Malaya J. Mat. 5(3)(2017) 550–555

Malaya Journal of Matematik
www.malayajournal.org

MJM
an international journal of mathematical sciences with computer applications...

ISSN : 2319-3786

Existence results of Boundary Value Problem for Implicit Impulsive Fractional Differential Equations

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Abstract

In this paper, we study the existence of solutions for boundary value problem for implicit fractional differential equations with impulsive conditions. We prove the existence results by applying fixed point theorem and finally an example is included to show the applicability of our results.

Keywords: Implicit Impulsive Fractional Differential Equations, boundary Conditions, fixed point.

2010 MSC: 26A23, 34G20, 34A37.

1 Introduction

The study of impulsive functional differential and integro-differential systems is linked to their utility in simulating processes and phenomena subject to short-time perturbations during their evolution. The perturbations are performed discretely and their duration is negligible in comparison with the total duration of the processes and phenomena. Now impulsive partial neutral functional differential and integro-differential systems have become an important object of investigation in recent years stimulated by their numerous applications to problems arising in mechanics, electrical engineering, medicine, biology, ecology, etc. With regard to this matter, we refer the reader to see the monographs by Benchohra *et al* [5], Lakshmikantham *et al* [15], Samoilenko and Perestyuk [20], and the papers [1–3, 13, 14] and the references therein.

On the other hand, the nonlinear fractional differential and integro-differential equations have been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics, and economics. It draws a great application in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic models. Actually, fractional differential equations are considered as an alternative model to integer differential equations. Some works have done on the qualitative properties of solutions for these equations. For more details on this theory and on its applications, we suggest the reader to refer [4, 9–12, 16–19].

Inspired by the work [22], we study the existence of solutions for boundary value problem for implicit fractional differential equations with impulsive conditions of the form

$$\begin{aligned} {}^cD^{\alpha}x(t) &= f(t, x(t), {}^cD^{\alpha}x(t)), \quad t \in J', \alpha \in (1,2). \\ \Delta x(t_k) &= y_k,\ \Delta x'(t_k) = \bar{y}_k,\ y_k, \bar{y}_k \in \mathbb{R}\ k = 1,2,...,m \\ x(0) &= 0,\ x'(1) = 0 \end{aligned} \tag{1.1}$$

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where ${}^cD^\alpha$ is the Caputo fractional derivative, $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function.

In this paper is organized as follows. Section 2 has definitions and elementary results of the fractional calculus. In section 3, boundary value problem for implicit impulsive fractional differential equations is attained and proved the theorems on the existence and uniqueness of a solution to the problem (1.1). In section 4, an illustrative example is provided in support of the results of a problem (1.1).

2 Preliminaries

In this section, we recall some basic theorems, definitions and preliminary facts need to problem (1.1).

Define $PC(J,\mathbb{R}) = \{x : J \to \mathbb{R} : x \in C(t_k, t_{k+1}], \mathbb{R},\ k = 0,1,...,m$ and there exist $x(t_k^-)\&x(t_k^+), k = 1,2,...,m$ with $x(t_k^- = x(t_k)\}$ and we define the norm $||x||_{PC} = \sup\{|x(t)| : t \in J\}$. $PC^{'}(J,\mathbb{R})$ is also a Banach space, denote $PC^{'}(J,\mathbb{R}) = \{x \in PC(J,\mathbb{R}) : \dot{x} \in PC(J,\mathbb{R})\}$ and set $||x||_{PC'} = ||x||_{PC} + ||\dot{x}||_{PC}$.

Definition 2.1. *The fractional order integral of the function $h \in L^1([0,T],\mathbb{R}_+)$ of order $\alpha \in \mathbb{R}_+$ is defined by*

$$I^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s) ds,$$

where Γ is the Euler's gamma function defined by $\Gamma(\alpha) = \int_0^\infty t^{\alpha-1} e^{-t} dt, \alpha > 0$

Definition 2.2. *For a function $h \in AC^n(J)$, the Caputo's fractional-order derivative of order α is defined by*

$$({}^cD_0^\alpha)(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} h^{(n)}(s) ds,$$

where $n = [\alpha] + 1$ and $[\alpha]$ denotes the integer part of the real number α.

Definition 2.3. *A function $x \in PC^{'}(J,\mathbb{R})$ is said to be a solution of the problem (1), if $x(t) = x_k(t)$ for $t \in (t_k, t_{k+1})$ and $x_k \in C([0 = t_0 < t_1 < ... < t_m < t_{m+1} = T], \mathbb{R})$ satisfies ${}^cD^\alpha x_k(t) = f(t, x_k(t), {}^cD^\alpha x_k(t))$, almost everywhere on $(0, t_{k+1})$ with the restriction of $x_k(t)$ on $[0, t_k)$ is just $x_{k-1}(t)$ amd the conditions $\Delta x(t_k) = y_k$, $\Delta x'(t_k) = \bar{y}_k$, $y_k, \bar{y}_k \in \mathbb{R}$ $k = 1,2,...,m$ with $x(0) = 0$, $x'(1) = 0$.*

Lemma 2.1 (Lemma 9,[7] and lemma 4.1 [22])**.** *Let $\alpha \in (1,2)$. Then the problem*

$$\begin{aligned} {}^cD^\alpha x(t) &= f(t, x(t), {}^cD^\alpha x(t)), \quad t \in J^{'}, \\ \Delta x(t_k) &= y_k,\ \Delta x'(t_k) = \bar{y}_k,\ y_k, \bar{y}_k \in \mathbb{R}\ k = 1,2,...,m \\ x(0) &= 0,\ x'(1) = 0 \end{aligned}$$

is equivalent to the problem

$$\begin{aligned} (x)(t) &= I^\alpha g(t) + \sum_{i=1}^k \bar{y}_i (t - t_i) + \sum_{i=1}^k y_i \\ &\quad - \left(\frac{1}{\Gamma(\alpha-1)} \int_0^1 (1-t)^{\alpha-2)} g(t) dt + \sum_{k=1}^m \bar{y}_k \right) t \text{ for } t \in (t_k, t_{k+1}], k = 0,1,2,...m \end{aligned}$$

where $g \in PC(J,\mathbb{R})$ satisfies the functional equation $g(t) = f(t,(t), g(t))$.

Theorem 2.1 ([22],Theorem 2.12)**.** *Let X be a Banach space and $\psi \subset PC(J,X)$. If the following conditions are satisfied a) ψ is uniformly bounded subset of $PC(J,X)$. b) ψ is equicontinuous in $(t_k, t_{k+1}), k = 0,1,...,m$. c) $\psi(t) = \{x(t) : x \in \psi, t \in J\}$, $\psi(t_k^-) = \{x(t_k^-) : x \in \psi\}$ and $\psi(t_k^+) = \{x(t_k^+) : x \in \psi\}$ is relatively compact subset of X, then ψ is a relatively compact subset of $PC(J,X)$.*

Theorem 2.2 (Krasnoselskii's theorem)**.** *. Let $\mathbb{H}$ be a closed convex and nonempty subset of a Banach space X. If A, B be the operators such that a) $Ax + By \in \mathbb{H}$ whenever $x, y \in \mathbb{H}$, b) A is compact and continuous, c) B is a contraction mapping, then there exists a $z \in A$ such that $z = Az + Bz$.*

3 Main Results

In this section, we present and prove the main results of this paper. In order to prove the existence and uniqueness results we need the following assumptions :

(A_1) The function $f : J \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function.

(A_2) There exists a postive constant $L_1 > 0$ and $0 < L_2 < 1$ such that

$$|f(t,u,v) - f(t,\bar{u},\bar{v})| \le L_1|u - \bar{u}| + L_2|v - \bar{v}| \quad \forall t \in J,\ u,v,\bar{u},\bar{v} \in \mathbb{R}$$

(A_3) There exist $m, n \in C(J, \mathbb{R}_+)$ with $l^* = \sup_{t\in J} l(t) < 1$ such that

$$|f(t,u,w)| \le m(t) + n(t)|u| + l(t)|w| \quad \text{for} \quad t \in J \ \text{and} \ u,w \in \mathbb{R}$$

where $l \in L^{\frac{1}{\sigma}}(J,\mathbb{R})$ and $\sigma \in (0, \alpha - 1)$.

Theorem 3.3. *Assume* $(A_1) - (A_2)$. *If*

$$\frac{L_1 T^\alpha}{2(1-L_2)\Gamma(1+\alpha)} < 1$$

then there exists a unique solution for BVPs (1.1) *on* J.

Proof. Set

$$B_r = \{x \in PC^{'}(J,\mathbb{R}) : ||x||_{PC'} \le r\},$$

$$r \ge 2\left[\frac{1+\alpha}{\Gamma(1+\alpha)}L_3 + \sum_{i=1}^{m}|\bar{y}_i| + 2\sum_{i=1}^{m}|y_i|\right]$$

and

$$L_3 = sup_{t\in J}|f(t,0,0)|.$$

Define an operator $F : B_r \to PC^{'}(J,\mathbb{R})$ by

$$(Fx)(t) = I^\alpha g(t) + \sum_{i=1}^{k}\bar{y}_i(t-t_i) + \sum_{i=1}^{k} y_i - \left(\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-t)^{\alpha-2)}g(t)dt + \sum_{k=1}^{m}\bar{y}_k\right)t$$

where $g(t) = f(t, y(t), g(t))$ for $t \in (t_k, t_{k+1}], k = 0,1,2,...m$. where $g \in PC(J,\mathbb{R})$ satisfies the functional equation.

First can be checked that $FB_r \subset B_r$. For $x \in B_r, t \in J^{'}$,

$$\begin{aligned}|(Fx)(t)| &= \left|I^\alpha g(t) + \sum_{i=1}^{k}\bar{y}_i(t-t_i) + \sum_{i=1}^{k} y_i - \left(\frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-t)^{\alpha-2)}g(t)dt + \sum_{k=1}^{m}\bar{y}_k\right)t\right| \\ &\le \frac{1+\alpha}{\Gamma(1+\alpha)}L_3 + \sum_{i=1}^{m}|\bar{y}_i| + 2\sum_{i=1}^{m}|y_i| \\ &\le r\end{aligned}$$

Second can be checked that F is a contraction mapping. we have that

$$|(Fu_1)(t) - (Fu_2)(t)| = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{(\alpha-1)}|g(t)-h(t)|dt - \frac{1}{\Gamma(\alpha-1)}\int_0^1 (1-t)^{(\alpha-2)}|g(t)-h(t)|dt \quad (1.2)$$

By (A_2)

$$\begin{aligned}|g(t) - h(t)| &= |f(t,u_1(t),g(t)) - f(t,u_2(t),h(t))| \\ &\le L_1|u_1(t) - u_2(t)| + L_2|g(t) - h(t)| \\ &\le \frac{L_1|u_1(t) - u_2(t)|}{1-L_2}\end{aligned}$$

Therefore equation (1.2) implies

$$\begin{aligned}|(Fu_1)(t)-(Fu_2)(t)| &\leq \frac{L_1}{(1-L_2)\Gamma(\alpha)}\int_0^t (t-s)^{(\alpha-1)}|u_1(t)-u_2(t)|dt\\ &\leq \frac{L_1T^\alpha}{(1-L_2)\Gamma(1+\alpha)}||u_1-u_2||_{PC}\\ &\leq \frac{1}{2}||u_1-u_2||_{PC}\end{aligned}$$

and

$$||Fu_1-Fu_2||_{PC}\leq \frac{1}{2}||u_1-u_2||_{PC}.$$

Thus, the conclusion of theorem follows by the contraction mapping principle, F has a unique fixed point which is a unique solution of problem (1.1). □

Theorem 3.4. *Assume $(A_1)-(A_3)$ holds. Then the problem (1.1) has at least one solution on J.*

Proof. Let us choose

$$r\geq \frac{||l||_{L^{\frac{1}{\sigma}}}(J)}{\Gamma(\alpha)\left(\frac{\alpha-\sigma}{1-\sigma}\right)^{1-\sigma}}+\frac{||l||_{L^{\frac{1}{\sigma}}}(J)}{\Gamma(\alpha-1)\left(\frac{\alpha-\sigma-1}{1-\sigma}\right)^{1-\sigma}}+2\sum_{i=1}^{m}|\bar{y}_i|+\sum_{i=1}^{m}|y_i|$$

and define the operator ϕ and χ on B_r as

$$(\phi x)(t)=I^\alpha g(t)-\left(\frac{1}{\Gamma(\alpha-1)}\int_0^1(1-t)^{\alpha-2}g(t)dt\right)t$$

$$(\chi x)(t)=\sum_{i=1}^{k}\bar{y}_i(t-t_i)+\sum_{i=1}^{k}y_i-\sum_{i=1}^{m}\bar{y}_it$$

For any $\lambda_1,\lambda_2\in B_r$, one can show $\phi\lambda_1+\chi\lambda_2\in B_r$. It is obviously that ϕ is a contraction with the constant zero.

On the otherhand, the continuity of f implies that the operator ϕ is continuous.

Also, ϕ is uniformly bounded on B_r.

$$||\phi x||_{PC}\leq \frac{||l||_{L^{\frac{1}{\sigma}}}(J)}{\Gamma(\alpha)\left(\frac{\alpha-\sigma}{1-\sigma}\right)^{1-\sigma}}+\frac{||l||_{L^{\frac{1}{\sigma}}}(J)}{\Gamma(\alpha-1)\left(\frac{\alpha-\sigma-1}{1-\sigma}\right)^{1-\sigma}}\leq r.$$

Next, one can prove the compactness of the operator ϕ on B_r and for any $t_k<\lambda_1<\lambda_2<t_{k+1}$ via conclude that $\phi: C(J,\mathbb{R})\to C(J,\mathbb{R})$ Arzela-Ascoli theorem again.

Thus all the assumption of krasnoselskii's fixed point theorem are satisfied, which implies that the problem (1.1) has at least one solution on J. □

4 Example

Consider the following boundary value problem for implicit fractional differential equations with impulsive conditions

$$\begin{aligned}{}^cD^{\frac{3}{2}}x(t) &= \frac{1}{2e^t(1+|x(t)|+|{}^cD^{\frac{3}{2}}x(t)|},\\ \Delta x(1/4) &= y_1,\ \Delta x'(1/4)=\bar{y}_1,\\ x(0) &= 0,\ x'(1)=0\end{aligned}$$

Set $f(t,u,v)=\frac{1}{2e^t(1+|u|+|v|}, t\in[0,1],\ u,v\in\mathbb{R}$. Clearly, the function f is jointly continuous. For any $u,v,\bar{u},\bar{v}\in\mathbb{R}$

$$|f(t,u,v)-f(t,\bar{u},\bar{v})|\leq\frac{1}{4e}(|u-\bar{u}|+|v-\bar{v}|)$$

Set $L_1 = L_2 = \frac{1}{4e}, \alpha = \frac{3}{2}$ and

$$\frac{L_1 T^\alpha}{2(1-L_2)\Gamma(1+\alpha)} = \frac{1}{2(4e-1)\Gamma(\frac{5}{2})} < 1$$

Thus all the assumptions of theorem (1.1) are satisfied. Thus the problem (1.1) as a unique solution.

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Received: January 22, 2017; *Accepted*: May 23, 2017

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