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The Generalized difference of $d\left(\chi^{2I}\right)$ of fuzzy real numbers over p metric spaces defined by Musielak Orlicz function

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Abstract

In this article we introduce the sequence spaces $\left[\chi^{2q}_{f\mu}, \left\|\left(d\left(x_1\right), d\left(x_2\right), \cdots, d\left(x_{n-1}\right)\right)\right\|_p\right]^{I(F)}$ and $\left[\Lambda^{2q}_{f\mu}, \left\|\left(d\left(x_1\right), d\left(x_2\right), \cdots, d\left(x_{n-1}\right)\right)\right\|_p\right]^{I(F)}$, associated with the differential operator of sequence space defined by Musielak. We study some basic topological and algebraic properties of these spaces. We also investigate some inclusion relations related to these spaces.

Keywords: Analytic sequence, double sequences, χ^2 space, difference sequence space,Musielak - Orlicz function, $p-$ metric space, Ideal; ideal convergent; fuzzy number; multiplier,differential operator.

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1 Introduction

Throughout w, Γ and Λ denote the classes of all, entire and analytic scalar valued single sequences, respectively.

We write w^2 for the set of all complex sequences (x_{mn}), where $m, n \in \mathbb{N}$, the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Let (x_{mn}) be a double sequence of real or complex numbers. Then the series $\sum_{m,n=1}^{\infty} x_{mn}$ is called a double series. The double series $\sum_{m,n=1}^{\infty} x_{mn}$ give one space is said to be convergent if and only if the double sequence $(S_{mn})$is convergent, where

$$S_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n = 1, 2, 3, ...) .$$

A double sequence $x = (x_{mn})$is said to be double analytic if

$$sup_{m,n} |x_{mn}|^{\frac{1}{m+n}} < \infty.$$

The vector space of all double analytic sequences are usually denoted by Λ^2. A sequence $x = (x_{mn})$ is called double entire sequence if

$$|x_{mn}|^{\frac{1}{m+n}} \to 0 \text{ as } m, n \to \infty.$$

The vector space of all double entire sequences are usually denoted by Γ^2. Let the set of sequences with this property be denoted by Λ^2 and Γ^2 is a metric space with the metric

$$d(x, y) = sup_{m,n} \left\{ |x_{mn} - y_{mn}|^{\frac{1}{m+n}} : m, n : 1, 2, 3, ... \right\}, \tag{1.1}$$

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forall $x = \{x_{mn}\}$ and $y = \{y_{mn}\}$ $in\, \Gamma^2$. Let $\phi = \{finite\ sequences\}$.

Consider a double sequence $x = (x_{mn})$. The $(m,n)^{th}$ section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij}\delta_{ij}$ for all $m,n \in \mathbb{N}\times\mathbb{N}$,

$$\delta_{mn} = \begin{pmatrix} 0 & 0 & ...0 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \\ . & & & & \\ . & & & & \\ . & & & & \\ 0 & 0 & ...1 & 0 & ... \\ 0 & 0 & ...0 & 0 & ... \end{pmatrix}$$

with 1 in the $(m,n)^{th}$ position and zero otherwise.

A double sequence $x = (x_{mn})$ is called double gai sequence if $\left((m+n)!\,|x_{mn}|\right)^{\frac{1}{m+n}} \to 0$ as $m,n \to \infty$. The double gai sequences will be denoted by χ^2.

Let M and Φ are mutually complementary Orlicz functions. Then, we have:

(i) For all $u, y \geq 0$,

$$uy \leq M(u) + \Phi(y), (Young's\, inequality) [See [1]] \tag{1.2}$$

(ii) For all $u \geq 0$,

$$u\eta(u) = M(u) + \Phi(\eta(u)). \tag{1.3}$$

(iii) For all $u \geq 0$, and $0 < \lambda < 1$,

$$M(\lambda u) \leq \lambda M(u) \tag{1.4}$$

Lindenstrauss and Tzafriri [2] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{x \in w : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) < \infty, for\, some\, \rho > 0\right\},$$

The space ℓ_M with the norm

$$\|x\| = inf\left\{\rho > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{\rho}\right) \leq 1\right\},$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^p \ (1 \leq p < \infty)$, the spaces ℓ_M coincide with the classical sequence space ℓ_p.

A sequence $f = (f_{mn})$ of Orlicz function is called a Musielak-Orlicz function. A sequence $g = (g_{mn})$ defined by

$$g_{mn}(v) = sup\{|v|\, u - (f_{mn})(u) : u \geq 0\}, m,n = 1,2,\cdots$$

is called the complementary function of a Musielak-Orlicz function f. For a given Musielak Orlicz function f, the Musielak-Orlicz sequence space t_f is defined as follows

$$t_f = \left\{x \in w^2 : M_f\left(|x_{mn}|\right)^{1/m+n} \to 0\, as\, m,n \to \infty\right\},$$

where M_f is a convex modular defined by

$$M_f(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(|x_{mn}|\right)^{1/m+n}, x = (x_{mn}) \in t_f.$$

We consider t_f equipped with the Luxemburg metric

$$d(x,y) = sup_{mn}\left\{inf\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn}\left(\frac{|x_{mn}|^{1/m+n}}{mn}\right)\right) \leq 1\right\}$$

If X is a sequence space, we give the following definitions:

(i)$X^{'}$= the continuous dual of X;

(ii)$X^{\alpha} = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \, for \, each \, x \in X\right\};$

(iii)$X^{\beta} = \left\{a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn} \, is \, convegent, \, foreach \, x \in X\right\};$

(iv)$X^{\gamma} = \left\{a = (a_{mn}) : sup_{mn\geq 1} \left|\sum_{m,n=1}^{M,N} a_{mn}x_{mn}\right| < \infty, foreachx \in X\right\};$

(v)$let \, X \, be \, an \, FK - space \supset \phi; \, then \, X^{f} = \left\{f(\Im_{mn}) : f \in X^{'}\right\};$

(vi)$X^{\delta} = \left\{a = (a_{mn}) : sup_{mn} |a_{mn}x_{mn}|^{1/m+n} < \infty, \, foreach \, x \in X\right\};$

$X^{\alpha}, X^{\beta}, X^{\gamma}$ are called $\alpha - (orK\ddot{o}the - Toeplitz)$dual of $X, \beta - (or \, generalized - K\ddot{o}the - Toeplitz) \, dual \, of \, X, \gamma - \, dual \, of \, X, \, \delta - \, dual \, of \, X \, respectively.X^{\alpha}$ is defined by Gupta and Kamptan . It is clear that $X^{\alpha} \subset X^{\beta}$ and $X^{\alpha} \subset X^{\gamma}$, but $X^{\beta} \subset X^{\gamma}$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference sequence spaces (for single sequences) was introduced by Kizmaz as follows

$$Z(\Delta) = \{x = (x_k) \in w : (\Delta x_k) \in Z\}$$

for $Z = c, c_0$ and ℓ_{∞}, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$.

Here c, c_0 and ℓ_{∞} denote the classes of convergent,null and bounded scalar valued single sequences respectively. The difference sequence space bv_p of the classical space ℓ_p is introduced and studied in the case $1 \leq p \leq \infty$ by Başar and Altay and in the case $0 < p < 1$ by Altay and Başar. The spaces $c(\Delta), c_0(\Delta), \ell_{\infty}(\Delta)$ and bv_p are Banach spaces normed by

$$\|x\| = |x_1| + sup_{k\geq 1} |\Delta x_k| \text{ and } \|x\|_{bv_p} = \left(\sum_{k=1}^{\infty} |x_k|^p\right)^{1/p}, (1 \leq p < \infty).$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z\right\}$$

where $Z = \Lambda^2, \chi^2$ and $\Delta x_{mn} = (x_{mn} - x_{mn+1}) - (x_{m+1n} - x_{m+1n+1}) = x_{mn} - x_{mn+1} - x_{m+1n} + x_{m+1n+1}$ for all $m, n \in \mathbb{N}$.

2 Definition and Preliminaries

Let $n \in \mathbb{N}$ and X be a real vector space of dimension m, where $n \leq m$. A real valued function $d_p(x_1, \ldots, x_n) = \|(d_1(x_1), \ldots, d_n(x_n))\|_p$ on X satisfying the following four conditions:

(i) $\|(d_1(x_1), \ldots, d_n(x_n))\|_p = 0$ if and and only if $d_1(x_1), \ldots, d_n(x_n)$ are linearly dependent,

(ii) $\|(d_1(x_1), \ldots, d_n(x_n))\|_p$ is invariant under permutation,

(iii) $\|(\alpha d_1(x_1), \ldots, \alpha d_n(x_n))\|_p = |\alpha| \, \|(d_1(x_1), \ldots, d_n(x_n))\|_p, \alpha \in \mathbb{R}$

(iv) $d_p((x_1, y_1), (x_2, y_2) \cdots (x_n, y_n)) = (d_X(x_1, x_2, \cdots x_n)^p + d_Y(y_1, y_2, \cdots y_n)^p)^{1/p} \, for 1 \leq p < \infty$; (or)

(v) $d((x_1, y_1), (x_2, y_2), \cdots (x_n, y_n)) := \sup\{d_X(x_1, x_2, \cdots x_n), d_Y(y_1, y_2, \cdots y_n)\}$,

for $x_1, x_2, \cdots x_n \in X, y_1, y_2, \cdots y_n \in Y$ is called the p product metric of the Cartesian product of n metric spaces is the p norm of the n-vector of the norms of the n subspaces.

A trivial example of p product metric of n metric space is the p norm space is $X = \mathbb{R}$ equipped with the following Euclidean metric in the product space is the p norm:

$$\|(d_1(x_1), \ldots, d_n(x_n))\|_E = sup\left(|det(d_{mn}(x_{mn}))|\right) = sup\left(\begin{vmatrix} d_{11}(x_{11}) & d_{12}(x_{12}) & \ldots & d_{1n}(x_{1n}) \\ d_{21}(x_{21}) & d_{22}(x_{22}) & \ldots & d_{2n}(x_{1n}) \\ . & & & \\ . & & & \\ . & & & \\ d_{n1}(x_{n1}) & d_{n2}(x_{n2}) & \ldots & d_{nn}(x_{nn}) \end{vmatrix}\right)$$

where $x_i = (x_{i1}, \cdots x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \cdots n$.
If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the $p-$ metric. Any complete $p-$ metric space is said to be $p-$ Banach metric space.

2.1 Definition

Let X be a linear metric space. A function $\rho : X \to \mathbb{R}$ is called paranorm, if
(1) $\rho(x) \geq 0$, for all $x \in X$;
(2) $\rho(-x) = \rho(x)$, for all $x \in X$;
(3) $\rho(x+y) \leq \rho(x) + \rho(y)$, for all $x, y \in X$;
(4) If (σ_{mn}) is a sequence of scalars with $\sigma_{mn} \to \sigma$ as $m, n \to \infty$ and (x_{mn}) is a sequence of vectors with $\rho(x_{mn} - x) \to 0$ as $m, n \to \infty$, then $\rho(\sigma_{mn} x_{mn} - \sigma x) \to 0$ as $m, n \to \infty$.
A paranorm w for which $\rho(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, w) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm.

The notion of ideal convergence was introduced first by Kostyrko et al. as a generalization of statistical convergence which was further studied in topological spaces by Kumar et al. and also more applications of ideals can be deals with various authors by B.Hazarika [3-14] .

2.2 Definition

A family $I \subset 2^{Y \times Y}$ of subsets of a non empty set Y is said to be an ideal in Y if
(1) $\phi \in I$
(2) $A, B \in I$ imply $A \bigcup B \in I$
(3) $A \in I, B \subset A$ imply $B \in I$.

while an admissible ideal I of Y further satisfies $\{x\} \in I$ for each $x \in Y$. Given $I \subset 2^{\mathbb{N} \times \mathbb{N}}$ be a non trivial ideal in $\mathbb{N} \times \mathbb{N}$. A sequence $(x_{mn})_{m,n \in \mathbb{N} \times \mathbb{N}}$ in X is said to be $I-$ convergent to $0 \in X$, if for each $\epsilon > 0$ the set $A(\epsilon) = \left\{m, n \in \mathbb{N} \times \mathbb{N} : \|(d_1(x_1), \ldots, d_n(x_n)) - 0\|_p \geq \epsilon\right\}$ belongs to I.

2.3 Definition

A non-empty family of sets $F \subset 2^{X \times X}$ is a filter on X if and only if
(1) $\phi \in F$
(2) for each $A, B \in F$, we have imply $A \bigcap B \in F$
(3) each $A \in F$ and each $A \subset B$, we have $B \in F$.

2.4 Definition

An ideal I is called non-trivial ideal if $I \neq \phi$ and $X \notin I$. Clearly $I \subset 2^{X \times X}$ is a non-trivial ideal if and only if $F = F(I) = \{X - A : A \in I\}$ is a filter on X.

2.5 Definition

A non-trivial ideal $I \subset 2^{X \times X}$ is called (i) admissible if and only if $\{\{x\} : x \in X\} \subset I$. (ii) maximal if there cannot exists any non-trivial ideal $J \neq I$ containing I as a subset.

If we take $I = I_f = \{A \subseteq \mathbb{N} \times \mathbb{N} : A \; is \; a \; finite \; subset\}$. Then I_f is a non-trivial admissible ideal of $\mathbb{N}$ and the corresponding convergence coincides with the usual convergence. If we take $I = I_\delta = \{A \subseteq \mathbb{N} \times \mathbb{N} : \delta(A) = 0\}$ where $\delta(A)$ denote the asyptotic density of the set A. Then I_δ is a non-trivial admissible ideal of $\mathbb{N} \times \mathbb{N}$ and the corresponding convergence coincides with the statistical convergence.

Let D denote the set of all closed and bounded intervals $X = [x_1, x_2]$ on the real line $\mathbb{R} \times \mathbb{N}$. For $X, Y \in D$, we define $X \leq Y$ if and only if $x_1 \leq y_1$ and $x_2 \leq y_2$, $d(X, Y) = max\{|x_1 - y_1|, |x_2 - y_2|\}$, where $X = [x_1, x_2]$ and $Y = [y_1, y_2]$.

Then it can be easily seen that d defines a metric on D and (D, d) is a complete metric space. Also the relation $\leq$ is a partial order on D. A fuzzy number X is a fuzzy subset of the real line $\mathbb{R} \times \mathbb{R}$ i.e. a mapping $X : R \to J (= [0, 1])$ associating each real number t with its grade of membership $X(t)$.

2.6 Definition

A fuzzy number X is said to be (i) convex if $X(t) \geq X(s) \wedge X(r) = min\{X(s), X(r)\}$, where $s < t < r$. (ii) normal if there exists $t_0 \in \mathbb{R} \times \mathbb{R}$ such that $X(t_0) = 1$. (iii) upper semi-continuous if for each $\epsilon > 0, X^{-1}([0, a+\epsilon])$ for all $a \in [0,1]$ is open in the usual topology of $\mathbb{R} \times \mathbb{R}$.

Let $R(J)$ denote the set of all fuzzy numbers which are upper semicontinuous and have compact support, i.e. if $X \in \mathbb{R}(J) \times \mathbb{R}(J)$ the for any $\alpha \in [0,1], [X]^\alpha$ is compact, where $[X]^\alpha = \{t \in \mathbb{R} \times \mathbb{R} : X(t) \geq \alpha,\ if\ \alpha \in [0,1]\}$, $[X]^0$ =closure of $(\{t \in \mathbb{R} \times \mathbb{R} : X(t) > \alpha, if \alpha = 0\})$.

The set $\mathbb{R}$ of real numbers can be embedded $\mathbb{R}(J)$ if we define $\bar{r} \in \mathbb{R}(J) \times \mathbb{R}(J)$ by

$$\bar{r}(t) = \begin{cases} 1, & \text{if } t = r : \\ 0, & \text{if } t \neq r \end{cases}$$

The absolute value, $|X|$ of $X \in \mathbb{R}(J)$ is defined by

$$|X|(t) = \begin{cases} max\{X(t), X(-t)\}, & \text{if } t \geq 0; \\ 0, & \text{if } t < 0 \end{cases}$$

Define a mapping $\bar{d} : \mathbb{R}(J) \times \mathbb{R}(J) \to \mathbb{R}^+ \cup \{0\}$by

$$\bar{d}(X, Y) = sup_{0 \leq \alpha \leq 1} d\left([X]^\alpha, [Y]^\alpha\right).$$

It is known that $\left(\mathbb{R}(J), \bar{d}\right)$ is a complete metric space.

2.7 Definition

A metric on $\mathbb{R}(J) \times \mathbb{R}(J)$ is said to be translation invariant if $\bar{d}(X+Z, Y+Z) = \bar{d}(X, Y)$, for $X, Y, Z \in \mathbb{R}(J) \times \mathbb{R}(J)$.

2.8 Definition

A sequence $X = (X_{mn})$ of fuzzy numbers is said to be convergent to a fuzzy number X_0 if for every $\epsilon > 0$, there exists a positive integer n_0 such that $\bar{d}(X_{mn}, X_0) < \epsilon$ for all $m, n \geq n_0$.

2.9 Definition

A sequence $X = (X_{mn})$ of fuzzy numbers is said to be (i) I-convergent to a fuzzy number X_0 if for each $\epsilon > 0$ such that

$$A = \left\{m, n \in \mathbb{N} : \bar{d}(X_{mn}, X_0) \geq \epsilon\right\} \in I.$$

The fuzzy number X_0 is called I-limit of the sequence (X_{mn}) of fuzzy numbers and we write $I - limX_{mn} = X_0$. (ii) I-bounded if there exists $M > 0$ such that

$$\{m, n \in \mathbb{N} : d(X_{mn}, \bar{0}) > M\} \in I.$$

2.10 Definition

A sequence space E_F of fuzzy numbers is said to be (i) solid (or normal) if $(Y_{mn}) \in E_F$ whenever $(X_{mn}) \in E_F$ and $\bar{d}(Y_{mn}, \bar{0}) \leq \bar{d}(X_{mn}, \bar{0})$ for all $m, n \in \mathbb{N}$. (ii) symmetric if $(X_{mn}) \in E_F$ implies $\left(X_{\pi(mn)}\right) \in E_F$ where π is a permutation of $\mathbb{N} \times \mathbb{N}$.

Let $K = \{k_1 < k_2 < ...\} \subseteq \mathbb{N}$ and E be a sequence space. A K-step space of E is a sequence space

$$\lambda^E_{mn} = \left\{\left(X_{m_p n_p}\right) \in w^2 : (m_p n_p) \in E\right\}.$$

A canonical preimage of a sequence $\left\{\left(x_{m_p n_p}\right)\right\} \in \lambda^E_K$ is a sequence $\{y_{mn}\} \in w^2$ defined as

$$y_{mn} = \begin{cases} x_{mn}, & \text{if } m, n \in E \\ 0, & \text{otherwise.} \end{cases}$$

A canonical preimage of a step space λ^E_K is a set of canonical preimages of all elements in λ^E_K, i.e. y is in canonical preimage of λ^E_K if and only if y is canonical preimage of some $x \in \lambda^E_K$.

2.11 Definition

A sequence space E_F is said to be monotone if E_F contains the canonical pre-images of all its step spaces.

The following well-known inequality will be used throughout the article. Let $p = (p_{mn})$ be any sequence of positive real numbers with $0 \leq p_{mn} \leq sup_{mn} p_{mn} = G, D = max\{1, 2G-1\}$ then

$$|a_{mn} + b_{mn}|^{p_{mn}} \leq D\left(|a_{mn}|^{p_{mn}} + |b_{mn}|^{p_{mn}}\right) \text{ for all } m, n \in \mathbb{N} \times \mathbb{N} \text{ and } a_{mn}, b_{mn} \in \mathbb{C} \times \mathbb{C}.$$

Also $|a_{mn}|^{p_{mn}} \leq max\left\{1, |a|^G\right\}$ for all $a \in \mathbb{C} \times \mathbb{C}$.

First we procure some known results; those will help in establishing the results of this article.

2.12 Lemma

A sequence space E_F is normal implies E_F is monotone. (For the crisp set case, one may refer to Kamthan and Gupta [1], page 53).

2.13 Lemma

If $I \subset 2^{\mathbb{N}\times\mathbb{N}}$ is a maximal ideal, then for each $A \subset \mathbb{N} \times \mathbb{N}$ we have either $A \in I$ or $\mathbb{N} \times \mathbb{N} - A \in I$.

3 Some new integrated sequence spaces of fuzzy numbers

The main aim of this article to introduce the following sequence spaces and examine topological and algebraic properties of the resulting sequence spaces. Let $p = (p_{mn})$ be a sequence of positive real numbers for all $m, n \in \mathbb{N} \times \mathbb{N}$. $f = (f_{mn})$ be a Musielak-Orlicz function, $\left(X, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)$ be a $p-$metric space, and (λ_{mn}) be a sequence of non-zero scalars, $\mu_{mn}(X) = \bar{d}\left(\lambda_{mn}\left((m+n)!\Delta^m X_{mn}\right)^{1/m+n}, \bar{0}\right)$ and $\eta_{mn}(X) = \bar{d}\left(\lambda_{mn}\left(\Delta^m X_{mn}\right)^{1/m+n}, \bar{0}\right)$ are sequence spaces of fuzzy numbers, we define the following sequence spaces as follows:

$$\left[\chi^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I(F)} =$$
$$\left\{(X_{mn}) \in w^{2F} : \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} \geq \epsilon\right\} \in I\right\},$$

$$\left[\Lambda^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I(F)} =$$
$$\left\{(X_{mn}) \in w^{2F} : \exists K > 0 \ni \left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \left[f_{mn}\left(\|\eta_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} \geq K\right\} \in I\right\}.$$

3.1 Theorem

Let $f = (f_{mn})$ be a Musielak-Orlicz function, $q = (q_{mn})$ be a double analytic sequence of strictly positive real numbers, the sequence spaces $\left[\chi^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I(F)}$ and $\left[\Lambda^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I(F)}$ are linear spaces.

Proof: We prove the result only for the space $\left[\chi^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I(F)}$. The other spaces can be treated, similarly. Let $X = (X_{mn})$ and $Y = (Y_{mn})$ be two elements $\left[\chi^{2q}_{f\mu}, \|(d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right]^{I(F)}$. We have

$$A_{\frac{\epsilon}{2}} =$$
$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \left[f_{mn}\left(\|\mu_{mn}(x), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} \geq \frac{\epsilon}{2}\right\} \in I$$

and

$$B_{\frac{\epsilon}{2}} =$$
$$\left\{(r,s) \in \mathbb{N} \times \mathbb{N} : \left[f_{mn}\left(\|\mu_{mn}(y), (d(x_1), d(x_2), \cdots, d(x_{n-1}))\|_p\right)\right]^{q_{mn}} \geq \frac{\epsilon}{2}\right\} \in I.$$

Let α and β be two scalars. By the Musielak continuity of the function $f = (f_{mn})$ the following inequality

holds:
$\left[f_{mn}\left(\left\|\frac{\mu_{mn}(\alpha x+\beta y)}{|\alpha|+|\beta|},\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}} \leq$
$D\left[\frac{|\alpha|}{|\alpha|+|\beta|}f_{mn}\left(\left\|\mu_{mn}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}+$
$D\left[\frac{|\beta|}{|\alpha|+|\beta|}f_{mn}\left(\left\|\mu_{mn}\left(y\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}} \leq$
$D\left[f_{mn}\left(\left\|\mu_{mn}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}+$
$D\left[f_{mn}\left(\left\|\mu_{mn}\left(y\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}$. From the above relation we obtain the following:
$\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:\left[f_{mn}\left(\left\|\frac{\mu_{mn}(\alpha x+\beta y)}{|\alpha|+|\beta|},\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}\geq\epsilon\right\}\subseteq$
$\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:DK\left[f_{mn}\left(\left\|\mu_{mn}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}\geq\frac{\epsilon}{2}\right\}\cup$
$\left\{(r,s)\in\mathbb{N}\times\mathbb{N}:DK\left[f_{mn}\left(\left\|\mu_{mn}\left(y\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}\geq\frac{\epsilon}{2}\right\}\in I$. This completes the proof.

3.2 Remark

It is easy to verify $\left[\Lambda^{2q}_{f\mu},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]^{I(F)}$ is a linear space

3.3 Theorem

The classes of sequences $\left[\chi^{2q}_{f\mu},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]^{F}$ and $\left[\Lambda^{2q}_{f\mu},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]^{F}$ are paranormed spaces paranormed by g, defined by

$$g\left(X\right)=inf\left\{\frac{q_{mn}}{H}:sup_{mn}f_{mn}\left(\left\|\mu_{mn}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\leq 1\right\}$$

where $H=max\left\{1,sup_{mn}q_{mn}\right\}$.
Proof: Clearly $g\left(X\right)\geq 0, g\left(-X\right)=g\left(X\right)$ and $g\left(X+Y\right)\leq g\left(X\right)+g\left(Y\right)$. Next we show the continuity of the product. Let α be fixed and $g\left(X\right)\to 0$. Then it is obvious that $g\left(\alpha X\right)\to 0$. Next let $\alpha\to 0$ and X be fixed. Since f_{mn} are continuous, we have $f_{mn}\left(\alpha\left\|\mu_{mn}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\to 0$, as $\alpha\to 0$. Thus we have

$$inf\left\{\frac{q_{mn}}{H}:sup_{mn}f_{mn}\left(\left\|\mu_{mn}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\leq 1\right\}\to 0, \text{ as } \alpha\to 0.$$

Hence $g\left(\alpha X\right)\to 0$ as $\alpha\to 0$. Therefore g is a paranorm.

3.4 Proposition

$\left[\chi^{2q}_{f\mu},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]^{I(F)}\subset$
$\left[\Lambda^{2q}_{f\mu},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]^{I(F)}$ and the inclusion is proper
Proof: Let $I\left(F\right)=I, f_{mn}\left(\left\|\mu_{mn}\left(x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)=\left(-1\right)^{m+n},\lambda_{mn}=q_{mn}=m=1$ then $\mu\left(x\right)=\left[\Lambda^{2q}_{f\mu},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]^{I(F)}$ but $\left(x_{mn}\right)\notin\left[\chi^{2q}_{f\mu},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]^{I(F)}$

3.5 Theorem

The spaces $\left[\chi^{2q}_{f\mu},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]^{I(F)}$ and
$\left[\Lambda^{2q}_{f\mu},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]^{I(F)}$ are neither solid nor monotone in general
Proof: Let $\left(x_{mn}\right)$ be a given sequence and $\left(\alpha_{mn}\right)$ be a sequence of scalars such that $\left|\alpha_{mn}\right|\leq 1$, for all $m,n\in\mathbb{N}$. Then we have
$\left[f_{mn}\left(\left\|\mu_{mn}\left(\alpha x\right),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}\leq$

$\left[f_{mn}\left(\left\|\mu_{mn}(x),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}$, for all $m,n\in\mathbb{N}\times\mathbb{N}$.
If $\Delta_{mn}=1$ then solidness follows above inequality. The monotonicity follows by lemma 2.12.
The first part of the proof follows from the following example:
Example: Let $I(F)=I$, $\left[f_{mn}\left(\left\|\mu_{mn}(x),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}=$
$\left[f\left(\left\|\mu_{mn}(x),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}=$
$\left[\left(\left\|\mu_{mn}(x),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}$, $m=1,\lambda_{mn}=1$ for all $m,n\in\mathbb{N}$, $q_{mn}=1$ for m,n odd, $q_{mn}=3$ for m,n even, $(x_{mn})=(mn)^{m+n}$ for all $m,n\in\mathbb{N}\times\mathbb{N}$ belongs to $\left[\Lambda_{\mu}^{2q},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]^I$. For E, a sequence space, consider its step space E_J defined by $(y_{mn})\in E_J$ implies $y_{mn}=0$ for all m,n odd and $y_{mn}=x_{mn}$ for m,n even. Then $(y_{mn})\in\left[\Lambda_{\mu}^{2q},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]_J^I$. Hence the spaces are not monotone. Hence are not solid.

3.6 Theroem

The spaces $\left[\chi_{\mu}^{2q},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]^I$ and
$\left[\Lambda_{\mu}^{2q},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]^I$ are not convergence free
Example: Let $I(F)=I$, $\left[f_{mn}\left(\left\|\mu_{mn}(x),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}=$
$\left[f\left(\left\|\mu_{mn}(x),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}=$
$\left[\left(\left\|\mu_{mn}(x),\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right)\right]^{q_{mn}}$, $m=1,\lambda_{mn}=1$ for all $m,n\in\mathbb{N}$, $q_{mn}=1$ for m,n odd, $q_{mn}=2$ for m,n even, consider the sequence $(x_{mn})=(mn)^{-(m+n)}$ for all $m,n\in\mathbb{N}$ belongs to each of $\left[\chi_{\mu}^{2q},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]^I$. and $\left[\Lambda_{\mu}^{2q},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]^I$. Consider the sequence (y_{mn}) defined by $(y_{mn})^{1/m+n}=m^2n^2$, for all $m,n\in\mathbb{N}$. Then (y_{mn}) neither belongs to
$\left[\chi_{\mu}^{2q},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]^I$ nor $\left[\Lambda_{\mu}^{2q},\left\|\left(d\left(x_1\right),d\left(x_2\right),\cdots,d\left(x_{n-1}\right)\right)\right\|_p\right]^I$. Hence the spaces are not convergence free.

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