( t \in [0, T] \), then for \( t_2 > t_1 \)

\[
x(t_2) = g(t_2) + \int_0^{t_2} f_1(t_2, s, x(\phi(s))) \, dq s
\]

\[
\leq g(t_1) + \int_0^{t_1} f_1(t_1, s, x(\phi(s))) \, dq s
\]

\[
= x(t_1).
\]

Hence,

\[
x(t_2) \leq x(t_1).
\]

Also, If \( f_1, g \) are monotonic nondecreasing functions in \( t \in [0, T] \), then for \( t_2 > t_1 \)

\[
x(t_2) = g(t_2) + \int_0^{t_2} f_1(t_2, s, x(\phi(s))) \, dq s
\]

\[
\geq g(t_1) + \int_0^{t_1} f_1(t_1, s, x(\phi(s))) \, dq s
\]

\[
= x(t_1).
\]
Hence
\[ x(t_2) \geq x(t_1). \]
\[ \square \]

Now, we study the existence and uniqueness of the solution of the \( q \)-functional integral equation
\[ x(t) = g(t) + f_2(t, \int_0^t g(s, x(\phi(s))) \, dq_s), \quad t \in [0, T] \]

Consider the \( q \)-functional integral equation (1.2) under the following assumptions

(i) \( g : [0, T] \rightarrow \mathbb{R} \) is continuous.

(ii) \( f_2 : [0, T] \times \mathbb{R} \rightarrow \mathbb{R} \) is continuous.

(iii) \( f_2 \) satisfies the Lipschitz condition
\[ |f_2(t, x(t)) - f_2(t, y(t))| \leq k |x(t) - y(t)|. \]

(iv) \( g \) satisfies the Lipschitz condition
\[ |g(s, x(t)) - g(s, y(t))| \leq l |x(t) - y(t)|. \]

For the existence of a unique continuous solution of the \( q \)-functional integral equation (1.2), we have the following theorem.

**Theorem 3.6.** Let the assumptions (i)-(iv) be satisfied. If \( klT < 1 \), then the \( q \)-functional integral equation (1.2) has a unique solution \( x \in C[0, T] \).

**Proof.** Define the operator \( F \) associated with the \( q \)-functional integral equation (1.2) by
\[ Fx(t) = g(t) + f_2(t, \int_0^t g(s, x(\phi(s))) \, dq_s). \]

To show that \( F : C[0, T] \rightarrow C[0, T] \), let \( x \in C[0, T], t_1, t_2 \in [0, T] \), then
\[
|Fx(t_2) - Fx(t_1)| = |(g(t_2) - g(t_1)) + (f_2(t_2, \int_0^{t_2} g(s, x(\phi(s))) \, dq_s) - f_2(t_1, \int_0^{t_1} g(s, x(\phi(s))) \, dq_s))|
\[
\leq |g(t_2) - g(t_1)| + |f_2(t_2, \int_0^{t_2} g(s, x(\phi(s))) \, dq_s) - f_2(t_1, \int_0^{t_1} g(s, x(\phi(s))) \, dq_s)|
\[
\leq |g(t_2) - g(t_1)| + |f_2(t_2, \int_0^{t_2} g(s, x(\phi(s))) \, dq_s) - f_2(t_1, \int_0^{t_2} g(s, x(\phi(s))) \, dq_s)|
\[
\leq |g(t_2) - g(t_1)| + |f_2(t_2, \int_0^{t_2} g(s, x(\phi(s))) \, dq_s) - f_2(t_1, \int_0^{t_1} g(s, x(\phi(s))) \, dq_s)|
\][
\[+ |\int_0^{t_2} g(s, x(\phi(s))) \, dq_s) - f_2(t_1, \int_0^{t_1} g(s, x(\phi(s))) \, dq_s)|
\]

applying Theorem (2.3) and Lemma (2.1), then we deduce that
\[ F : C[0, T] \rightarrow C[0, T]. \]
Let \( x, y \in C[0, T] \), we have

\[
|Fx(t) - Fy(t)| = |g(t) + f_2(t, \int_0^t g(s, x(\phi(s))) \, dq_s) - g(t) - f_2(t, \int_0^t g(s, y(\phi(s))) \, dq_s)| \\
= |f_2(t, \int_0^t g(s, x(\phi(s))) \, dq_s) - f_2(t, \int_0^t g(s, y(\phi(s))) \, dq_s)| \\
\leq k |\int_0^t g(s, x(\phi(s))) \, dq_s - \int_0^t g(s, y(\phi(s))) \, dq_s| \\
\leq kl \int_0^t |x(\phi(s)) - y(\phi(s))| \, dq_s \\
\leq klT \|x - y\|.
\]

This means that \( F \) is contraction. Then \( F \) has a fixed point \( x \in C[0, T] \) which proves that there exists a unique solution of the \( q \)-functional integral equation (1.2). \( \square \)

**References**


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