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On a subclass of bi-univalent functions related to shell-like curves connected with Fibonacci number

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Abstract

In this paper we defined a new subclass of bi-univalent functions related to shell-like curves connected with Fibonacci number using the Frasin differential operator. We find some coefficient bounds and solve the linear functional $|a_3 - \mu a_2^2|$. Also we obtained various results proved by several authors as particular cases.

Keywords

Bi-Univalent, Shell-like, Fibonacci Number, Differential operator.

AMS Subject Classification

30C45.

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Contents

1. Introduction

We denote by A the class of regular functions defined in the open unit disk $\Delta = \{z/|z| < 1\}$ with the normalization conditions $f(0) = f'(0) - 1 = 0$ and the Taylor series expansion,

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1.1}$$

Consider S to be the class of univalent functions in A. For any two analytic functions $f(z)$ and $g(z)$ in Δ. We say that $f(z)$ is subordinate to $g(z)$ [9], (symbolically, f $\prec$ g) if there exists a function $\phi(z)$ analytic in Δ satisfying $\phi(0) = 0$ and $|\phi(z)| < 1$ such that

$$f(z) = g(\phi(z)), (|z| < 1).$$

By the Koebe-one quater theorem[4](Theorem.2.3 pg.31) , we know that "The range of every function of the class S contains a disk $\{w : |w| < 1/4\}$". Hence there exists inverse f^{-1} for every function $f \in S$, defined by

$$f^{-1}(f(z)) = z, (z \in \Delta)\ ;\ \ and$$

$$f(f^{-1}(w)) = w, (|w| < r_0(f) : r_0(f) \geq 1/4).$$

Where the inverse of f is given by,

$$\begin{aligned} f^{-1}(w) &= w - a_2 w^2 + (2a_2^2 w^2 - a_3) w^3 \\ &\quad - (5a_2^2 - 5a_2 a_3 + a_4) w^4 + \ldots \\ &=: g(w). \end{aligned}$$

A function $f \in A$ is said to be bi-univalent if both f and f^{-1}(its inverse) are univalent in Δ. We denote by $\sum$ the class of bi-univalent and analytic functions in Δ of the form (1.1). Using the binomial series,

$$(1-\lambda)^m = \sum_{j=0}^{m} \binom{m}{j} (-1)^j \lambda^j,$$

$m \in \mathbb{N} = 1, 2, \ldots$ *and* $j \in \mathbb{N}_0 = 0, 1, 2, \ldots$

Frasin [5] defined the following differential operator for function $f \in A$,

$$\begin{aligned} D^0 f(z) &= f(z) \\ D^1_{m,\lambda} f(z) &= (1-\lambda)^m f(z) + (1-(1-\lambda)^m) z f'(z) \\ &= D_{m,\lambda} f(z)\ ,\ (\lambda > 0; m \in \mathbb{N}). \end{aligned}$$

In general,

$$\begin{aligned} D^n_{m,\lambda} &= D_{m,\lambda}(D^{n-1}_{m,\lambda} f(z)), n \in \mathbb{N}_0 \\ &= z + \sum_{k=2}^{\infty} [1 + (k-1) c_j^m(\lambda)]^n a_k z^k \end{aligned}$$

where, $c_j^m(\lambda) = \sum_{j=1}^{m} \binom{m}{j}(-1)^{j+1}\lambda^j$.

Remarks:

1. For $m = 1$, we get the Al-oboudi differential operator, $D^n_{1,\lambda}$ [1].
2. For $m = \lambda = 1$, we get the Salagean differential operator, D^n [11].

For $f \in A$ the class SL of shell-like functions which is the subclass of the class S^* of starlike functions was first introduced by Sokol[12] in 1999 as below,

Definition 1.1. *[12] A function $f \in A$ having the series expansion (1.1) is said to be in the class SL of starlike shell-like functions if it satisfies the following conditions:*

$$\frac{zf'(z)}{f(z)} \prec \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2}$$

where $\tau = \frac{(1-\sqrt{5})}{(2)} \simeq -0.618$.

In the year 2011, Dziok *et al.*[2], introduced the class KSL of convex functions related to a shell-like curves as follows:

Definition 1.2. *[2] A function $f \in A$ of the form (1.1) belongs to the class KSL of convex shell-like functions if it satisfies the following condition:*

$$1+\frac{zf''(z)}{f'(z)} \prec \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2}$$

where $\tau = \frac{(1-\sqrt{5})}{(2)} \simeq -0.618$.

Again Dziok *et al.* [3] in the year 2011, defined the following class SLM_α of α-convex shell-like functions.

Definition 1.3. *[3] A function $f \in A$ of the form (1.1) belongs to the class SLM_α of α-convex shell-like functions if it satisfies the following condition:*

$$(1-\alpha)\{\frac{zf'(z)}{f(z)}\}+\alpha\{1+\frac{zf''(z)}{f'(z)}\} \prec \tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2}$$

where $\tau = \frac{(1-\sqrt{5})}{(2)} \simeq -0.618$.

We note that $SLM_0 \equiv SL$ and $SLM_1 \equiv KSL$. We consider $\tau = \frac{(1-\sqrt{5})}{(2)} \simeq -0.618$ and $\tilde{p}(z) = \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2}$ throughout this paper.

The function $\tilde{p}(z)$ does not belongs to the class S. Since $\tilde{p}(z)$ is univalent in the disc $|z| < \tau^2 \simeq 0.38$. We can observe the following from $\tilde{p}(z)$ [6]; $\tilde{p}(0) = \tilde{p}(\frac{-1}{2\tau}) = 1$; $\tilde{p}$ takes the unit circle to a curve described by $(10x-\sqrt{5})y^2 = (\sqrt{5}x-1)^2$, which is translated and revolved trisectrix of Maclaurin. The curve $\tilde{p}(re^{it})$ is a closed curve without any loops for $0 < r \leq r_0 = \tau^2 \simeq 0.38$. For $r_0 < r < 1$, it has a loop, and for $r = 1$ it has a vertical asymptote. In the year 2016, Raina and Sokol [10] proved the following,

$$\begin{aligned}\tilde{p}(z) &= \frac{1+\tau^2 z^2}{1-\tau z-\tau^2 z^2} \\ &= 1+\sum_{n=1}^{\infty}(u_{n-1}+u_{n+1})\tau^n z^n\end{aligned}$$

where $u_n = \frac{(1-\tau)^n-\tau^n}{\sqrt{5}}$, such that

$$u_n = u_{n-2}+u_{n-1} \quad for \ \ n = 2,3,... \tag{1.2}$$

By simple calculation we can decompose all the higher powers τ^n as a linear combination of τ and 1. The resulting recurrence relationships yield Fibonacci number u_n,

$$\tau^n = u_n\tau+u_{n-1}.$$

Thus $\tilde{p}(z)$ is related to Fibonacci number. So we can rewrite $\tilde{p}(z)$ as ,

$$\tilde{p}(z) = 1+\sum_{n=1}^{\infty}\tilde{p}_n\tau^n z^n \tag{1.3}$$

where $\tilde{p}_n = (u_{n-1}+u_{n+1})$. Now using (1.2) in (1.3) we have,

$$\tilde{p}(z) = 1+\tau z+3\tau^2 z^2+4\tau^3 z^3+... \tag{1.4}$$

Motivated by the works of earlier authors we define a new subclass of bi-univalent functions related to shell-like curves connected to Fibonacci number using Frasin differential operator.

Definition 1.4. *A function $f(z) \in \sum$ given by (1.1) is said to be in the class $\alpha - SLM_{\sum}(n,m,\lambda,\tilde{p}(z))$ if the following conditions are satisfied,*

$$(1-\alpha)\frac{D^{n+1}_{m,\lambda}f(z)}{D^n_{m,\lambda}f(z)}+\alpha\frac{(D^{n+1}_{m,\lambda}f(z))'}{(D^n_{m,\lambda}f(z))'} \prec \tilde{p}(z) \tag{1.5}$$

$$(m \in \mathbb{N}, n \in \mathbb{N}_0, 0 \leq \alpha \leq 1, z \in \Delta)$$

and

$$(1-\alpha)\frac{D^{n+1}_{m,\lambda}g(w)}{D^n_{m,\lambda}g(w)}+\alpha\frac{(D^{n+1}_{m,\lambda}g(w))'}{(D^n_{m,\lambda}g(w))'} \prec \tilde{p}(w) \tag{1.6}$$

$$(m \in \mathbb{N}, n \in \mathbb{N}_0, 0 \leq \alpha \leq 1, w \in \Delta)$$

Remarks

1. $\alpha - SLM_{\sum}(1,n,\lambda,\tilde{p}(z)) = SLM^{\lambda}_{\alpha,\sum}(n,\tilde{p}(z))$, the class of bi-univalent functions defined by Gurmeet singh *et al.* [7].
2. $\alpha - SLM_{\sum}(1,0,1,\tilde{p}(z)) = SLM_{\alpha,\sum}(\tilde{p}(z))$, the class of bi-univalent functions defined by Guney *et al.* [6].

We consider $\mathbb{P}$ to be the class of Caratheodary functions. i.e., for $p(z) \in \mathbb{P}$, $\Re\{p(z)\} > 0$, $p(z)$ is analytic in Δ and have the series expansion

$$p(z) = 1+\sum_{n=1}^{\infty}p_n z^n \ \ , \ \ z \in \Delta.$$

Lemma 1.5. *If $p(z) \in \mathbb{P}$, then $|p_n| \leq 2$ for each $n = 1,2,...$*

2. Coefficient estimate for the functions in the class $\alpha - SLM_{\Sigma}(n,m,\lambda,\tilde{p}(z))$

Theorem 2.1. *If $f(z)$ is in the class $\alpha - SLM_{\Sigma}(n,m,\lambda,\tilde{p}(z))$ then,*

$$|a_2| \leq \frac{|\tau|}{\sqrt{(c_j^m(\lambda)[\tau\varsigma+\psi])}}$$

and

$$|a_3| \leq \frac{|\tau|[\psi-\tau(1+3\alpha)c_j^m(\lambda)(1+c_j^m(\lambda))^{2n}]}{2\Psi[\tau\varsigma+\psi]}$$

where
$\varsigma = 2(1+2\alpha)(1+2c_j^m(\lambda))^n-(1+3\alpha)(1+c_j^m(\lambda))^{2n}$,
$\psi = (1+\alpha)^2 c_j^m(\lambda)(1+c_j^m(\lambda))^{2n}(1-3\tau)$ *and*
$\Psi = (1+2\alpha)(c_j^m(\lambda))^2(1+2c_j^m(\lambda))^n$.

Proof. Since $f \in \alpha - SLM_{\Sigma}(n,m,\lambda,\tilde{p}(z))$, from the definition 1.4, we have

$$(1-\alpha)\frac{D_{m,\lambda}^{n+1}f(z)}{D_{m,\lambda}^{n}f(z)}+\alpha\frac{(D_{m,\lambda}^{n+1}f(z))'}{(D_{m,\lambda}^{n}f(z))'}=\tilde{p}(r(z)) \quad (2.1)$$

and

$$(1-\alpha)\frac{D_{m,\lambda}^{n+1}g(w)}{D_{m,\lambda}^{n}g(w)}+\alpha\frac{(D_{m,\lambda}^{n+1}g(w))'}{(D_{m,\lambda}^{n}g(w))'}=\tilde{p}(s(w)) \quad (2.2)$$

where $r(z)$ and $s(w)$ are analytic functions in Δ with $r(0)=s(0)=0$ and $|r(z)|<1$ and $|s(w)|<1$.
Now define the function,

$$h(z)=\frac{1+r(z)}{1-r(z)}=1+r_1z+r_2z^2+\ldots.$$

Then,

$$\tilde{p}(r(z))=1+\frac{r_1}{2}\tau z+\frac{1}{2}(r_2-\frac{r_1^2}{2}+\frac{3r_1^2}{2}\tau)\tau z^2+\ldots \quad (2.3)$$

Similarly we define the function,

$$k(w)=\frac{1+s(z)}{1-s(z)}=1+s_1z+s_2z^2+\ldots.$$

Then,

$$\tilde{p}(s(w))=1+\frac{s_1}{2}\tau w+\frac{1}{2}(s_2-\frac{s_1^2}{2}+\frac{3s_1^2}{2}\tau)\tau w^2+\ldots \quad (2.4)$$

and by considering the LHS of (2.1), we have

$$(1-\alpha)\frac{D_{m,\lambda}^{n+1}f(z)}{D_{m,\lambda}^{n}f(z)}+\alpha\frac{(D_{m,\lambda}^{n+1}f(z))'}{(D_{m,\lambda}^{n}f(z))'}$$

$$= 1+(1+\alpha)c_j^m(\lambda)(1+c_j^m(\lambda))^n a_2 z+[2(1+2\alpha)c_j^m(\lambda)(1+2c_j^m(\lambda))^n a_3-(1+3\alpha)c_j^m(\lambda)(1+c_j^m(\lambda))^{2n}a_2^2]z^2+\ldots$$

and

$$(1-\alpha)\frac{D_{m,\lambda}^{n+1}g(w)}{D_{m,\lambda}^{n}g(w)}+\alpha\frac{(D_{m,\lambda}^{n+1}g(w))'}{(D_{m,\lambda}^{n}g(w))'}$$

$$= 1-(1+\alpha)c_j^m(\lambda)(1+c_j^m(\lambda))^n a_2 w+[2(1+2\alpha)c_j^m(\lambda)(1+2c_j^m(\lambda))^n(2a_2^2-a_3)-(1+3\alpha)c_j^m(\lambda)(1+c_j^m(\lambda))^{2n}a_2^2]w^2+\ldots$$

Using (2.3), (2.4), and the above two equations in (2.1) and (2.2) and equating the coefficients of z, z^2, w and w^2 we have the following equations,

$$c_j^m(\lambda)(1+c_j^m(\lambda))^n(1+\alpha)a_2=\frac{r_1}{2}\tau \quad (2.5)$$

$$2(1+2\alpha)c_j^m(\lambda)(1+2c_j^m(\lambda))^n a_3-(1+3\alpha)c_j^m(\lambda)(1+c_j^m(\lambda))^{2n}a_2^2=(r_2-\frac{r_1^2}{2})\frac{\tau}{2}+\frac{3r_1^2}{4}\tau^2 \quad (2.6)$$

$$-c_j^m(\lambda)(1+c_j^m(\lambda))^n(1+\alpha)a_2=\frac{s_1}{2}\tau \quad (2.7)$$

$$2(1+2\alpha)c_j^m(\lambda)(1+2c_j^m(\lambda))^n(2a_2^2-a_3)-(1+3\alpha)c_j^m(\lambda)(1+c_j^m(\lambda))^{2n}a_2^2=(s_2-\frac{s_1^2}{2})\frac{\tau}{2}+\frac{3s_1^2}{4}\tau^2 \quad (2.8)$$

from (2.5)and (2.6),

$$r_1=-s_1, \quad (2.9)$$

also

$$2[c_j^m(\lambda)]^2[1+c_j^m(\lambda)]^{2n}(1+\alpha)^2a_2^2=\frac{1}{4}(r_1^2+s_1^2)\tau^2$$

$$r_1^2+s_1^2=\frac{8[c_j^m(\lambda)]^2[1+c_j^m(\lambda)]^{2n}(1+\alpha)^2}{\tau^2}. \quad (2.10)$$

Adding (2.6) and (2.8), we get

$$a_2^2[4c_j^m(\lambda)(1+2\alpha)(1+2c_j^m(\lambda))^n-2c_j^m(\lambda)(1+c_j^m(\lambda))^{2n}(1+3\alpha)]=(r_2+s_2)\frac{\tau}{2}-\frac{1}{4}(r_1^2+s_1^2)\tau+\frac{3}{4}(r_1^2+s_1^2)\tau^2. \quad (2.11)$$

Using (2.10) in the above equation we get

$$a_2^2=\frac{(r_2+s_2)\tau^2}{4(c_j^m(\lambda)[\tau\varsigma+\psi])} \quad (2.12)$$

where $\varsigma=2(1+2\alpha)(1+2c_j^m(\lambda))^n-(1+3\alpha)(1+c_j^m(\lambda))^{2n}$ and $\psi=(1+\alpha)^2c_j^m(\lambda)(1+c_j^m(\lambda))^{2n}(1-3\tau)$.

By using Lemma.1.5 and triangular inequality we get the required inequality for $|a_2|$.
To estimate $|a_3|$ first we subtract (2.8) from (2.6) and then by using (2.9), we get

$$4c_j^m(\lambda)(1+2\alpha)(1+2c_j^m(\lambda))^n(a_3-a_2^2)=(r_2-s_2)\frac{\tau}{2}. \quad (2.13)$$

Now by using (2.12) in the above equation we get the coefficient bound for $|a_3|$. □

For $m=1$ in theorem2.1 we get the following corollary,

Corollary 2.2. *If $f(z)\in SLM^{\lambda}_{\alpha,\Sigma}(n,\tilde{p}(z))$, then*

$$|a_2|\leq\frac{|\tau|}{\sqrt{\xi}}$$

and

$$|a_3|\leq\frac{|\tau|(1+\lambda)^{2n}[\lambda^2(1+\alpha)^2(1-3\tau)-\tau(1+3\alpha)\lambda]}{2\xi\lambda(1+2\alpha)(1+2\lambda)^n}$$

where $\xi=\lambda[\tau\{2(1+2\alpha)(1+2\lambda)^n-(1+3\alpha)(1+\lambda)^{2n}\}+(1+\alpha)^2\lambda(1+\lambda)^{2n}(1-3\tau)]$ which agrees with the results of Gurmeet singh et al.[7] Theorem.6.

For $m=\lambda=1$ in theorem2.1 gives the following corollary,

Corollary 2.3. *If $f(z)\in SLM_{\alpha,\Sigma}(n,\tilde{p}(z))$, then*

$$|a_2|\leq\frac{|\tau|}{\sqrt{4^n(1+\alpha)^2+[2(1+2\alpha)3^n-\eta 4^n]\tau}}$$

and

$$|a_3|\leq\frac{|\tau|4^n[(1+\alpha)^2-\eta\tau]}{2(1+2\alpha)3^n[4^n(1+\alpha)^2+[2(1+2\alpha)3^n-\eta 4^n]\tau]}$$

where $\eta=3\alpha^2+9\alpha+4$ which agrees with the results of Gurmeet singh et al.[7] Corollary.7.

On substituting $m=\lambda=1$ and $n=0$ in theorem2.1 gives the following corollary,

Corollary 2.4. *If $f(z)\in SL_{\alpha,\Sigma}(\tilde{p}(z))$, then*

$$|a_2|\leq\frac{|\tau|}{\sqrt{(1+\alpha)^2-(1+\alpha)(2+3\alpha)\tau}}$$

and

$$|a_3|\leq\frac{|\tau|[(1+\alpha)^2-(3\alpha^2+9\alpha+4)\tau]}{2(1+2\alpha)[(1+\alpha)^2-(1+\alpha)(2+3\alpha)\tau])}$$

which agrees with the results of Guney et al.[6] Corollary.1.

On substituting $m=\lambda=1$ and $n=\alpha=0$ in theorem2.1 gives the following corollary,

Corollary 2.5. *If $f(z)\in SL_{\Sigma}(\tilde{p}(z))$, then*

$$|a_2|\leq\frac{|\tau|}{\sqrt{1-2\tau}}$$

and

$$|a_3|\leq\frac{|\tau|(1-4\tau)}{2(1-2\tau)}$$

which agrees with the results of Guney et al.[6] Corollary.1.

Also for $m=\lambda=\alpha=1$ and $n=0$ in theorem2.1 gives the following corollary,

Corollary 2.6. *If $f(z)\in KSL_{\Sigma}(\tilde{p}(z))$, then*

$$|a_2|\leq\frac{|\tau|}{\sqrt{4-10\tau}}$$

and

$$|a_3|\leq\frac{|\tau|(1-4\tau)}{3(2-5\tau)}$$

which agrees with the results of Guney et al.[6] Corollary.2.

3. Fekete-Szego Inequality for the function class $\alpha-SLM_{\Sigma}(n,m,\lambda,\tilde{p}(z))$

Theorem 3.1. *If $f(z)$ is in the class $\alpha-SLM_{\Sigma}(n,m,\lambda,\tilde{p}(z))$ then,*

$$|a_3-\mu a_2^2|\leq\begin{cases}\frac{|\tau|}{2c_j^m(\lambda)(1+2\alpha)(1+2c_j^m(\lambda))^n} & ,|\mu-1|\leq\frac{Y}{X}\\ \frac{|1-\mu|\tau^2}{Y} & ,|\mu-1|\geq\frac{Y}{X}\end{cases} \quad (3.1)$$

where

$$\begin{aligned}Y&=c_j^m(\lambda)(\tau[2(1+2\alpha)(1+2c_j^m(\lambda))^n\\&-(1+3\alpha)(1+c_j^m(\lambda))^{2n}]\\&+c_j^m(\lambda)(1+c_j^m(\lambda))^{2n}(1+\alpha)^2(1-3\tau))\end{aligned}$$

and $X=2|\tau|c_j^m(\lambda)(1+2\alpha)(1+2c_j^m(\lambda))^n$

Proof. From (2.12) and (2.13), we have

$$a_3-\mu a_2^2=\frac{\tau(c_2-d_2)}{8c_j^m(\lambda)(1+2\alpha)(1+2c_j^m(\lambda))^n}+(c_2+d_2)\chi(\mu) \quad (3.2)$$

where

$$\chi(\mu)=\frac{(1-\mu)\tau^2}{4c_j^m(\lambda)(\tau\varsigma+\phi)}.$$

with $\varsigma=2(1+2\alpha)(1+2c_j^m(\lambda))^n-(1+3\alpha)(1+c_j^m(\lambda))^{2n}$ and $\phi=c_j^m(\lambda)(1+c_j^m(\lambda))^{2n}(1+\alpha)^2(1-3\tau)$.

The above equation can be expressed as,

$$a_3-\mu a_2^2=[\chi(\mu)+\frac{\tau}{8c_j^m(\lambda)(1+2\alpha)(1+2c_j^m(\lambda))^n}]c_2 \\ +[\chi(\mu)-\frac{\tau}{8c_j^m(\lambda)(1+2\alpha)(1+2c_j^m(\lambda))^n}]d_2. \tag{3.3}$$

Taking modulus on the above equation, we obtain,

$$|a_3-\mu a_2^2|\leq\begin{cases}\frac{|\tau|}{2\delta} & ,\ 0\leq|\chi(\mu)|\leq\frac{|\tau|}{8\delta}\\ 4|\chi(\mu)| & ,\ |\chi(\mu)|\geq\frac{|\tau|}{8\delta}.\end{cases} \tag{3.4}$$

where $\delta=c_j^m(\lambda)(1+2\alpha)(1+2c_j^m(\lambda))^n$. Using the above equation we can get the desired bound for the Fekete-Szego problem. □

By varying the parameters in Theorem.3.1 we get the following corollaries.
When we consider $m=1$ in Theorem3.1 we get the following corollary, which is proved by Gurmeet Singh *et al.*[8] in Theorem.11.

Corollary 3.2. *If $f(z)\in SLM_{\alpha,\Sigma}^{\lambda}(n,\tilde{p}(z))$, then*

$$|a_3-\mu a_2^2|\leq\begin{cases}\frac{|\tau|}{2\lambda(1+2\alpha)(1+2\lambda)^n} & ,|\mu-1|\leq\frac{X}{2|\tau|\lambda(1+2\alpha)(1+2\lambda)^n}\\ \frac{|1-\mu|\tau^2}{X} & ,|\mu-1|\geq\frac{X}{2|\tau|\lambda(1+2\alpha)(1+2\lambda)^n}\end{cases} \tag{3.5}$$

where

$$X = \lambda(\tau[2(1+2\alpha)(1+2\lambda)^n-(1+3\alpha)(1+\lambda)^{2n}] \\ +\lambda(1+\lambda)^{2n}(1+\alpha)^2(1-3\tau)).$$

If we consider $m=\lambda=1$ in Theorem3.1 we get the following corollary, which is proved by Gurmeet Singh *et al.*[8] in Corollary.12.

Corollary 3.3. *If $f(z)\in SLM_{\alpha,\Sigma}(n,\tilde{p}(z))$, then*

$$|a_3-\mu a_2^2|\leq\begin{cases}\frac{|\tau|}{2(1+2\alpha)3^n} & ,|\mu-1|\leq\frac{Z}{2|\tau|(1+2\alpha)3^n}\\ \frac{|1-\mu|\tau^2}{Z} & ,|\mu-1|\geq\frac{Z}{2|\tau|(1+2\alpha)3^n}\end{cases} \tag{3.6}$$

where

$$Z=(2(1+2\alpha)3^n-(3\alpha^2+9\alpha+4)4^n)\tau+(1+\alpha)^2 4^n.$$

If we consider $m=\lambda=1$ and $n=0$ in Theorem3.1 we get the following corollary, which is proved by Guney *et al.*[6] in Theorem.11.

Corollary 3.4. *If $f(z)\in SLM_{\alpha,\Sigma}(\tilde{p}(z))$, then*

$$|a_3-\mu a_2^2|\leq\begin{cases}\frac{|\tau|}{2(1+2\alpha)} & ,|\mu-1|\leq\frac{P}{2|\tau|(1+2\alpha)}\\ \frac{|1-\mu|\tau^2}{P} & ,|\mu-1|\geq\frac{P}{2|\tau|(1+2\alpha)}\end{cases} \tag{3.7}$$

where $P=(1+\alpha)[(1+\alpha)-(2+3\alpha)\tau]$

If we consider $m=\lambda=1$ and $n=\alpha=0$ in Theorem3.1 we get the following corollary, which is proved by Guney *et al.*[6] in corollary.4.

Corollary 3.5. *If $f(z)\in SLM_{\Sigma}(\tilde{p}(z))$, then*

$$|a_3-\mu a_2^2|\leq\begin{cases}\frac{|\tau|}{2} & ,|\mu-1|\leq\frac{1-2\tau}{2|\tau|}\\ \frac{|1-\mu|\tau^2}{1-2\tau} & ,|\mu-1|\geq\frac{1-2\tau}{2|\tau|}.\end{cases}$$

If we consider $m=\lambda=\alpha=1$ and $n=0$ in Theorem3.1 we get the following corollary, which is proved by Guney *et al.*[6] in corollary.5.

Corollary 3.6. *If $f(z)\in KSL_{\Sigma}(\tilde{p}(z))$, then*

$$|a_3-\mu a_2^2|\leq\begin{cases}\frac{|\tau|}{6} & ,|\mu-1|\leq\frac{(2-5\tau)}{3|\tau|}\\ \frac{|1-\mu|\tau^2}{2(2-5\tau)} & ,|\mu-1|\geq\frac{(2-5\tau)}{3|\tau|}.\end{cases}$$

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