



Lucky χ -polynomial of graphs of order 5

Johan Kok

Abstract

The concept of Lucky k -polynomials was recently introduced for null and complete split graphs. This paper extends on the introductory work and presents Lucky χ -polynomials ($k = \chi(G)$) for graphs of order 5. The methodical work done for graphs of order 5 serves mainly to set out the fundamental method to be used for all other classes of graphs. Finally, further problems for research related to this concept are presented.

Keywords

Chromatic completion number, chromatic completion graph, chromatic completion edge, bad edge, Lucky k -polynomial, Lucky χ -polynomial.

AMS Subject Classification

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1. Introduction

For general notation and concepts in graphs see [1,2,7]. Unless stated otherwise, all graphs will be finite and simple graphs. The set of vertices and the set of edges of a graph G are denoted by, $V(G)$ and $E(G)$ respectively. The number of vertices is called the order of G say, n and the number of edges is called the size of G denoted by, $\varepsilon(G)$. If G has order $n \geq 1$ and has no edges ($\varepsilon(G) = 0$) then G is called a null graph denoted by, \mathfrak{N}_n .

For a set of distinct colours $\mathcal{C} = \{c_1, c_2, c_3, \dots, c_\ell\}$ a vertex colouring of a graph G is an assignment $\varphi : V(G) \mapsto \mathcal{C}$. A vertex colouring is said to be a *proper vertex colouring* of

a graph G if no two distinct adjacent vertices have the same colour. The cardinality of a minimum set of distinct colours in a proper vertex colouring of G is called the *chromatic number* of G and is denoted $\chi(G)$. We call such a colouring a χ -colouring or a *chromatic colouring* of G . A chromatic colouring of G is denoted by $\varphi_\chi(G)$. Generally a graph G of order n is k -colourable for $\chi(G) \leq k \leq n$. The number of times a colour c_i is allocated to vertices of a graph G is denoted by $\theta_G(c_i)$ or if the context is clear simply, $\theta(c_i)$.

Generally the set, $c(V(G)) \subseteq \mathcal{C}$. A set $\{c_i \in \mathcal{C} : c(v) = c_i$ for at least one $v \in V(G)\}$ is called a colour class of the colouring of G . If \mathcal{C} is the chromatic set it can be agreed that $c(G)$ means set $c(V(G))$ hence, $c(G) \Rightarrow \mathcal{C}$ and $|c(G)| = |\mathcal{C}|$. For the set of vertices $X \subseteq V(G)$, the subgraph induced by X is denoted by, $\langle X \rangle$. The colouring of $\langle X \rangle$ permitted by $\varphi : V(G) \mapsto \mathcal{C}$ is denoted by, $c(\langle X \rangle)$.

In this paper, Section 2 deals with the introduction to Lucky χ -polynomials ($k = \chi(G)$). Section 3 presents Lucky χ -polynomials of all graphs of order 5. Section 4 concludes the paper and presents problems for further research.

2. Lucky χ -Polynomials

In a proper colouring of G all edges are good i.e. $uv \Leftrightarrow c(u) \neq c(v)$. For any proper colouring $\varphi(G)$ of a graph G the addition of all good edges, if any, is called the chromatic completion of G in respect of $\varphi(G)$. The additional edges are

called *chromatic completion edges*. The set of such chromatic completion edges is denoted by, $E_\varphi(G)$. The resultant graph G_φ is called a *chromatic completion graph* of G . See [3].

The *chromatic completion number* of a graph G denoted by, $\zeta(G)$ is the maximum number of good edges that can be added to G over all chromatic colourings (χ -colourings). Hence, $\zeta(G) = \max\{|E_\chi(G)| : \text{over all } \varphi_\chi(G)\}$.

A χ -colouring which yields $\zeta(G)$ is called a *Lucky χ -colouring* or simply, a *Lucky colouring*¹ and is denoted by, $\varphi_{\mathcal{L}}(G)$. The resultant graph $G_{\mathcal{L}}$ is called a *minimal chromatic completion graph* of G . It is trivially true that $G \subseteq G_{\mathcal{L}}$. Furthermore, the graph induced by the set of completion edges, $\langle E_\chi \rangle$ is a subgraph of the complement graph, \overline{G} . See [3] for an introduction to chromatic completion of a graph. Also see [4] for the notion of stability in respect of chromatic completion.

In an improper colouring an edge uv for which, $c(u) = c(v)$ is called a *bad edge*. See [6] for an introduction to defect colourings of graphs. It is observed that the number of edges of \overline{G} which are omitted from E_χ is the minimum number of bad edges in a *bad chromatic completion* of a graph G .

In [5] the introduction to Lucky k -polynomial for null graphs was presented. We recall the next definition, Theorem 2.1 and Lemma 2.1 from [3].

Definition 2.1. [3] For two positive integers $2 \leq \ell \leq n$ the division, $\frac{n}{\ell} = \lfloor \frac{n}{\ell} \rfloor + r$, with r some positive integer and $\ell > r \geq 0$. Hence, $n = \underbrace{\lfloor \frac{n}{\ell} \rfloor + \lfloor \frac{n}{\ell} \rfloor + \dots + \lfloor \frac{n}{\ell} \rfloor}_{(\ell-r)\text{-terms}} + \underbrace{\lceil \frac{n}{\ell} \rceil + \lceil \frac{n}{\ell} \rceil + \dots + \lceil \frac{n}{\ell} \rceil}_{(r \geq 0)\text{-terms}}$.

This specific ℓ -partition, $(\underbrace{\lfloor \frac{n}{\ell} \rfloor, \lfloor \frac{n}{\ell} \rfloor, \dots, \lfloor \frac{n}{\ell} \rfloor}_{(\ell-r)\text{-terms}}, \underbrace{\lceil \frac{n}{\ell} \rceil, \lceil \frac{n}{\ell} \rceil, \dots, \lceil \frac{n}{\ell} \rceil}_{(r \geq 0)\text{-terms}})$ is called a *completion ℓ -partition* of n .

To ease the formulation of the next result let, $t_i = \lfloor \frac{n}{\ell} \rfloor$, $i = 1, 2, 3, \dots, (\ell - r)$ and $t'_j = \lceil \frac{n}{\ell} \rceil$, $j = 1, 2, 3, \dots, r$. Call, $\mathcal{L} = \sum_{i=1}^{\ell-r-1} \prod_{k=i+1}^{\ell-r} t_{ik} + \sum_{i=1}^{\ell-r} \prod_{j=1}^r t_{ij} + \sum_{j=1}^{r-1} \prod_{k=j+1}^r t'_j t'_{jk}$, the *ℓ -completion sum-product* of n .

Theorem 2.1. (Lucky's Theorem)²[3] For a positive integer $n \geq 2$ and $2 \leq p \leq n$ let integers, $1 \leq a_1, a_2, a_3, \dots, a_{p-r}, a'_1, a'_2, a'_3, \dots, a'_r \leq n - 1$ be such that $n = \sum_{i=1}^{p-r} a_i + \sum_{j=1}^r a'_j$ then, the ℓ -completion sum-product

¹Note that for many graphs a Lucky colouring is equivalent to an equitable χ -colouring (or k -colouring). Since it is not generally the case the alias is meant to associate the paper with Lucky's Theorem and the notion of chromatic completion in [3].

²Dedicated to late Lucky Mahlalela who was a disabled, freelance traffic pointsman in the City of Tshwane. Sadly he was brutally murdered.

$$\mathcal{L} = \max\left\{ \sum_{i=1}^{p-r-1} \prod_{k=i+1}^{p-r} a_i a_k + \sum_{i=1}^{p-r} \prod_{j=1}^r a_i a'_j + \sum_{j=1}^{r-1} \prod_{k=j+1}^r a'_j a'_k \right\} \text{ over all possible, } n = \sum_{i=1}^{p-r} a_i + \sum_{j=1}^r a'_j.$$

From Theorem 2.1 the next lemma followed which prescribes a particular colouring convention.

Lemma 2.1. [3] If a subset of m vertices say, $X \subseteq V(G)$ can be chromatically coloured by t distinct colours then allocate colours as follows:

1. For t vertex subsets each of cardinality $s = \lfloor \frac{m}{t} \rfloor$ allocate a distinct colour followed by:
2. Colour one additional vertex (from the $r \geq 0$ which are uncoloured), each in a distinct colour,

if the graph structure permits such colour allocation. This chromatic colouring permits the maximum number of chromatic completion edges between the vertices in X amongst all possible chromatic colourings of X .

It is known from [3] that for a graph which does not permit a colour allocation as prescribed in Lemma 2.1, an optimal near-completion ℓ -partition of the vertex set exists which yields the maximum chromatic completion edges. Note that the colouring in accordance with Lemma 2.1 is essentially a special case of an optimal near-completion ℓ -partition of the vertex set $V(G)$ in that, the vertex partition yielding the colour clusters complies with Theorem 2.1. If $\varphi(G)$ assigns k colours in accordance with an optimal near-completion k -partition of the vertex set (inclusive of Lemma 2.1) the colouring is called a *Lucky k -colouring*. The next result follows immediately.

Theorem 2.2. (a) If $\varphi(G)$ assigned k colours in accordance with an optimal near-completion k -partition of the vertex set (inclusive of Lemma 2.1) then the resultant graph G_φ has $\chi(G_\varphi) = k$.

(b) G_φ is a super graph (order n) of G such that, $\varepsilon(G_\varphi) = \max\{\varepsilon(H) : \text{over all super graphs } H \text{ (order } n) \text{ of } G \text{ such that, } \chi(H) = k\}$.

Proof. (a) Assume it is possible to replace any colour cluster with the colour of another cluster in G_φ to obtain a proper $(k - 1)$ -colouring, then at least one bad edge arises which is a contradiction. Hence, since the number of chromatic completion edges is a maximum over all possible proper k -colourings the contradiction implies that G_φ has $\chi(G_\varphi) = k$. (b) A direct consequence of Definition 2.1, Lemma 2.1 or an optimal near-completion k -partition read together with the definition of $\zeta(G)$. \square

Henceforth, a chromatic colouring ($k = \chi(G)$) in accordance with either Lemma 2.1 or an optimal near-completion χ -partition will be called a *Lucky χ -colouring* or simply a *Lucky colouring* denoted by, $\varphi_{\mathcal{L}}(G)$.



Also, for a set of colours \mathcal{C} , $|\mathcal{C}| = \lambda \geq \chi(G) \geq 2$ a graph G of order n can always be coloured properly in $\mathcal{P}_G(\lambda, n)$ distinct ways. The polynomial $\mathcal{P}_G(\lambda, n)$ is called the chromatic polynomial of G .

From [5] it is known that for $\chi(G) \leq n \leq \lambda$ colours the number of distinct proper k -colourings, $\chi(G) \leq k \leq n$ in accordance to either, Lemma 2.1 or an optimal near-completion k -partition is determined by a polynomial, called the *Lucky k -polynomial*, $\mathcal{L}_G(\lambda, k)$.

Let the product be abbreviated to $\lambda^{(n)} = \lambda(\lambda - 1)(\lambda - 2) \cdots (\lambda - n + 1)$. Hence, if $\lambda = n$ then, $\lambda^{(n)} = \lambda^{(\lambda)} = \lambda!$.

Theorem 2.3. Any 2-colourable graph G has $\mathcal{L}_G(\lambda, \chi) = \lambda(\lambda - 1)$.

Proof. Since G is connected by assumption and $\chi(G) = 2$, it implies that G is 2-chromatic. Therefore, any optimal near-completion 2-partition of $V(G)$ say, $\{X_1, X_2\}$ is unique. Hence, the result. \square

In respect of the independence number, $\alpha(G)$ the next result follows.

Theorem 2.4. For any graph G , $\alpha(G) \geq \max\{\theta(c_i) : i = 1, 2, 3, \dots, \chi(G) \text{ overall Lucky } \chi\text{-colourings}\} \geq \max\{\theta(c_j) : j = 1, 2, 3, \dots, \ell \text{ overall Lucky } \ell\text{-colourings}\}$.

Proof. It follows easily that for $\chi(G) \leq k_1 \leq k_2 \leq \lambda$ and the optimal near-completion partition (by Lucky's theorem or otherwise), corresponding to a k_1 - and a k_2 -colouring yields vertex partitions say, $\{X_1, X_2, X_3, \dots, X_{k_1}\}$ and $\{Y_1, Y_2, Y_3, \dots, Y_{k_2}\}$ such that, $\max\{|X_i| : i = 1, 2, 3, \dots, k_1\} \geq \max\{|Y_j| : j = 1, 2, 3, \dots, k_2\}$. Therefore, any such χ -colouring yields a colour cluster which corresponds to an absolute maximum independent set of G permitted by a proper colouring. Therefore the result. \square

In [5] a complete split graph was defined as a split graph such that each vertex in the independent set is adjacent to all the vertices of the clique (the clique is a smallest clique which permits a maximum independent set). Note that a complete graph K_n is also a complete split graph i.e. any subset of $n - 1$ vertices induces a smallest clique and the corresponding 1-element subset is a maximum independent set.

Theorem 2.5. If G is a complete split graph then, $\alpha(G) = \max\{\theta(c_i) : i = 1, 2, 3, \dots, \chi(G) \text{ over all Lucky } \chi\text{-colourings}\}$.

Proof. If G is a complete split graph the result follows by similar reasoning found in the proof of Theorem 2.5 read together with Lemma 3.2 in [5]. \square

3. Lucky χ -Polynomials of Graphs of Order 5

Thus far connected graphs were considered. Now the requirement is relaxed and up to isomorphism, all 34 graphs of

order 5 will be considered. Following from [5] we have that for $G = \mathfrak{N}_5$ or K_5 , $\mathcal{L}_{\mathfrak{N}_5}(\lambda, 1) = \lambda$ and $\mathcal{L}_{K_5}(\lambda, 5) = \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)(\lambda - 4)$ for $\chi(G) \leq \lambda$.

The path graph $P_1 \cong K_1$. A path graph (simply, path) of order $n \geq 2$ has vertices which are consecutively labeled v_i , $i = 1, 2, 3, \dots, n$ with the edge set $\{v_i v_{i+1} : 1 \leq i \leq n - 1\}$. For path P_1 it follows easily that $\mathcal{L}_{P_1}(\lambda, \chi) = \mathcal{L}_{P_1}(\lambda, 1) = \lambda$. For 2-colourable graphs (therefore for paths of order $n \geq 2$) it follows from Theorem 2.4 that, $\mathcal{L}_{P_1}(\lambda, \chi) = \mathcal{L}_{P_1}(\lambda, 2) = \lambda(\lambda - 1)$, $i \geq 2$.

Theorem 2.3 also provides the result for the star $S_{1,4}$, the 4-pan (or banner) and a fork (or chair).

3.1 Bell partitions

We recursively obtain the bell partitions for the set $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$ as follows:

For the set $\{v_1\}$ we obtain the Bell partition, $\{\{v_1\}\}$.

For the set $\{v_1, v_2\}$ we obtain the Bell partitions, $\{\{v_1, v_2\}\}, \{\{v_1\}, \{v_2\}\}$.

For the set $\{v_1, v_2, v_3\}$ we obtain the Bell partitions,

$$\{\{v_1, v_2, v_3\}\}, \{\{v_1, v_2\}, \{v_3\}\}, \{\{v_1\}, \{v_2\}, \{v_3\}\}, \{\{v_1, v_3\}, \{v_2\}\}, \{\{v_1\}, \{v_2, v_3\}\}$$

For the set $\{v_1, v_2, v_3, v_4\}$ we obtain the Bell partitions, $\{\{v_1, v_2, v_3, v_4\}\}, \{\{v_1, v_2, v_3\}, \{v_4\}\}, \{\{v_1, v_2\}, \{v_3\}, \{v_4\}\}, \{\{v_1, v_2, v_4\}, \{v_3\}\}, \{\{v_1, v_2\}, \{v_3, v_4\}\}, \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}\}, \{\{v_1, v_4\}, \{v_2\}, \{v_3\}\}, \{\{v_1\}, \{v_2, v_4\}, \{v_3\}\}, \{\{v_1\}, \{v_2\}, \{v_3, v_4\}\}, \{\{v_1, v_3\}, \{v_2\}, \{v_4\}\}, \{\{v_1, v_3, v_4\}, \{v_2\}\}, \{\{v_1, v_3\}, \{v_2, v_4\}\}, \{\{v_1\}, \{v_2, v_3\}, \{v_4\}\}, \{\{v_1, v_4\}, \{v_2, v_3\}\}, \{\{v_1\}, \{v_2, v_3, v_4\}\}$.

For the set $\{v_1, v_2, v_3, v_4, v_5\}$ we obtain the Bell partitions,

$$\begin{aligned} & \{\{v_1, v_2, v_3, v_4, v_5\}\}, \{\{v_1, v_2, v_3, v_4\}, \{v_5\}\}, \\ & \{\{v_1, v_2, v_3, v_5\}, \{v_4\}\}, \{\{v_1, v_2, v_3\}, \{v_4, v_5\}\}, \\ & \{\{v_1, v_2, v_3\}, \{v_4\}, \{v_5\}\}, \{\{v_1, v_2, v_5\}, \{v_3\}, \{v_4\}\}, \\ & \{\{v_1, v_2\}, \{v_3, v_5\}, \{v_4\}\}, \{\{v_1, v_2\}, \{v_3\}, \{v_4, v_5\}\}, \\ & \{\{v_1, v_2\}, \{v_3\}, \{v_4\}, \{v_5\}\}, \{\{v_1, v_2, v_4, v_5\}, \{v_3\}\}, \\ & \{\{v_1, v_2, v_4\}, \{v_3, v_5\}\}, \{\{v_1, v_2, v_4\}, \{v_3\}, \{v_5\}\}, \\ & \{\{v_1, v_2, v_5\}, \{v_3, v_4\}\}, \{\{v_1, v_2\}, \{v_3, v_4, v_5\}\}, \\ & \{\{v_1, v_2\}, \{v_3, v_4\}, \{v_5\}\}, \{\{v_1, v_5\}, \{v_2\}, \{v_3\}, \{v_4\}\}, \\ & \{\{v_1\}, \{v_2, v_5\}, \{v_3\}, \{v_4\}\}, \{\{v_1\}, \{v_2\}, \{v_3, v_5\}, \{v_4\}\}, \\ & \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4, v_5\}\}, \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}\}, \\ & \{\{v_1, v_4, v_5\}, \{v_2\}, \{v_3\}\}, \{\{v_1, v_4\}, \{v_2, v_5\}, \{v_3\}\}, \\ & \{\{v_1, v_4\}, \{v_2\}, \{v_3, v_5\}\}, \{\{v_1, v_4\}, \{v_2\}, \{v_3\}, \{v_5\}\}, \\ & \{\{v_1, v_5\}, \{v_2, v_4\}, \{v_3\}\}, \{\{v_1\}, \{v_2, v_4, v_5\}, \{v_3\}\}, \\ & \{\{v_1\}, \{v_2, v_4\}, \{v_3, v_5\}\}, \{\{v_1\}, \{v_2, v_4\}, \{v_3\}, \{v_5\}\}, \\ & \{\{v_1, v_5\}, \{v_2\}, \{v_3, v_4\}\}, \{\{v_1\}, \{v_2, v_5\}, \{v_3, v_4\}\}, \\ & \{\{v_1\}, \{v_2\}, \{v_3, v_4, v_5\}\}, \{\{v_1\}, \{v_2\}, \{v_3, v_4\}, \{v_5\}\}, \\ & \{\{v_1, v_3, v_5\}, \{v_2\}, \{v_4\}\}, \{\{v_1, v_3\}, \{v_2, v_5\}, \{v_4\}\}, \\ & \{\{v_1, v_3\}, \{v_2\}, \{v_4, v_5\}\}, \{\{v_1, v_3\}, \{v_2\}, \{v_4\}, \{v_5\}\}, \\ & \{\{v_1, v_3, v_4, v_5\}, \{v_2\}\}, \{\{v_1, v_3, v_4\}, \{v_2, v_5\}\}, \\ & \{\{v_1, v_3, v_4\}, \{v_2\}, \{v_5\}\}, \{\{v_1, v_3, v_5\}, \{v_2, v_4\}\}, \end{aligned}$$



$\{\{v_1, v_3\}, \{v_2, v_4, v_5\}\}, \{\{v_1, v_3\}, \{v_2, v_4\}, \{v_5\}\},$
 $\{\{v_1, v_5\}, \{v_2, v_3\}, \{v_4\}\}, \{\{v_1\}, \{v_2, v_3, v_5\}, \{v_4\}\},$
 $\{\{v_1\}, \{v_2, v_3\}, \{v_4, v_5\}\}, \{\{v_1\}, \{v_2, v_3\}, \{v_4\}, \{v_5\}\},$
 $\{\{v_1, v_4, v_5\}, \{v_2, v_3\}\}, \{\{v_1, v_4\}, \{v_2, v_3, v_5\}\},$
 $\{\{v_1, v_4\}, \{v_2, v_3\}, \{v_5\}\}, \{\{v_1, v_5\}, \{v_2, v_3, v_4\}\},$
 $\{\{v_1\}, \{v_2, v_3, v_4, v_5\}\}, \{\{v_1\}, \{v_2, v_3, v_4\}, \{v_5\}\}.$

3.2 Category 1: $\chi(G) = 2$

If adjacency within a graph permits then consider partitions of the form, $\{\{3\text{-element}\}, \{2\text{-element}\}\}$. Hence select,

$\{\{v_1, v_2, v_3\}, \{v_4, v_5\}\}, \{\{v_1, v_2, v_4\}, \{v_3, v_5\}\},$
 $\{\{v_1, v_2, v_5\}, \{v_3, v_4\}\}, \{\{v_1, v_2\}, \{v_3, v_4, v_5\}\},$
 $\{\{v_1, v_3, v_4\}, \{v_2, v_5\}\}, \{\{v_1, v_3, v_5\}, \{v_2, v_4\}\},$
 $\{\{v_1, v_3\}, \{v_2, v_4, v_5\}\}, \{\{v_1, v_4, v_5\}, \{v_2, v_3\}\},$
 $\{\{v_1, v_4\}, \{v_2, v_3, v_5\}\}, \{\{v_1, v_5\}, \{v_2, v_3, v_4\}\}.$

Without loss of generality consider the graph $G_1 = P_2 \cup 3K_1$ and assume P_2 is on vertices v_1, v_2 .

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence select,

$\{\{v_1, v_3, v_4\}, \{v_2, v_5\}\}, \{\{v_1, v_3, v_5\}, \{v_2, v_4\}\},$
 $\{\{v_1, v_3\}, \{v_2, v_4, v_5\}\}, \{\{v_1, v_4, v_5\}, \{v_2, v_3\}\},$
 $\{\{v_1, v_4\},$
 $\{v_2, v_3, v_5\}\}, \{\{v_1, v_5\}, \{v_2, v_3, v_4\}\}.$

Therefore, $\mathcal{L}_{G_1}(\lambda, \chi) = \mathcal{L}_{G_1}(\lambda, 2) = 6\lambda(\lambda - 1)$.

Without loss of generality consider the graph $G_2 = P_3 \cup 2K_1$ and assume P_3 is on vertices v_1, v_2, v_3 .

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence select,

$\{\{v_1, v_3, v_4\}, \{v_2, v_5\}\}, \{\{v_1, v_3, v_5\}, \{v_2, v_4\}\},$
 $\{\{v_1, v_3\}, \{v_2, v_4, v_5\}\}.$

Therefore, $\mathcal{L}_{G_2}(\lambda, \chi) = \mathcal{L}_{G_2}(\lambda, 2) = 3\lambda(\lambda - 1)$.

Without loss of generality consider the graph $G_3 = 2P_2 \cup K_1$ and assume first P_2 is on vertices v_1, v_2 and second P_2 is on vertices v_3, v_4 .

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence select,

$\{\{v_1, v_3, v_5\}, \{v_2, v_4\}\}, \{\{v_1, v_3\}, \{v_2, v_4, v_5\}\},$
 $\{\{v_1, v_4, v_5\}, \{v_2, v_3\}\}, \{\{v_1, v_4\}, \{v_2, v_3, v_5\}\}.$

Therefore, $\mathcal{L}_{G_3}(\lambda, \chi) = \mathcal{L}_{G_3}(\lambda, 2) = 4\lambda(\lambda - 1)$.

Without loss of generality consider the graph $G_4 = S_{1,3} \cup K_1$ and assume that the star has central vertex v_1 with pendant vertices v_2, v_3, v_4 .

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence

select,

$\{\{v_1, v_5\}, \{v_2, v_3, v_4\}\}.$

Therefore, $\mathcal{L}_{G_4}(\lambda, \chi) = \mathcal{L}_{G_4}(\lambda, 2) = \lambda(\lambda - 1)$.

Without loss of generality consider the graph $G_5 = P_3 \cup P_2$ and assume that P_3 has vertices v_1, v_2, v_3 .

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence select,

$\{\{v_1, v_3, v_4\}, \{v_2, v_5\}\}, \{\{v_1, v_3, v_5\}, \{v_2, v_4\}\}.$

Therefore, $\mathcal{L}_{G_5}(\lambda, \chi) = \mathcal{L}_{G_5}(\lambda, 2) = 2\lambda(\lambda - 1)$.

Without loss of generality consider the graph $G_6 = P_4 \cup K_1$ and assume that P_4 has vertices v_1, v_2, v_3, v_4 .

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence select,

$\{\{v_1, v_3, v_5\}, \{v_2, v_4\}\}, \{\{v_1, v_3\}, \{v_2, v_4, v_5\}\}.$

Therefore, $\mathcal{L}_{G_6}(\lambda, \chi) = \mathcal{L}_{G_6}(\lambda, 2) = 2\lambda(\lambda - 1)$.

Without loss of generality consider the graph $G_7 = C_4 \cup K_1$ and assume that C_4 has vertices v_1, v_2, v_3, v_4 .

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence select,

$\{\{v_1, v_3, v_5\}, \{v_2, v_4\}\}, \{\{v_1, v_3\}, \{v_2, v_4, v_5\}\}.$

Therefore, $\mathcal{L}_{G_7}(\lambda, \chi) = \mathcal{L}_{G_7}(\lambda, 2) = 2\lambda(\lambda - 1)$.

Without loss of generality consider the cycle C_4 on vertices v_1, v_2, v_3, v_4 . Obtain G_8 by adding vertex v_5 with the edges v_2v_5, v_4v_5 .

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence select, $\{\{v_1, v_3, v_5\}, \{v_2, v_4\}\}.$

Therefore, $\mathcal{L}_{G_8}(\lambda, \chi) = \mathcal{L}_{G_8}(\lambda, 3) = \lambda(\lambda - 1)$.

Theorem 3.1. All graphs G of order 5 with $\chi(G) = 2$ except $S_{1,4}$, have equitable chromatic number equal to chromatic number.

Proof. The result is a direct consequence of the analysis above. \square

3.3 Category 2: $\chi(G) = 3$

If adjacency within a graph permits only consider partitions of the form, $\{\{2\text{-element}\}, \{2\text{-element}\}, \{1\text{-element}\}\}$. Hence select,

$\{\{v_1, v_2\}, \{v_3, v_5\}, \{v_4\}\}, \{\{v_1, v_2\}, \{v_3\}, \{v_4, v_5\}\},$
 $\{\{v_1, v_2\}, \{v_3, v_4\}, \{v_5\}\}, \{\{v_1, v_4\}, \{v_2, v_5\}, \{v_3\}\},$
 $\{\{v_1, v_4\}, \{v_2\}, \{v_3, v_5\}\}, \{\{v_1, v_5\}, \{v_2, v_4\}, \{v_3\}\},$
 $\{\{v_1\}, \{v_2, v_4\}, \{v_3, v_5\}\}, \{\{v_1, v_5\}, \{v_2\}, \{v_3, v_4\}\},$
 $\{\{v_1\}, \{v_2, v_5\}, \{v_3, v_4\}\}, \{\{v_1, v_3\}, \{v_2, v_5\}, \{v_4\}\},$



$$\begin{aligned} & \{\{v_1, v_3\}, \{v_2\}, \{v_4, v_5\}\}, \{\{v_1, v_3\}, \{v_2, v_4\}, \{v_5\}\}, \\ & \{\{v_1, v_5\}, \{v_2, v_3\}, \{v_4\}\}, \{\{v_1\}, \{v_2, v_3\}, \{v_4, v_5\}\}, \\ & \{\{v_1, v_4\}, \{v_2, v_3\}, \{v_5\}\}. \end{aligned}$$

The cycle graph $C_1 \cong K_1$ and $C_2 \cong P_2$. A cycle graph (simply, cycle) of order $n \geq 3$ has vertices which are consecutively labeled $v_i, i = 1, 2, 3, \dots, n$ with the edge set $\{v_i v_{i+1} : 1 \leq i \leq n-1\} \cup \{v_n v_1\}$. For cycles C_1, C_2, C_3 it follows easily that, $\mathcal{L}_{C_1}(\lambda, \chi) = \mathcal{L}_{C_1}(\lambda, 1) = \lambda$, $\mathcal{L}_{C_2}(\lambda, \chi) = \mathcal{L}_{C_2}(\lambda, 2) = \lambda(\lambda - 1)$ and $\mathcal{L}_{C_3}(\lambda, \chi) = \mathcal{L}_{C_3}(\lambda, 3) = \lambda(\lambda - 1)(\lambda - 2)$. Since all even cycles are connected 2-colourable graphs, Theorem 2.4 provides the result.

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence select,

$$\begin{aligned} & \{\{v_1, v_4\}, \{v_2, v_5\}, \{v_3\}\}, \{\{v_1, v_4\}, \{v_2\}, \{v_3, v_5\}\}, \\ & \{\{v_1\}, \{v_2, v_4\}, \{v_3, v_5\}\}, \{\{v_1, v_3\}, \{v_2, v_5\}, \{v_4\}\}, \\ & \{\{v_1, v_3\}, \{v_2, v_4\}, \{v_5\}\}. \end{aligned}$$

Therefore, $\mathcal{L}_{C_5}(\lambda, \chi) = \mathcal{L}_{C_5}(\lambda, 3) = 5\lambda(\lambda - 1)(\lambda - 2)$.

Without loss of generality consider the graph $G_9 = C_4 + K_1$ and assume that C_4 has vertices v_1, v_2, v_3, v_4 .

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence select, $\{\{v_1, v_3\}, \{v_2, v_4\}, \{v_5\}\}$.

Therefore, $\mathcal{L}_{G_9}(\lambda, \chi) = \mathcal{L}_{G_9}(\lambda, 3) = \lambda(\lambda - 1)(\lambda - 2)$.

Without loss of generality consider the graph $G_{10} = C_3 \cup 2K_1$ and assume that C_3 has vertices v_1, v_2, v_3 .

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence select,

$$\begin{aligned} & \{\{v_1, v_4\}, \{v_2, v_5\}, \{v_3\}\}, \{\{v_1, v_4\}, \{v_2\}, \{v_3, v_5\}\}, \\ & \{\{v_1, v_5\}, \{v_2, v_4\}, \{v_3\}\}, \{\{v_1\}, \{v_2, v_4\}, \{v_3, v_5\}\}, \\ & \{\{v_1, v_5\}, \{v_2\}, \{v_3, v_4\}\}, \{\{v_1\}, \{v_2, v_5\}, \{v_3, v_4\}\}. \end{aligned}$$

Therefore, $\mathcal{L}_{G_{10}}(\lambda, \chi) = \mathcal{L}_{G_{10}}(\lambda, 3) = 6\lambda(\lambda - 1)(\lambda - 2)$.

Without loss of generality consider the graph $G_{11} = (C_4 + K_1) - v_1 v_5$ and assume that C_4 has vertices v_1, v_2, v_3, v_4 .

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence select, $\{\{v_1, v_5\}, \{v_2, v_4\}, \{v_3\}\}, \{\{v_1, v_3\}, \{v_2, v_4\}, \{v_5\}\}$.

Therefore, $\mathcal{L}_{G_{11}}(\lambda, \chi) = \mathcal{L}_{G_{11}}(\lambda, 3) = 2\lambda(\lambda - 1)(\lambda - 2)$.

Without loss of generality consider the graph $G_{12} = P_4 + K_1$ (a 3-fan) and assume that P_4 has vertices v_1, v_2, v_3, v_4 .

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence select, $\{\{v_1, v_3\}, \{v_2, v_4\}, \{v_5\}\}$.

Therefore, $\mathcal{L}_{G_{12}}(\lambda, \chi) = \mathcal{L}_{G_{12}}(\lambda, 3) = \lambda(\lambda - 1)(\lambda - 2)$.

Without loss of generality consider the cycle C_3 on vertices v_1, v_2, v_3 and form graph H by adding a pendant vertex v_4 to v_1 . Now consider the graph $G_{14} = H \cup v_5$. Only consider partitions of the form,

$\{\{2\text{-element}\}, \{2\text{-element}\}, \{1\text{-element}\}\}$ as above.

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence select,

$$\begin{aligned} & \{\{v_1, v_5\}, \{v_2, v_4\}, \{v_3\}\}, \{\{v_1\}, \{v_2, v_4\}, \{v_3, v_5\}\}, \\ & \{\{v_1, v_5\}, \{v_2\}, \{v_3, v_4\}\}, \{\{v_1\}, \{v_2, v_5\}, \{v_3, v_4\}\}. \end{aligned}$$

Therefore, $\mathcal{L}_{G_{14}}(\lambda, \chi) = \mathcal{L}_{G_{14}}(\lambda, 3) = 4\lambda(\lambda - 1)(\lambda - 2)$.

Without loss of generality consider the wheel $W_{1,4}$ with central vertex v_1 . Let $G_{15} = W_{1,4} - \{v_2 v_5, v_3 v_4\}$ (or butterfly graph). Only consider partitions of the form,

$\{\{2\text{-element}\}, \{2\text{-element}\}, \{1\text{-element}\}\}$ as above.

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence select,

$$\{\{v_1\}, \{v_2, v_4\}, \{v_3, v_5\}\}, \{\{v_1\}, \{v_2, v_5\}, \{v_3, v_4\}\}.$$

Therefore, $\mathcal{L}_{G_{15}}(\lambda, \chi) = \mathcal{L}_{G_{15}}(\lambda, 3) = 2\lambda(\lambda - 1)(\lambda - 2)$.

Without loss of generality consider the complete graph K_4 on vertices v_1, v_2, v_3, v_4 and form graph H by adding a pendant vertex v_5 to v_1 . Now consider the graph $G_{16} = H - v_1 v_3$. Only consider partitions of the form,

$\{\{2\text{-element}\}, \{2\text{-element}\}, \{1\text{-element}\}\}$ as above.

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence select,

$$\{\{v_1, v_3\}, \{v_2, v_5\}, \{v_4\}\}, \{\{v_1, v_3\}, \{v_2\}, \{v_4, v_5\}\}.$$

Therefore, $\mathcal{L}_{G_{16}}(\lambda, \chi) = \mathcal{L}_{G_{16}}(\lambda, 3) = 2\lambda(\lambda - 1)(\lambda - 2)$.

Without loss of generality consider the complete graph K_4 on vertices v_1, v_2, v_3, v_4 and form graph H by adding a pendant vertex v_5 to v_1 . Now consider the graph $G_{17} = H - v_2 v_4$. Only consider partitions of the form,

$\{\{2\text{-element}\}, \{2\text{-element}\}, \{1\text{-element}\}\}$ as above.

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence select, $\{\{v_1\}, \{v_2, v_4\}, \{v_3, v_5\}\}$.

Therefore, $\mathcal{L}_{G_{17}}(\lambda, \chi) = \mathcal{L}_{G_{17}}(\lambda, 3) = \lambda(\lambda - 1)(\lambda - 2)$.

Without loss of generality consider the complete graph $G_{18} = C_3 \cup P_2$ with C_3 on vertices on vertices v_1, v_2, v_3 . Only consider partitions of the form,

$\{\{2\text{-element}\}, \{2\text{-element}\}, \{1\text{-element}\}\}$ as above.

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence



select,

$$\begin{aligned} & \{\{v_1, v_4\}, \{v_2, v_5\}, \{v_3\}\}, \{\{v_1, v_4\}, \{v_2\}, \{v_3, v_5\}\}, \\ & \{\{v_1, v_5\}, \{v_2, v_4\}, \{v_3\}\}, \{\{v_1\}, \{v_2, v_4\}, \{v_3, v_5\}\}, \\ & \{\{v_1, v_5\}, \{v_2\}, \{v_3, v_4\}\}, \{\{v_1\}, \{v_2, v_5\}, \{v_3, v_4\}\}. \end{aligned}$$

Therefore, $\mathcal{L}_{G_{18}}(\lambda, \chi) = \mathcal{L}_{G_{18}}(\lambda, 3) = 5\lambda(\lambda - 1)(\lambda - 2)$.

Without loss of generality consider the cycle C_3 on vertices v_1, v_2, v_3 . Obtain G_{19} by adding pendant vertex v_4 to v_1 and pendant vertex v_5 to v_2 . Only consider partitions of the form, $\{\{2\text{-element}\}, \{2\text{-element}\}, \{1\text{-element}\}\}$ as above.

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence select,

$$\begin{aligned} & \{\{v_1, v_5\}, \{v_2, v_4\}, \{v_3\}\}, \{\{v_1\}, \{v_2, v_4\}, \{v_3, v_5\}\}, \\ & \{\{v_1, v_5\}, \{v_2\}, \{v_3, v_4\}\}. \end{aligned}$$

Therefore, $\mathcal{L}_{G_{19}}(\lambda, \chi) = \mathcal{L}_{G_{19}}(\lambda, 3) = 3\lambda(\lambda - 1)(\lambda - 2)$.

Without loss of generality consider the cycle C_4 on vertices v_1, v_2, v_3, v_4 . Obtain G_{20} by adding vertex v_5 with the edges v_1v_5, v_2v_5 (or house). Only consider partitions of the form, $\{\{2\text{-element}\}, \{2\text{-element}\}, \{1\text{-element}\}\}$ as above.

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence select,

$$\begin{aligned} & \{\{v_1, v_3\}, \{v_2, v_4\}, \{v_5\}\}, \{\{v_1, v_3\}, \{v_4, v_5\}, \{v_2\}\}, \\ & \{\{v_1\}, \{v_2, v_4\}, \{v_3, v_5\}\}. \end{aligned}$$

Therefore, $\mathcal{L}_{G_{20}}(\lambda, \chi) = \mathcal{L}_{G_{20}}(\lambda, 3) = 3\lambda(\lambda - 1)(\lambda - 2)$.

Without loss of generality consider the path P_5 on vertices v_1, v_2, v_3, v_4, v_5 . Obtain G_{21} by adding edge v_3v_5 . Only consider partitions of the form, $\{\{2\text{-element}\}, \{2\text{-element}\}, \{1\text{-element}\}\}$ as above.

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence select,

$$\begin{aligned} & \{\{v_1, v_4\}, \{v_2, v_5\}, \{v_3\}\}, \{\{v_1, v_5\}, \{v_2, v_4\}, \{v_3\}\}, \\ & \{\{v_1, v_3\}, \{v_2, v_5\}, \{v_4\}\}, \{\{v_1, v_3\}, \{v_2, v_4\}, \{v_5\}\}. \end{aligned}$$

Therefore, $\mathcal{L}_{G_{21}}(\lambda, \chi) = \mathcal{L}_{G_{21}}(\lambda, 3) = 4\lambda(\lambda - 1)(\lambda - 2)$.

Without loss of generality consider the complete graph K_4 on vertices v_1, v_2, v_3, v_4 . Obtain H by deleting edge v_1v_3 . Obtain graph $G_{22} = H \cup K_1$. Only consider partitions of the form, $\{\{2\text{-element}\}, \{2\text{-element}\}, \{1\text{-element}\}\}$ as above.

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence select, $\{\{v_1, v_3\}, \{v_2, v_5\}, \{v_4\}\}, \{\{v_1, v_3\}, \{v_2\}, \{v_4, v_5\}\}$.

Therefore, $\mathcal{L}_{G_{22}}(\lambda, \chi) = \mathcal{L}_{G_{22}}(\lambda, 3) = 2\lambda(\lambda - 1)(\lambda - 2)$.

Without loss of generality consider the cycle C_3 on vertices v_1, v_2, v_3 . Obtain G_{23} by adding two pendant vertices v_4, v_5 to v_1 . Only consider partitions of the form, $\{\{2\text{-element}\}, \{2\text{-element}\}, \{1\text{-element}\}\}$ as above.

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence select, $\{\{v_1\}, \{v_2, v_4\}, \{v_3, v_5\}\}, \{\{v_1\}, \{v_2, v_5\}, \{v_3, v_4\}\}$. Therefore, $\mathcal{L}_{G_{23}}(\lambda, \chi) = \mathcal{L}_{G_{23}}(\lambda, 3) = 2\lambda(\lambda - 1)(\lambda - 2)$.

Without loss of generality consider the graph $G_{13} = K_5 - \{v_1v_3, v_1v_4, v_3v_4\}$. Only consider partitions of the form, $\{\{3\text{-element}\}, \{1\text{-element}\}, \{1\text{-element}\}\}$.

Hence, select

$$\begin{aligned} & \{\{v_1, v_2, v_3\}, \{v_4\}, \{v_5\}\}, \{\{v_1, v_2, v_5\}, \{v_3\}, \{v_4\}\}, \\ & \{\{v_1, v_2, v_4\}, \{v_3\}, \{v_5\}\}, \{\{v_1, v_4, v_5\}, \{v_2\}, \{v_3\}\}, \\ & \{\{v_1\}, \{v_2, v_4, v_5\}, \{v_3\}\}, \{\{v_1\}, \{v_2\}, \{v_3, v_4, v_5\}\}, \\ & \{\{v_1, v_3, v_5\}, \{v_2\}, \{v_4\}\}, \{\{v_1, v_3, v_4\}, \{v_2\}, \{v_5\}\}, \\ & \{\{v_1\}, \{v_2, v_3, v_5\}, \{v_4\}\}, \{\{v_1\}, \{v_2, v_3, v_4\}, \{v_5\}\}. \end{aligned}$$

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence select, $\{\{v_1, v_3, v_4\}, \{v_2\}, \{v_5\}\}$. Therefore, $\mathcal{L}_{G_{13}}(\lambda, \chi) = \mathcal{L}_{G_{13}}(\lambda, 3) = \lambda(\lambda - 1)(\lambda - 2)$.

Theorem 3.2. All graphs G of order 5 with $\chi(G) = 3$ except for G_{13} , have equitable chromatic number equal to the chromatic number.

Proof. The result is a direct consequence of the analysis above. \square

3.4 Category 3: $\chi(G) = 4$

If adjacency within a graph permits only consider partitions of the form, $\{\{2\text{-element}\}, \{1\text{-element}\}, \{1\text{-element}\}, \{1\text{-element}\}\}$. Hence select,

$$\begin{aligned} & \{\{v_1, v_2\}, \{v_3\}, \{v_4\}, \{v_5\}\}, \{\{v_1, v_5\}, \{v_2\}, \{v_3\}, \{v_4\}\}, \\ & \{\{v_1\}, \{v_2, v_5\}, \{v_3\}, \{v_4\}\}, \{\{v_1\}, \{v_2\}, \{v_3, v_5\}, \{v_4\}\}, \\ & \{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4, v_5\}\}, \{\{v_1, v_4\}, \{v_2\}, \{v_3\}, \{v_5\}\}, \\ & \{\{v_1\}, \{v_2, v_4\}, \{v_3\}, \{v_5\}\}, \{\{v_1\}, \{v_2\}, \{v_3, v_4\}, \{v_5\}\}, \\ & \{\{v_1, v_3\}, \{v_2\}, \{v_4\}, \{v_5\}\}, \{\{v_1\}, \{v_2, v_3\}, \{v_4\}, \{v_5\}\}. \end{aligned}$$

Without loss of generality consider the graph $G_{24} = K_5 - v_1v_2$.

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence, $\{\{v_1, v_2\}, \{v_3\}, \{v_4\}, \{v_5\}\}$. Therefore, $\mathcal{L}_{G_{24}}(\lambda, \chi) = \mathcal{L}_{G_{24}}(\lambda, 4) = \lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)$.

Without loss of generality consider the complete graph K_4 on vertices v_1, v_2, v_3, v_4 . Obtain G_{25} by adding vertex v_5



together with the edges v_1v_5, v_2v_5 .

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence, $\{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4, v_5\}\}, \{\{v_1\}, \{v_2\}, \{v_4\}, \{v_3, v_5\}\}$. Therefore, $\mathcal{L}_{G_{25}}(\lambda, \chi) = \mathcal{L}_{G_{25}}(\lambda, 4) = 2\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)$.

Without loss of generality consider the graph $G_{26} = K_4 \cup K_1$.

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence select, $\{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4, v_5\}\}, \{\{v_1\}, \{v_2\}, \{v_4\}, \{v_3, v_5\}\}, \{\{v_1\}, \{v_2, v_5\}, \{v_3\}, \{v_4\}\}, \{\{v_1, v_5\}, \{v_2\}, \{v_3\}, \{v_4\}\}$. Therefore, $\mathcal{L}_{G_{26}}(\lambda, \chi) = \mathcal{L}_{G_{26}}(\lambda, 4) = 4\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)$.

Without loss of generality consider the complete graph K_4 on vertices v_1, v_2, v_3, v_4 . Obtain G_{27} by adding pendant vertex v_5 to v_1 .

Now select permissible partitions i.e. those partitions for which some subsets do not contain adjacent vertices. Hence select, $\{\{v_1\}, \{v_2\}, \{v_3\}, \{v_4, v_5\}\}, \{\{v_1\}, \{v_2\}, \{v_4\}, \{v_3, v_5\}\}, \{\{v_1\}, \{v_2, v_5\}, \{v_3\}, \{v_4\}\}$. Therefore, $\mathcal{L}_{G_{27}}(\lambda, \chi) = \mathcal{L}_{G_{27}}(\lambda, 4) = 3\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)$.

Theorem 3.3. All graphs G of order 5 with $\chi(G) = 4$ have equitable chromatic number equal to chromatic number.

Proof. The result is a direct consequence of the analysis above. \square

4. Conclusion

Note that as observed in [5], the step before partitions with adjacency conflict are eliminated, the number of optimal near-completion ℓ -partitions equals the corresponding Stirling number of the second kind. Obviously with a Stirling of the second kind partition generator the tedious step of generating all corresponding Bell partitions can be eliminated.

Problem 1. If a Bell number or Stirling of the second kind partition generator can be found utilise such tool to find the Lucky k -polynomials for classical graph structures such as cycles, sun graphs, $S_n, n \geq 3$, wheel graphs, $W_{1,n}, n \geq 3$, helm graphs, $H_{1,n}, n \geq 3$ and so on for all permissible Lucky k -colourings in respect of $\lambda \geq 2$.

Problem 2. If possible characterize graphs for which the vertex partitions have the form prescribed by Lucky's theorem.

For the determination of Lucky k -polynomials it was conjectured that the *deletion-contraction principle* applies to Lucky polynomials. That is:

$$\mathcal{L}_G(\lambda, n) = \mathcal{L}_{G-v_iv_j}(\lambda, n) - \mathcal{L}_{G \setminus v_iv_j}(\lambda, n - 1), v_iv_j \in E(G).$$

Fundamentally a graph G of order $n > 0$ is defined to be, an ordered triple $(V(G), E(G), \psi_G)$, with $V(G)$ a non-empty set of vertices, a set $E(G)$ disjoint from $V(G)$, of edges and an incidence function ψ_G that associates with each edge of G and unordered pair of, not necessarily distinct vertices of G . Hence, any given graph G of order $n > 0$ can be reconstructed from the null graph \mathfrak{N}_n by the addition of edges in accordance to ψ_G .

For the determination of Lucky k -polynomials it was conjectured that the dual to the deletion-contraction principle, i.e. the *edge-addition-contraction principle* applies to Lucky polynomials. That is:

$$\mathcal{L}_G(\lambda, n) = \mathcal{L}_{G+v_iv_j}(\lambda, n) + \mathcal{L}_{G \setminus v_iv_j}(\lambda, n - 1), v_iv_j \notin E(G).$$

Analysis of the Lucky χ -polynomials of all graphs of order 5 indicates conclusively that in general, both the conjectures are invalid. However, a specialised conjecture may be proved or disproved.

Conjecture. For the determination of Lucky k -polynomials the deletion contraction principle or the edge addition contraction principle holds if and only if, $\chi(G) = \chi(G - v_iv_j) = \chi(G \setminus v_iv_j)$, or $\chi(G) = \chi(G + v_iv_j) = \chi(G \setminus v_iv_j)$.

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