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Weighted value sharing and uniqueness of entire and meromorphic functions

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Abstract

In this paper, we study the uniqueness of entire and meromorphic functions sharing a nonzero value and obtain some results improving the results obtained by Harina P. Waghamore and Tanuja A (2014).

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1 Introduction

In the present paper, meromorphic functions are always regarded as meromorphic in the entire complex plane. We use the standard notation of the Nevanlinna value-distribution theory, such as $T(r,f), N(r,f), \overline{N}(r,f), m(r,f)$ etc., as explained in Hayman [4], Yang [6], and Yi and Yang [9]. We denote by $S(r,f)$ any function such that $S(r,f) = o(T(r,f))$ as $r \to \infty$, possibly outside a set r of finite linear measure.

Let a be a finite complex number and let k be a positive integer. By $E_{k)}(a,f)$, we denote the set of zeros of $f-a$ with multiplicities at most k, where each zero is counted according to its multiplicity. Also let $\overline{E}_{k)}(a,f)$ be the set of zeros of $f-a$ whose multiplicities are not greater than k and each zero is counted only once. In addition, by $N_{(k}\left(r,\frac{1}{f-a}\right)\left(or\overline{N}_{(k}\left(r,\frac{1}{f-a}\right)\right)$, we denote the counting function with respect to the set $E_{k)}(a,f)(or\overline{E}_{k)}(a,f))$.

We set

$$N_k\left(r,\frac{1}{f-a}\right) = \overline{N}\left(r,\frac{1}{f-a}\right) + \overline{N}_{(2}\left(r,\frac{1}{f-a}\right) + ... + \overline{N}_{(k}\left(r,\frac{1}{f-a}\right)$$

and define

$$\Theta(a,f) = 1 - \limsup_{r\longrightarrow\infty} \frac{\overline{N}\left(r,\frac{1}{f-a}\right)}{T(r,f)}$$

and

$$\delta_k(a,f) = 1 - \limsup_{r\to\infty} \frac{N_k\left(r,\frac{1}{f-a}\right)}{T(r,f)},$$

Let f and g be two nonconstant meromorphic functions and let a be a finite complex number. We say that f and g share the value a CM(counting multiplicities) if f and g have the same a-points with the same multiplicities. We also say that f and g share the value a IM(ignoring multiplicities) if we do not consider the multiplicities. We denote by $\overline{N}_L\left(r,\frac{1}{f-a}\right)$ the counting function for a-points of both f and g at which f has larger multiplicity than g (in the case where the multiplicities are not counted). Similarly, we have the

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notation $\overline{N}_L(r, \frac{1}{g-a})$. Further, by $N_0(r, \frac{1}{F'})$, we denote the counting function of those zeros of F' that are not zeros of $F(F-1)$.

In 1996, Fang proved the following result.

Theorem A([1]). Let f and g be two non-constant entire functions and let n, k be two positive integers with $n > 2k+4$. If $(f^n)^{(k)}$and $(g^n)^{(k)}$ share the value 1 CM, then either $f(z) = c_1e^{cz}$ and $g(z) = c_2e^{-cz}$ where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$ or $f = tg$ for a constant t such that $t^n = 1$.

In 1997, Yang and Hua obtained a unicity theorem corresponding to above result.

Theorem B([9]). Let f and g be two non-constant entire functions, $n \geq 6$ a positive integer. If $f^nf'$and g^ng' share the value 1 CM, then either $f(z) = c_1e^{cz}$ and $g(z) = c_2e^{-cz}$ where c_1, c_2 and c are three constants satisfying $(c_1c_2)^{n+1}c^2 = 1$ or $f = tg$ for a constant t such that $t^{n+1} = 1$.

In 2002, Fang proved the following result.

Theorem C([2]). Let f and g be two non-constant entire functions and let n, k be two positive integers with $n > 2k+8$. If $[f^n(f-1)]^k$ and $[g^n(g-1)^k]$ share the value 1 CM, then $f \equiv g$.

In 2008, Zhang and Lin, Zhang, Chen and Lin extended Theorem C and obtain the following results.

Theorem D([11]). Let f and g be two non-constant entire functions and let n, m and k be three positive integers with $n > 2k+m+4$, and λ, μ be constants such that $\mid \lambda \mid + \mid \mu \mid \neq 0$. If $[f^n(\mu f^m + \lambda)]^{(k)}$ and $[g^n(\mu g^m + \lambda)]^{(k)}$ share 1 CM, then

(i) when $\lambda\mu \neq 0$, $f \equiv g$.

(ii) when $\lambda\mu = 0$, either $f \equiv tg$, where t is a constant satisfying $t^{n+m} = 1$, or $f(z) = c_1e^{cz}$ and $g(z) = c_2e^{-cz}$ where c_1, c_2 and c are three constants satisfying $(-1)^k\lambda^2(c_1c_2)^{n+m}[(n+m)c]^{2k} = 1$ or $(-1)^k\mu^2(c_1c_2)^{n+m}[(n+m)c]^{2k} = 1$.

Theorem E([12]) . Let f and g be two non-constant entire functions and let n, m and k be three positive integers with $n > 2k+m+4$, and let $P(z) = a_mz^m + a_{m-1}z^{m-1} + ... + a_1z + a_0$, or $P(z) \equiv c_0$, where $a_0 \neq 0, a_1, a_2, ..., a_{m-1}, a_m \neq 0, c_0 \neq 0$ are complex constants. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share 1 CM, then

(i)when $P(z) = a_mz^m + a_{m-1}z^{m-1} + ... + a_1z + a_0$, either $f \equiv tg$ for a constant t such that $t^d = 1$, where $d = \{n+m, ..., n+m-i, ..., n\}, a_{m-i} \neq 0$ for some $i = 0, 1, ..., m$, or f and g satisfy the algebraic equation $R(f, g) \equiv 0$, where $R(w_1, w_2) = w_1^n \left(a_mw_1^m + a_{m-1}w_1^{m-1} + ... + a_1w_1 + a_0\right) - w_2^n \left(a_mw_2^m + a_{m-1}w_2^{m-1} + ... + a_1w_2 + a_0\right)$;

(ii) when $P(z) = c_0$, either $f(z) = c_1/\sqrt[n]{c_0}e^{cz}, g(z) = c_2/\sqrt[n]{c_0}e^{-cz}$, where c_1, c_2 and c are three constants satisfying $(-1)^k(c_1c_2)^n(nc)^{2k} = 1$, or $f = tg$ for a constant t such that $t^n = 1$.

In 2009, H.Y. Xu and T. B. Cao proved the following result.

Theorem F([6]). Let f and g be two non constant entire functions, and let n, m and k be three positive integers with $n \geq 5k+5m+8$. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share $(1, 0)$, then the conclusion of Theorem E still holds.

Theorem G([6]). Let f and g be two non constant entire functions, and let n, m and k be three positive integers with $n \geq \frac{9}{2}m + 4k + \frac{9}{2}$. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share $(1, 1)$, then the conclusion of Theorem E still holds.

Theorem H([6]). Let f and g be two non constant entire functions, and let n, m and k be three positive integers with $n \geq 3m+3k+5$. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share $(1, 2)$, then the conclusion of Theorem E still holds.

In 2014, Harina P. Waghamore and Tanuja A. by introducing the notion of multiplicity and extend the above theorem to meromorphic functions and obtain the following results.

Theorem I([3]). Let f and g be two non-constant meromorphic functions, whose zeros and poles are of multiplicities atleast s, where s is a positive integer. Let $P(f) = a_mf^m + a_{m-1}f^{m-1} + ... + a_1f + a_0, (a_m \neq 0)$, and $a_i(i = 0, 1, ..., m)$ is the first nonzero coefficient from the right, and let n, k, m be three positive integers. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share $(1, l)$ and one of the following conditions holds: (i) $l \geq 2$ and $s(n+m) > 3k+10$

(ii) $l = 1$ and $s(n+m) > 5k+13$

(iii) $l = 0$ and $s(n+m) > 9k+16$

then either $f = tg$ for a constant t such that $t^d = 1$, where $d = \{n+m, ..., n+m-i, ..., n\}, a_{m-i} \neq 0$ for some $i = 0, 1, ..., m$, or f and g satisfy algebraic equation $R(f, g) \equiv 0$, where $R(w_1, w_2) = w_1^nP(w_1) - w_2^nP(w_2)$.

Theorem J([3]). let f and g be two non-constant entire functions, whose zeros and poles are of multiplicities atleast s, where s is a positive integer. Let $P(f) = a_mf^m + a_{m-1}f^{m-1} + ... + a_1f + a_0, (a_m \neq 0)$, and $a_i(i = 0, 1, ..., m)$ is the first non zero coefficient from the right, and let n, k, m be three positive integers. If $[f^nP(f)]^{(k)}$ and $[g^nP(g)]^{(k)}$ share $(1, l)$ and one of the following conditions holds:

(i) $l \geq 2$ and $s(n+m) > 3k+5$

(ii) $l = 1$ and $s(n+m) > 4k+6$

(iii) $l = 0$ and $s(n+m) > 5k+8$
then either $f = tg$ for a constant t such that $t^d = 1$, where $d = (n+m, ..., n+m-i, ..., n), a_{m-i} \neq 0$ for some $i = 0, 1, ..., m$ or f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(w_1, w_2) = w_1^n P(w_1) - w_2^n P(w_2)$.

In the present paper, we always use $L(z)$ to denote an arbitrary polynomial of degree n, i.e.,

$$L(z) = a_n z^n + a_{n-1} z^{n-1} + ... + a_0 = a_n (z-c_1)^{l_1} (z-c_2)^{l_2} ... (z-c_s)^{l_s} \tag{1.1}$$

where $a_i, i = 0, 1, ..., n, a_n \neq 0$, and $c_j, j = 1, 2, ..., s$ are finite complex number constants; $c_1, c_2, ..., c_s$ are all distinct zeros of $L(z), l_1, l_2, ..., l_s$. s, n are all positive integers satisfying the equality

$$l_1 + l_2 + ... + l_s = n \ \ and \ \ l = max\{l_1, l_2 ... l_s.\} \tag{1.2}$$

In this paper, we study the existence of solutions for $[L(f)]^{(k)}$ and the corresponding uniqueness theorems. Thus, we obtain the following results as a generalization of the theorems presented above:

Theorem 1.1. Let f and g be two non constant meromorphic functions, and let n, k, l be three positive integers. If $[L(f)]^{(k)}$ and $[L(g)]^{(k)}$ share $(1, l)$, and one of the following conditions holds:
(i) $l \geq 2$ and $(k+8)l > (k+7)n + 3k + 8$
(ii) $l = 1$ and $(2k+10)l > (2k+9)n + 5k + 11$
(iii) $l = 0$ and $(4k+14)l > (4k+13)n + 9k + 14$.
then either $f = b_1 e^{bz} + c,\ g = b_2 e^{-bz} + c$ or f and g satisfy the algebraic equation $R(f,g) \equiv 0$. where b_1, b_2 and b are three constants such that
$(-1)^k (b_1 b_2)^n (nb)^{2k} = 1$ and $R(w_1, w_2) = L(w_1) - L(w_2)$.

Remark 1: If $l = n$, then for $l \geq 2$ we get $n > 3k+8$, for $l = 1$ we get $n > 5k+11$ and for $l = 0$ we get $n > 9k+14$.

Theorem 1.2. Let f and g be two non constant entire functions and let n, k and l be three positive integers. If $[L(f)]^{(k)}$ and $[L(g)]^{(k)}$ share $(1, l)$, and one of the following conditions holds:
(i) $l \geq 2$ and $4l > 3n + 3k + 4$
(ii) $l = 1$ and $11l > 9n + 8k + 9$
(iii) $l = 0$ and $6l > 5n + 5k + 7$
then either $f = b_1 e^{bz} + c,\ g = b_2 e^{-bz} + c$ or f and g satisfy the algebraic equation $R(f,g) \equiv 0$. where b_1, b_2 and b are three constants such that
$(-1)^k (b_1 b_2)^n (nb)^{2k} = 1$ and $R(w_1, w_2) = L(w_1) - L(w_2)$.

Remark 2: If $l = n$, then for $l \geq 2$ we get $n > 3k+4$, for $l = 1$ we get $n > \frac{8k+9}{2}$ and for $l = 0$ we get $n > 5k+7$.

Remark 3 : If $L(f) \equiv L(g)$, then we get

$$a_n f^n + a_{n-1} f^{n-1} + ... + a_1 f \equiv a_n g^n + a_{n-1} g^{n-1} + ... + a_1 g.$$

Let $h = \frac{f}{g}$. If h is a constant, then we substitute $f = gh$ in this equation and obtain $a_n g^n (h^n - 1) + a_{n-1} g^{n-1} (h^{n-1} - 1) + ... + a_1 g (h-1) \equiv 0$. This yields $h^d = 1, d = (n, ..., n-i, ...1)$, and $a_{n-i} \neq 0$ for some $i = 0, 1, ..., n-1$. Thus $f \equiv tg$ for a constant t such that $t^d = 1$. If h is not a constant, then by virtue of the equation presented above, we know that f and g satisfy the algebraic equation $R(f,g) \equiv 0$, where $R(w_1, w_2) = L(w_1) - L(w_2)$.

2 Lemmas

Lemma 2.1. *([4]) Let $f(z)$ be a nonconstant meromorphic function and let $a_1(z)$ and $a_2(z)$ be two meromorphic functions such that $T(r, a_i) = S(r, f), i = 1, 2$. Then $T(r,f) \leq \overline{N}(r,f) + \overline{N}(r, \frac{1}{f-a_1}) + \overline{N}(r, \frac{1}{f-a_2}) + S(r,f)$.*

Lemma 2.2. *([4]) Let f be a nonconstant meromorphic function, let k be a positive integer, and let c be a nonzero finite complex number. Then*

$$\begin{aligned} T(r,f) &\leq \overline{N}(r,f) + N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{f^{(k)} - c}\right) - N\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f) \\ &\leq \overline{N}(r,f) + N_{k+1}\left(r, \frac{1}{f}\right) + \overline{N}\left(r, \frac{1}{f^{(k)} - c}\right) - N_0\left(r, \frac{1}{f^{(k+1)}}\right) + S(r,f). \end{aligned}$$

where $N_0\left(r, \frac{1}{f^{(k+1)}}\right)$ is the counting function which only counts those points such that $f^{(k+1)} = 0$ but $f(f^{(k)} - c) \neq 0$.

Lemma 2.3. *([10]) Let f be a nonconstant meromorphic function and $P(f) = a_0 + a_1 f + ... + a_n f^n$, where $a_0, a_1, ..., a_n$ are constants and $a_n \neq 0$. Then*

$$T(r, P(f)) = nT(r, f) + S(r, f).$$

Lemma 2.4. *([5], [13])Let f be a non-constant meromorphic function and k be a positive integer, then*

$$\begin{aligned} N_p\left(r, \frac{1}{f^{(k)}}\right) &\leq N_{p+k}\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f) \\ &\leq (p+k)\overline{N}\left(r, \frac{1}{f}\right) + k\overline{N}(r, f) + S(r, f). \end{aligned}$$

and clearly $\overline{N}(r, \frac{1}{f^{(k)}}) = N_1(r, \frac{1}{f^{(k)}})$.

Lemma 2.5. *([6]) Let f and g be two nonconstant entire functions, and let k be a positive integer. If $f^{(k)}$ and $g^{(k)}$ share $(1, l)$ $(l = 0, 1, 2)$. Then*

(i) If $l = 0$,

$\Theta(0, f) + \delta_k(0, f) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) + \delta_{k+2}(0, f) + \delta_{k+2}(0, g) > 5$, *then either $f^{(k)} g^{(k)} = 1$ or $f \equiv g$;*

(ii) If $l = 1$,

$\frac{1}{2}\left[\Theta(0, f) + \delta_k(0, f) + \delta_{k+2}(0, f)\right] + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) + \Theta(0, g) + \delta_k(0, g) > \frac{9}{2}$, *then either $f^{(k)} g^{(k)} = 1$ or $f \equiv g$;*

(iii) If $l \geq 2$,

$\Theta(0, f) + \delta_k(0, f) + \delta_{k+1}(0, f) + \delta_{k+2}(0, g) > 3$, *then either $f^{(k)} g^{(k)} = 1$ or $f \equiv g$.*

Lemma 2.6. *([3]). Let f and g be two non-constant meromorphic functions, $k(\geq 1)$ and $(l \geq 0)$ be integers. If $f^{(k)}$ and $g^{(k)}$ share $(1, l)$ $(l = 0, 1, 2)$. Then*

(i) If $l \geq 2$,

$(k+2)\Theta(\infty, f) + 2\Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0.g) > k + 7$, *then either $f^{(k)} g^{(k)} = 1$ or $f \equiv g$;*

(ii) If $l = 1$,

$(2k+3)\Theta(\infty, f) + 2\Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) + \delta_{k+2}(0, f) > 2k + 9$, *then either $f^{(k)} g^{(k)} = 1$ or $f \equiv g$;*

(iii) If $l = 0$,

$(2k+3)\Theta(\infty, f) + (2k+4)\Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + 2\delta_{k+1}(0, f) + 3\delta_{k+1}(0, g) > 4k + 13$, *then either $f^{(k)} g^{(k)} = 1$ or $f \equiv g$.*

Lemma 2.7. *([2]) Let $f(z)$ be a nonconstant entire function and let $k(\geq 2)$ be a positive integer. If $ff^{(k)} \neq 0$, then $f = e^{az+b}$, where a and b are constants.*

3 Almost Contra Pre Generalized b - Continuous Functions

Proof of Theorem 1.1. Let $L(z)$ and l be given by (1.1) and (1.2), respectively. Without loss of generality, we can assume that $a_n = 1$, $l = l_1$ and $c = c_1$. This yields

$$\begin{aligned} \Theta(0, L(f)) &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, \frac{1}{L(f)})}{T(r, L(f))} \\ &\geq 1 - \limsup_{r\to\infty} \frac{\sum_{j=1}^{s} \overline{N}(r, \frac{1}{f - c_j})}{nT(r, f)} \geq 1 - \frac{s}{n} \geq \frac{l-1}{n} \end{aligned} \tag{3.3}$$

similarly, we get

$$\Theta(0, L(g)) \geq \frac{l-1}{n} \tag{3.4}$$

Moreover, we have

$$\begin{aligned} \Theta(\infty, L(f)) &= 1 - \limsup_{r\to\infty} \frac{\overline{N}(r, L(f))}{T(r, L(f))} \\ &\geq 1 - \limsup_{r\to\infty} \frac{\sum_{j=1}^{s} \overline{N}(r, f - c_j)}{nT(r, f)} \geq 1 - \frac{s}{n} \geq \frac{l-1}{n} \end{aligned} \tag{3.5}$$

similarly

$$\Theta(\infty, L(g)) \geq \frac{l-1}{n} \tag{3.6}$$

Also we have

$$\begin{aligned} \delta_k(0, L(f)) &= 1 - \limsup_{r\to\infty} \frac{N_k(r, \frac{1}{L(f)})}{T(r, L(f))} \\ &\geq 1 - \limsup_{r\to\infty} \frac{\sum_{j=1}^{s} N_k\left(r, \frac{1}{(f-c_j)^{l_1}}\right) + N_k\left(r, \frac{1}{(f-c)^{l}}\right)}{nT(r,f)} \\ &\geq 1 - \limsup_{r\to\infty} \frac{(s-1)T(r,f) + kT(r,f) + S(r,f)}{nT(r,f)} \\ &\geq 1 - \frac{s-1+k}{n} \geq \frac{l-k}{n} \end{aligned} \tag{3.7}$$

and similarly

$$\delta_k(0, L(g)) \geq \frac{l-k}{n} \tag{3.8}$$

$$\delta_{k+1}(0, L(f)) \geq \frac{l-k-1}{n} \tag{3.9}$$

$$\delta_{k+1}(0, L(g)) \geq \frac{l-k-1}{n} \tag{3.10}$$

$$\delta_{k+2}(0, L(f)) \geq \frac{l-k-2}{n} \tag{3.11}$$

$$\delta_{k+2}(0, L(g)) \geq \frac{l-k-2}{n} \tag{3.12}$$

Case i. If $l \geq 2$ and from (3.1) to (3.10) and also from Lemma 2.6, we get

$$\Delta = (k+2)\Theta(\infty, f) + 2\Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0.g) > k+7$$

and $(k+8)l > (k+7)n + 3k + 8$. we conclude that $h(z) \equiv 0$, i.e.,

$$\frac{f^{(k+2)}(z)}{f^{(k+1)}(z)} - 2\frac{f^{(k+1)}(z)}{f^{(k)}(z)-1} = \frac{g^{(k+2)}(z)}{g^{(k+1)}(z)} - 2\frac{g^{(k+1)}(z)}{g^{(k)}(z)-1}$$

Solving this equation, we obtain

$$\frac{1}{f^{(k)}-1} = \frac{bg^{(k)} + a - b}{g^{(k)}-1}.$$

we can write the above equation as

$$\frac{1}{L(f)^{(k)}-1} = \frac{bL(g)^{(k)} + a - b}{L(g)^{(k)}-1}. \tag{3.13}$$

Further, we consider the following three cases:
Case I. If $b \neq 0$ and $a = b$, then it follows from (3.11) that

$$\frac{1}{L(f)^{(k)}-1} = \frac{bL(g)^{(k)}}{L(g)^{(k)}-1}. \tag{3.14}$$

1.1. If $b \neq -1$, then it follows from (3.11) that $[L(f)^{(k)}][L(g)^{(k)}] \equiv 1$, i.e.,

$$[(f-c)^{l}(f-c)^{l_2}...(f-c_s)^{l_s}]^{(k)}[(g-c)^{l}(g-c)^{l_2}...(g-c_s)^{l_s}]^{(k)} = 1 \tag{3.15}$$

1.1.1. If $s = 1$, then we can rewrite (3.13) as follows:

$$[(f-c)^n]^{(k)}[(g-c)^n]^{(k)} = 1.$$

Since $(k+8)l > (k+7)n + 3k + 8$, $l = n$, we conclude that $n > 3k + 8$. Hence, $f - c \neq 0$ and $g - c \neq 0$. Thus, according to Lemma 2.7 we find

$$f = b_1 e^{bz} + c, \;\; g = b_2 e^{-bz} + c,$$

where b_1, b_2 and b are three constants such that $(-1)^k (b_1 b_2)^n (nb)^{2k} = 1$.

1.1.2. For $s \geq 2$, we note that $(k+8)l > (k+7)n + 3k + 8$. Hence, $l > 3k + 8$. Suppose that z_0 is an l-fold zero of $f - c$. We know that z_0 must be an $(l-k)$-fold zero of $[(f-c)^l (f-c)^{l_2} ... (f-c_s)^{l_s}]^{(k)}$. Note that it follows from (3.13) that g is an entire function. This is a contradiction. Hence, $f - c \neq 0$ and $g - c \neq 0$. Thus, we get $f = e^{\alpha(z)} + c$, where $\alpha(z)$ is a non constant entire function. Therefore,

$$[f^i]^{(k)} = [(e^{\alpha} + c)^i]^{(k)} = p_i(\alpha', \alpha'', ..., \alpha^{(k)}) e^{i\alpha}, \;\; i = 1, 2, ..., n, \tag{3.16}$$

where $p_i, i = 1, 2, ..., n$, are differential polynomials in $\alpha', \alpha'', ..., \alpha^{(k)}$. Clearly, if $p_i \not\equiv 0$ and $T(r, p_i) = S(r, f), i = 1, 2, ..., n$, then it follows from (3.13) and (3.14) that

$$N\left(r, \frac{1}{p_n e^{(n-1)\alpha} + ... + p_1}\right) = S(r, f).$$

In view of Lemmas 2.1 and 2.3 and the fact that $f = e^{\alpha} + c$, we get

$$\begin{aligned}
(n-1)T(r, f-c) &= T\left(r, p_n e^{(n-1)\alpha} + ... + p_1\right) + S(r, f) \\
&\leq \overline{N}\left(r, \frac{1}{p_n e^{(n-1)\alpha} + ... + p_1}\right) + \overline{N}\left(r, \frac{1}{p_n e^{(n-1)\alpha} + ... + p_2 e^{\alpha}}\right) + S(r, f) \\
&\leq \overline{N}\left(r, \frac{1}{p_n e^{(n-2)\alpha} + ... + p_2}\right) + S(r, f) \\
&\leq (n-2)T(r, f-c) + S(r, f),
\end{aligned}$$

which is a contradiction.

1.2. If $a = b \neq -1$, then relation (3.12) can be rewritten as

$$L(g)^{(k)} = \frac{-1}{b} . \frac{1}{L(f)^{(k)} - (1+b)/b}. \tag{3.17}$$

From (3.15), we get

$$\overline{N}\left(r, \frac{1}{L(f)^{(k)} - (1+b)/b}\right) = \overline{N}(r, g) = S(r, f). \tag{3.18}$$

By relation (3.16) and Lemma 2.2, we obtain

$$\begin{aligned}
nT(r, f) &= T(r, L(f)) + O(1) \\
&\leq N_{k+1}\left(r, \frac{1}{L(f)}\right) + N\left(r, \frac{1}{L(f)^{(k)} - (1+b)/b}\right) + S(r, f) \\
&\leq N_{k+1}\left(r, \frac{1}{(f-c)^l}\right) + N_{k+1}\left(r, \frac{1}{(f-c_2)^{l_2} ... (f-c_s)^{l_s}}\right) + S(r, f) \\
&\leq (k+s)T(r, f) \leq (k+n-l+1)T(r, f) + S(r, f),
\end{aligned}$$

which is a contradiction because $(k+8)l > (k+7)n + 3k + 8$.

Case II. $b \neq 0$ and $a \neq b$. We discuss the following sub cases:

2.1. Suppose that $b = -1$. Then $a \neq 0$ and relation (3.11) can be rewritten as

$$L(f)^{(k)} = \frac{a}{a + 1 - L(g)^{(k)}}. \tag{3.19}$$

It follows from (3.17) that

$$\overline{N}\left(r, \frac{1}{a + 1 - L(g)^{(k)}}\right) = \overline{N}(r, f) = S(r, g). \tag{3.20}$$

In view of (3.18) and Lemmas 2.2 and 2.4, we find

$$nT(r, g) = T(r, L(g)) + O(1) \leq N_{k+1}\left(r, \frac{1}{L(g)}\right) + S(r, g).$$

Further, by using the argument as in Case 1.2, we arrive at a contradiction.
2.2. Suppose that $b \neq -1$. Then relation (3.11) be rewritten as

$$L(f)^{(k)} - \frac{b+1}{b} = \frac{-a}{b^2} \cdot \frac{1}{L(g)^{(k)} + (a-b)/b}. \tag{3.21}$$

It follows from (3.19) that

$$\overline{N}(r, \frac{1}{L(g)^{(k)} - (b+1)/b}) = \overline{N}(r, g). \tag{3.22}$$

By using (3.20) and Lemmas 2.2 and 2.4, we arrive at a contradiction in exactly the same way as in Case 1.2.
Case III. $b = 0$ and $a \neq 0$. Then relation (3.11) can be rewritten as

$$L(g)^{(k)} = aL(f)^{(k)} + (1-a), \tag{3.23}$$

$$L(g) = aL(f) + (1-a)p_1(z), \tag{3.24}$$

where p_1 is a polynomial with $degp_1 \leq k$. If $a \neq 1$, then $(1-a)p_1 \not\equiv 0$. Together with (3.22) and Lemma 2.1, this yields

$$\begin{aligned} nT(r,g) = T(r, L(g)) + O(1) &\leq \overline{N}\left(r, \frac{1}{L(g)}\right) + \overline{N}\left(r, \frac{1}{L(f)}\right) + S(r,g) \\ &\leq \sum_{i=1}^{s} \overline{N}\left(r, \frac{1}{g - c_i}\right) + \sum_{j=1}^{s} \overline{N}\left(r, \frac{1}{f - c_j}\right) + S(r,g) \\ &\leq s[T(r,g) + T(r,f)] + S(r,g). \end{aligned} \tag{3.25}$$

Since $n = l + l_2 + ... + l_s$, we get $n - l = l_2 + ... + l_s \geq s - 1$, i.e., $n - l \geq s - 1, n - s \geq l - 1$. In view of the inequality $(k+8)l > (k+7)n + 3k + 8$, we conclude that

$$l - 1 > (k+7)(n-l) + 3k + 7 > (k+7)(s-1) + 3k + 7$$

and hence,

$$n - s \geq l - 1 > (k+7)(s-1) + 3k + 7,$$

i.e., $n - s > (k+7)(s-1) + 3k + 7$. Therefore,

$$s < \frac{n - 2k - 1}{k + 8}$$

and thus,

$$nT(r,g) < \frac{n-2k-1}{k+8}[T(r,g) + T(r,f)] + S(r,g). \tag{3.26}$$

On the other hand, it follows from (3.22) and Lemma 2.3 that

$$T(r,g) = T(r,f) + S(r,g).$$

Substituting this relation in (3.24), we conclude that

$$\frac{3n + 4k + 2}{k + 8} T(r,g) < S(r,g),$$

which is a contradiction.
Thus $a = 1$ and therefore, it follows from (3.22) that $L(f) = L(g)$.
Further, we consider the case where f and g are polynomials. Suppose that $f - c$ and $g - c$ have u and v pairwise distinct zeros, respectively. Then $f - c$ and $g - c$ admit the representations

$$f - c = k_1(z - a_1)^{n_1}(z - a_2)^{n_2}...(z - a_u)^{n_u},$$

$$g - c = k_2(z - b_1)^{m_1}(z - b_2)^{m_2}...(z - b_v)^{m_v},$$

and hence,

$$[f - c]^l = k_1^l(z - a_1)^{l_{n_1}}(z - a_2)^{l_{n_2}}...(z - a_u)^{l_{n_u}}, \tag{3.27}$$

$$[g-c]^l = k_2^l(z-b_1)^{l_{m_1}}(z-b_2)^{l_{m_2}}...(z-b_v)^{l_{m_v}}, \tag{3.28}$$

where k_1 and k_2 are nonzero constants, $n_i l > 3k+8$, $m_j l > 3k+8$, and $n_i, m_j,\ i = 1,2,...,u, j = 1,2,...,v$, are positive integers. Differentiating (3.22), we get

$$L(g)^{(k+1)} = aL(f)^{(k+1)}. \tag{3.29}$$

It follows from (3.25)(3.26) and (3.27) that

$$\begin{aligned}(z-a_1)^{l_{n_1}-k-1}&(z-a_2)^{l_{n_2}-k-1}...(z-a_u)^{l_{n_u}-k-1}\xi_1(z)\\ &= (z-b_1)^{l_{m_1}-k-1}(z-b_2)^{l_{m_2}-k-1}...(z-b_v)^{l_{m_v}-k-1}\xi_2(z),\end{aligned} \tag{3.30}$$

where $\xi_1(z)$ and $\xi_2(z)$ are polynomials, $deg\xi_1 = (n-l)\Sigma_{i=1}^u n_i + (u-1)(k+1)$, and $deg\xi_2 = (n-l)\Sigma_{j=1}^v m_j + (v-1)(k+1)$. Thus, in view of the fact that $(k+8)l > (k+7)n + 3k + 8$, we find $5l - 4n > (k+3)(n-l) + 3k + 8 > 3k + 8$. Then $(5l-4n)n_i > 3k+8,\ (5l-4n)m_j > 3k+8,\ i = 1,2,...,u, j = 1,2,...,v$. Hence,

$$\begin{aligned}&\Sigma_{i=1}^u[n_i l - (k+1)] - \Sigma_{i=1}^v n_i(n-l) = \Sigma_{i=1}^u[n_i(5l-4n) - (k+1)]\\ &> u(2k+7) > (u-1)(k+1),\end{aligned}$$

i.e.,

$$\Sigma_{i=1}^u[n_i l - (k+1)] > (n-l)\Sigma_{i=1}^v n_i + (u-1)(k+1).$$

Similarly,

$$\Sigma_{j=1}^v[m_j l - (k+1)] > (n-l)\Sigma_{j=1}^v m_j + (v-1)(k+1).$$

Thus, by using (3.28), we show that there exists z_0 such that $L(f(z_0)) = L(g(z_0)) = 0$, where the multiplicity of z_0 is greater than $3k+8$. Together with (3.22), this yields $p_1(z) \equiv 0$, which also proves the claim.
Therefore, it follows from (3.21) and (3.22) that $a = 1$ and, therefore, $L(f) \equiv L(g)$. **Case ii.** If $l = 1$

$$\Delta = (2k+3)\Theta(\infty, f) + 2\Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) + \delta_{k+2}(0, f) > 2k+9,$$

and $(2k+10)l > (2k+9)n + 5k + 11$, proceeding as in case i, we get a contradiction.
Case iii. If $l = 0$

$$\Delta = (2k+3)\Theta(\infty, f) + (2k+4)\Theta(\infty, g) + \Theta(0, f) + \Theta(0, g) + 2\delta_{k+1}(0, f) + 3\delta_{k+1}(0, g) > 4k+13,$$

and $(4k+14)l > (4k+13)n + 9k + 14$. proceeding as in case i, we get a contradiction.
This completes the proof of Theorem 1.1.
Proof of Theorem 1.2 Let $L(z)$ and l be given by (1.1) and (1.2), respectively. Without loss of generality, we can assume that $a_n = 1$, $l = l_1$ and $c = c_1$. This yields

$$\begin{aligned}\Theta(0, L(f)) &= 1 - \limsup_{r\to\infty}\frac{\overline{N}(r, \frac{1}{L(f)})}{T(r, L(f))}\\ &\geq 1 - \limsup_{r\to\infty}\frac{\sum_{j=1}^s \overline{N}(r, \frac{1}{f-c_j})}{nT(r, f)} \geq 1 - \frac{s}{n} \geq \frac{l-1}{n}\end{aligned} \tag{3.31}$$

similarly, we get

$$\Theta(0, L(g)) \geq \frac{l-1}{n} \tag{3.32}$$

Moreover, we have

$$\begin{aligned}\Theta(\infty, L(f)) &= 1 - \limsup_{r\to\infty}\frac{\overline{N}(r, L(f))}{T(r, L(f))}\\ &\geq 1 - \limsup_{r\to\infty}\frac{\sum_{j=1}^s \overline{N}(r, f-c_j)}{nT(r, f)} \geq 1 - \frac{s}{n} \geq \frac{l-1}{n}\end{aligned} \tag{3.33}$$

and also we have

$$\Theta(\infty, L(g)) \geq \frac{l-1}{n} \tag{3.34}$$

Also we have

$$\begin{aligned}
\delta_k(0, L(f)) &= 1 - \limsup_{r\to\infty} \frac{N_k(r, \frac{1}{L(f)})}{T(r, L(f))} \\
&\geq 1 - \limsup_{r\to\infty} \frac{\sum_{j=1}^{s} N_k(r, \frac{1}{(f-c_j)^{l_1}}) + N_k(r, \frac{1}{(f-c)^{l}})}{nT(r,f)} \\
&\geq 1 - \limsup_{r\to\infty} \frac{(s-1)T(r,f) + kT(r,f) + S(r,f)}{nT(r,f)} \\
&\geq 1 - \frac{s-1+k}{n} \geq \frac{l-k}{n}
\end{aligned} \tag{3.35}$$

and similarly

$$\delta_k(0, L(f)) \geq \frac{l-k}{n} \tag{3.36}$$

$$\delta_{k+1}(0, L(f)) \geq \frac{l-k-1}{n} \tag{3.37}$$

$$\delta_{k+1}(0, L(g)) \geq \frac{l-k-1}{n} \tag{3.38}$$

$$\delta_{k+2}(0, L(f)) \geq \frac{l-k-2}{n} \tag{3.39}$$

$$\delta_{k+2}(0, L(g)) \geq \frac{l-k-2}{n} \tag{3.40}$$

Case i. If $l = 0$. and from (3.1) to (3.10) and also from Lemma 2.6, we get $\Delta = \Theta(0, f) + \delta_k(0, f) + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) + \delta_{k+2}(0, f) + \delta_{k+2}(0, g) > 5$, and $6l > 5n + 5k + 7$.
Case ii. If $l = 1$,
$\Delta = \frac{1}{2}[\Theta(0, f) + \delta_k(0, f) + \delta_{k+2}(0, f)] + \delta_{k+1}(0, f) + \delta_{k+1}(0, g) + \Theta(0, g) + \delta_k(0, g) > \frac{9}{2}$, and $11l > 9n + 8k + 9$.
Case iii. If $l \geq 0$,
$\Delta = \Theta(0, f) + \delta_k(0, f) + \delta_{k+1}(0, f) + \delta_{k+2}(0, g) > 3$, and $4l > 3n + 3k + 4$.
proceeding as in the proof of Theorem 1.1, we get Theorem 1.2.
This completes the proof of Theorem 1.2.

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