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A relational reformulation of the Phelps–Cardwell lemma

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Abstract

By using some results on translation and superadditive relations, we give some relational reformulations of the Phelps–Cardwell lemma in terms of open and closed surroundings.

These reformulations have mainly been suggested by a unifying scheme for continuities of relations in relator spaces and a projective generation of translation relators by superadditive relations.

Keywords: Translation and superadditive relations, open and closed surroundings, Phelps–Cardwell lemma.

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1 Introduction

In 1960, by using the classical Hahn-Banach extension theorem, Phelps [9] proved the following lemma which has had some applications in [2] and [5, Proposition 8], and also in [4] which has not been available to the author.

Lemma 1.1. *Suppose that E is a real normed linear space and that $\varepsilon > 0$. Let U and S^* denote the closed unit ball of E and the unit sphere of the dual space E^*, respectively. If $f, g \in S^*$ are such that*

$$f^{-1}(0) \cap U \subset g^{-1}[-\varepsilon/2, \varepsilon/2],$$

then either $\| f - g \| \leq \varepsilon$ or $\| f + g \| \leq \varepsilon$.

Remark 1.1. The above inclusion in a detailed form means only that if $x \in E$ such that $f(x) = 0$ and $\| x \| \leq 1$, then $| g(x) | \leq \varepsilon/2$.

In 2006, by using a quite elementary, but rather tricky computation, Cardwell [3] proved the following partial generalization of Lemma 1.1.

Lemma 1.2. *Let X be a complex Banach space and let ε be such that $0 < \varepsilon < 1/2$. Let $\varphi, \psi \in X^*$ be such that $\| \varphi \| = \| \psi \| = 1$. Suppose that for all $x \in X$ with $\| x \| \leq 1$ and $\varphi(x) = 0$, it holds that $\| \psi(x) \| \leq \varepsilon$. Then there is some complex number α such that $| \alpha | = 1$ and $\| \varphi - \alpha \psi \| \leq 5 \varepsilon$.*

Remark 1.2. If in particular φ and ψ are real-valued, then by slightly modifying the proof of Lemma 1.2 one can choose α to be either 1 or -1. Thus, Lemma 1.1, with bound ε replaced by $(5/2)\varepsilon$, can also be proved in an elementary way.

In 2007, by modifying the original proof of Phelps, Aron et al. [1] proved the following improvement of Lemma 1.2.

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Lemma 1.3. *Let X be a complex Banach space and S_X its unit sphere. If $f, g : X \to \mathbb{C}$ are linear forms of norm one and $\varepsilon > 0$ such that*

$$S_X \cap \{ f(x) = 0 \} \subset S_X \cap \{ |g(x)| \leq \varepsilon \},$$

then $\| g - \alpha f \| \leq 2\varepsilon$ for some $|\alpha| = 1$.

Remark 1.3. Note that if $x \in X$ such that $0 \neq \| x \| \leq 1$ and $f(x) = 0$, then by taking $u = \| x \|^{-1} x$, we have $\| u \| = 1$ and $f(u) = \| x \|^{-1} f(x) = 0$. Therefore, if the condition of Lemma 1.3 holds, then $|g(u)| \leq \varepsilon$, and thus $|g(x)| = \big| g\big(\| x \| u \big) \big| = \| x \| \, |g(u)| \leq \varepsilon$ also holds. Hence, since $|g(0)| = 0 \leq \varepsilon$, we can note that the condition of Lemma 1.2 also holds.

Now, by using the closed surroundings $\bar{B}_r = \{ (x, y) : \ d(x, y) \leq r \}$, we shall prove the following relational reformulation of Lemma 1.3.

Lemma 1.4. *Let X be a normed space over $\mathbb{C}$, and assume that φ and ψ are linear functions of X to $\mathbb{C}$ such that $\| \varphi \| = 1$ and $\| \psi \| = 1$. Moreover, assume that $r > 0$ and $s > 0$ such that*

$$\Big(\bar{B}_r \cap \varphi^{-1} \Big)(0) \subset \Big(\psi^{-1} \circ \bar{B}_{rs} \Big)(0). \tag{1.1}$$

Then, there exists $\alpha \in \mathbb{C}$, with $|\alpha| = 1$, such that

$$(\varphi - \alpha\psi) \circ \bar{B}_r \subset \bar{B}_{2rs} \circ (\varphi - \alpha\psi). \tag{1.2}$$

Remark 1.4. We shall show that (1.1) is equivalent to the inclusions

$$\Big(B_r \cap \varphi^{-1} \Big)(0) \subset \Big(\psi^{-1} \circ \bar{B}_{rs} \Big)(0)$$

and

$$\Big((\bar{B}_r \setminus B_r) \cap \varphi^{-1} \Big)(0) \subset \Big(\psi^{-1} \circ \bar{B}_{rs} \Big)(0).$$

Moreover, we shall also show that (1.1) and (1.2) are equivalent to the inclusions

$$\Big(\psi \circ \big(\bar{B}_r \cap \varphi^{-1} \big) \Big)(0) \subset \bar{B}_{rs}(0)$$

and

$$\bar{B}_r \subset (\varphi - \alpha\psi)^{-1} \circ \bar{B}_{2rs} \circ (\varphi - \alpha\psi).$$

The relational reformulations of Lemma 1.3 have been mainly suggested by a unifying scheme for continuities of relations in relator spaces [15, Definition 4.1] and a basic theorem on translation and superadditive relations [13, Theorem 4.8] which allows of a projective generation of translation relators by superadditive relations.

2 A few basic facts on relations

A subset F of a product set $X \times Y$ is called a *relation on* X *to* Y. If in particular $F \subset X^2$, then we may simply say that F is a *relation on* X. Thus, in particular $\Delta_X = \{ (x, x) : \ x \in X \}$ is a relation on X.

If F is a relation on X to Y, then for any $x \in X$ and $A \subset X$ the sets $F(x) = \{ y \in Y : \ (x, y) \in F \}$ and $F[A] = \bigcup_{a \in A} F(a)$ are called the *images of* x *and* A *under* F, respectively.

Moreover, the sets $D_F = \{ x \in X : \ F(x) \neq \emptyset \}$ and $R_F = F[D_F]$ are called the *domain and range of* F, respectively. If in particular $D_F = X$, then we say that F is a *relation of* X *to* Y, or that F is a *total relation on* X *to* Y.

In particular, a relation f on X to Y is called a *function* if for each $x \in D_f$ there exists $y \in Y$ such that $f(x) = \{ y \}$. In this case, by identifying singletons with their elements, we may simply write $f(x) = y$ in place of $f(x) = \{ y \}$.

If F is a relation on X to Y, then the values $F(x)$, where $x \in X$, uniquely determine F since we have $F = \bigcup_{x \in X} \{x\} \times F(x)$. Therefore, the *inverse relation* F^{-1} of F can be naturally defined such that $F^{-1}(y) = \{ x \in X : \ y \in F(x) \}$ for all $y \in Y$.

Moreover, if in addition G is a relation on Y to Z, then the *composition relation* $G \circ F$ of G and F can be naturally defined such that $(G \circ F)(x) = F\big[G(x)\big]$ for all $x \in X$. Thus, we also have $(G \circ F)(A) = G\big[F(A)\big]$ for all $A \subset X$.

If d is a nonnegative function of X^2, then for each $r > 0$ we may naturally define two relations B_r^d and $\bar{B}_r^d$ on X such that

$$B_r^d(x) = \big\{ y \in X : \ d(x, y) < r \big\} \qquad \text{and} \qquad \bar{B}_r^d(x) = \big\{ y \in X : \ d(x, y) \leq r \big\}$$

for all $x \in X$.

In the distance space $X(d) = (X, d)$, the *r–sized open and closed surroundings* B_r^d *and* $\bar{B}_r^d$ are usually more convenient means, than the *open and closed subsets of* $X(d)$, or even the *distance function* d itself.

For instance, a function f of one *distance space* $X(d)$ to another $Y(\rho)$ can be easily seen to be *uniformly continuous* if and only if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$f \circ B_\delta^d \subset B_\varepsilon^\rho \circ f, \qquad \text{or equivalently} \qquad B_\delta^d \subset f^{-1} \circ B_\varepsilon^\rho \circ f.$$

To more nicely express this notion and some other more complicated ones, instead of the relator $\mathcal{R}_d = \{ B_r^d : \ r > 0 \}$, it is necessary to work with the various refinements and modifications of $\mathcal{R}_d$ considered in [8].

For instance, if $\mathcal{R}$ is a *relator (relational system) on* X *to* Y, then the relator

$$\mathcal{R}^\wedge = \big\{ U \subset X \times Y : \quad \forall \ x \in X : \quad \exists \ R \in \mathcal{R} : \quad R(x) \subset U(x) \big\}$$

may be naturally called the *topological refinement or closure of* $\mathcal{R}$.

Thus, a pair (F, G) of relations on one *relator space* $(X, Y)(\mathcal{R})$ to another $(Z, W)(\mathcal{S})$ may be naturally called *topologically upper semicontinuous*, resp. *topologically mildly continuous* if

$$\mathcal{S}^\wedge \circ F \subset \big(G \circ \mathcal{R}^\wedge \big)^\wedge, \qquad \text{resp.} \qquad G^{-1} \circ \mathcal{S}^\wedge \circ F \subset \mathcal{R}^\wedge.$$

3 A few basic facts on translation relations

Definition 3.1. A relation R on a groupoid X is called a *translation relation* if for any $x, y \in X$ we have

$$x + R(y) \subset R(x + y).$$

Remark 3.5. By using the notation $u\,R\,v$ instead of $v \in R(u)$, the above inclusion can be expressed by saying that $y R z$ implies $(x + y) R (x + z)$ for all $x \in X$. Thus, in particular, the usual inequality relations $<$ and $\leq$ on $\mathbb{R}$ are translation relations.

Remark 3.6. However, it is now more important to note that if p is a nonnegative function of a group X and

$$d(x, y) = p(-x + y)$$

for all $x, y \in X$, then the surroundings $B_r^p = B_r^{d_p}$ and $\bar{B}_r^p = \bar{B}_r^{d_p}$ are translation relations on X.

To check the translation property of B_r^p, note that if $x, y \in X$ and $z \in B_r^p(y)$, then $p(-y + z) = d(y, z) < r$, and thus

$$d(x + y, x + z) = p\big(-(x + y) + x + z\big) = p(-y - x + x + z) = p(-y + z) < r.$$

Therefore, $x + z \in B_r^p(x + y)$. Thus, $x + B_r^p(y) \subset B_r^p(x + y)$ also holds.

The above facts and the following theorem has been first established in [13].

Theorem 3.1. *For a relation R on a group X, the following assertions are equivalent:*

(1) R is a translation relation;

(2) $R(x) = x + R(0)$ for all $x \in X$;

(3) $R(x + y) = x + R(y)$ for all $x, y \in X$;

(4) $R(x + y) \subset x + R(y)$ for all $x, y \in X$.

Proof. For instance, if (4) holds, then

$$R(x) = R(x+0) \subset x + R(0) = x + R(-x+x) \subset x - x + R(x) = R(x)$$

for all $x \in X$. Therefore, (2) also holds. □

Remark 3.7. Now, in addition to Remark 3.6, we can also state that

$$B_r^p(x+y) = x + B_r^p(y) \qquad \text{and} \qquad \bar{B}_r^p(x+y) = x + \bar{B}_r^p(y)$$

for all $x, y \in X$.

Some further basic properties of the above surrounding can also be derived from the following theorems of [13].

Theorem 3.2. *If R is a translation relation on a groupoid X, then for any $A, B \subset X$ we have*

$$A + R[B] \subset R[A+B].$$

Moreover, if in particular X is a group, then the corresponding equality is also true.

Theorem 3.3. *If R is a translation relation on a groupoid X, then R^{-1} is also a translation relation on X. Moreover, if in particular X is a commutative group, then for any $A \subset X$ we have*

$$R^{-1}[A] = -R[-A].$$

Theorem 3.4. *If R and S are translation relation on a groupoid X, then $S \circ R$ is also a translation relation on X. Moreover, if in particular X is a commutative group, then for any $A, B \subset X$ we have*

$$(S \circ R)[A+B] = R[A] + S[B].$$

Remark 3.8. In this respect, it is also worth mentioning that the family of all translation relations on a groupoid is also closed under complementation, and arbitrary unions and intersections.

4 A few basic facts on superadditive relations

Definition 4.2. A relation F on one groupoid X to another Y is called *superadditive* if for any $x, y \in X$ we have

$$F(x) + F(y) \subset F(x+y).$$

Remark 4.9. By using the notation $u\,F\,v$ instead of $v \in F(u)$, the above inclusion can be expressed by saying that xFz and yFw implies $(x+y)F(z+w)$. Thus, in particular, the usual inequality relations $<$ and $\leq$ on $\mathbb{R}$ are superadditive relations.

Remark 4.10. It is clear that a reflexive and superadditive relation R on a groupoid X is a translation relation. Moreover, by [16, Theorem 3.14], a translation relation R on a commutative group X is superadditive if and only if it is transitive.

Definition 4.3. A relation F on a group X to a groupoid Y with zero is called *quasi-odd* if for any $x \in D_F$ we have

$$0 \in F(x) + F(-x).$$

Remark 4.11. Thus, a reflexive relation R on a group X is quasi-odd. Moreover, if F is an *odd relation* on one group X to another Y in the sense that $F(-x) = -F(x)$ for all $x \in X$, then F is in particular quasi-odd.

Now, as certain counterparts of Theorem 3.1, we can also prove the following two theorems.

Theorem 4.5. *If F is a quasi-odd and superadditive relation on a group X to a monoid Y, then*

$$F(x+y) = F(x) + F(y)$$

for all $x, y \in X$ with either $x \in D_F$ or $y \in D_F$.

Proof. If $x \in D_F$, then $0 \in F(x) + F(-x) \subset F(0)$. Moreover,

$$F(x+y) \subset F(x) + F(-x) + F(x+y) \subset F(x) + F(y)$$

for all $y \in X$. The case $x \in X$ and $y \in D_F$ can be treated quite similarly. □

Theorem 4.6. *If F is a quasi-odd and superadditive relation on one group X to another Y then there exists a function f on X to Y such that for all $x \in X$ we have*

$$F(x) = f(x) + F(0) \qquad \text{and} \qquad F(x) = F(0) + f(x).$$

Proof. Now, for any $x \in D_F$, we have $0 \in F(x) + F(-x)$. Therefore, there exist $y \in F(x)$ and $z \in F(-x)$ such that $0 = y + z$. Hence, we can already infer that $y = -z \in -F(-x)$. Therefore, $y \in F(x) \cap \big(-F(-x)\big)$, and thus $F(x) \cap \big(-F(-x)\big) \neq \emptyset$. Hence, by the Axiom of Choice, it is clear that there exists a function f of D_F to Y such that $f(x) \in F(x) \cap (-F(-x)$, and thus $f(x) \in F(x)$ and $f(x) \in -F(-x)$ for all $x \in D_F$.

Now, if $x \in D_F$, then we can see that $f(x) + F(0) \subset F(x) + F(0) \subset F(x)$. Moreover, since $-f(x) \in F(-x)$, we can also see that

$$F(x) \subset f(x) - f(x) + F(x) \subset f(x) + F(-x) + F(x) \subset f(x) + F(0).$$

Therefore, $F(x) = f(x) + F(0)$. Hence, since $F(x) = \emptyset$ and $f(x) = \emptyset$ if $x \in X \setminus D_F$, it is clear that the first part of the required assertion is true. The second part can be proved quite similarly. □

Remark 4.12. Note that if F is a relation on one group X to another Y and f is a selection function of F such that either F or f is odd, then we also have $-f(x) \in F(-x)$ for all $x \in D_F$. Therefore, if in particular F is superadditive, then by the above argument we also have $F(x) = f(x) + F(0)$ for all $x \in X$.

Various conditions in order that a relation F could have an additive selection function f have been given by several authors dealing with relational generalizations of the Hahn-Banach extension theorems and the Hyers-Ulam stability theorems. (For a rapid overview on the subjects, see [18] and the reference therein.)

The close relationship between translation and supperadditive relations can also be clarified by the following generalization of [13, Theorem 4.8].

Theorem 4.7. *If F and G are superadditive relations of one groupoid X to another Y such that $G \subset F$, and S is a translation relation on Y, then $R = G^{-1} \circ S \circ F$ is a translation relation on X.*

Proof. If $x, y \in X$ and $z \in R(y)$, then by the corresponding definitions we also have

$$z \in \big(G^{-1} \circ S \circ F\big)(y) = G^{-1}\big[S\big[F(y)\big]\big].$$

Thus, there exists $w \in S\big[F(y)\big]$ such that $z \in G^{-1}(w)$, and hence $w \in G(z)$. Consequently, we also have $G(z) \cap S\big[F(y)\big] \neq \emptyset$. Hence, since $G(x) \neq \emptyset$, it follows that

$$\big(G(x) + G(z)\big) \cap \big(G(x) + S\big[F(y)\big]\big) \neq \emptyset.$$

Now, by using that $G(x) + G(z) \subset G(x+z)$ and

$$G(x) + S\big[F(y)\big] \subset F(x) + S\big[F(y)\big] \subset S\big[F(x) + F(y)\big] \subset S\big[F(x+y)\big],$$

we can see that

$$G(x+z) \cap S\big[F(x+y)\big] \neq \emptyset.$$

Thus, there exists $\omega \in S\big[F(x+y)\big]$ such that $\omega \in G(x+z)$, and hence $x + z \in G^{-1}(\omega)$. Consequently, we also have

$$x + z \in G^{-1}\big[S\big[F(x+y)\big]\big] = \big(G^{-1} \circ S \circ F\big)(x+y) = R(x+y).$$

Therefore, the inclusion $x + R(y) \subset R(x+y)$ is also true. □

Finally, we note that, analogously to the corresponding results of Section 3, the following theorems can also be proved.

Theorem 4.8. *If F is a superadditive relation on one groupoid to another Y, then for any A, $B \subset X$ we have*

$$F[A] + F[B] \subset F[A+B].$$

Remark 4.13. If in particular X and Y are groups, and F is in addition quasi-odd, then the corresponding equality is also true with either $A \subset D_F$ or $B \subset D_F$.

Theorem 4.9. *If F is a superadditive relation on one groupoid to another Y, then F^{-1} is a superadditive relation on Y to X.*

Theorem 4.10. *If F is a superadditive relation on one groupoid to another Y and G is a superadditive relation on Y to another groupoid Z, then $G \circ F$ is a superadditive relation on X to Z.*

Remark 4.14. In this respect, it is also worth noticing that if F and G are superadditive relations on a groupoid X to a commutative semigroup Y, then their pointwise sum $F+G$ is also a superadditive relation on X to Y.

Thus, in particular if f is an additive function on X to Y and Z is a subsemigroup of Y, then the relation $f+Z$, defined such that $(f+Z)(x) = f(x)+Z$ for all $x \in X$, is an additive relation on X to Y. Note that, by Theorems 3.1 and 4.6, some translation and superadditive relations are of the latter form.

5 A relational reformulation of Lemma 1.3

Now, by using our former results on translation and superadditive relations, we can prove the following

Lemma 5.5. *Let X be a normed space over $\mathbb{C}$, and assume that φ and ψ are linear functions of X to $\mathbb{C}$ such that*

$$\|\varphi\| = 1 \qquad \text{and} \qquad \|\psi\| = 1.$$

Moreover, assume that $r > 0$ and $s > 0$ such that

$$\left(\bar{B}_r \cap \varphi^{-1}\right)(0) \subset \left(\psi^{-1} \circ \bar{B}_{rs}\right)(0). \tag{5.3}$$

Then, there exists $\alpha \in \mathbb{C}$, with $|\alpha| = 1$, such that

$$(\varphi - \alpha\psi) \circ \bar{B}_r \subset \bar{B}_{2rs} \circ (\varphi - \alpha\psi). \tag{5.4}$$

Proof. If $x \in X$ such that

$$\|x\| = 1 \qquad \text{and} \qquad \varphi(x) = 0,$$

then we also have

$$\|rx\| = r\|x\| = r \qquad \text{and} \qquad \varphi(rx) = r\varphi(x) = 0.$$

Hence, we can already infer that

$$rx \in \bar{B}_r(0) \qquad \text{and} \qquad rx \in \varphi^{-1}(0),$$

and thus

$$rx \in \bar{B}_r(0) \cap \varphi^{-1}(0) = \left(\bar{B}_r \cap \varphi^{-1}\right)(0) \subset \left(\psi^{-1} \circ \bar{B}_{rs}\right)(0) = \psi^{-1}\left[\bar{B}_{rs}(0)\right].$$

This implies that $\psi(rx) \in \bar{B}_{rs}(0)$. Therefore,

$$r|\psi(x)| = |r\psi(x)| = |\psi(rx)| \leq rs, \qquad \text{and thus} \qquad |\psi(x)| \leq s.$$

Now, by Lemma 1.3, we can state that here exists $\alpha \in \mathbb{C}$, with $|\alpha| = 1$, such that under the notation

$$f = \varphi - \alpha\psi$$

we have $\|f\| \leq 2s$. This implies that

$$|f(x)| \leq \|f\|\,\|x\| \leq 2s\|x\|$$

for all $x \in X$. Hence, if in particular

$$x \in \bar{B}_r(0), \qquad \text{and thus} \qquad \|x\| \leq r,$$

we can see that

$$|f(x)| \leq 2rs, \qquad \text{and thus} \qquad f(x) \in \bar{B}_{rs}(0).$$

Therefore, we also have

$$x \in f^{-1}\big[\bar{B}_{2rs}(0)\big] = f^{-1}\big[\bar{B}_{2rs}\big(f(0)\big)\big] = f^{-1}\big[\big(\bar{B}_{2rs} \circ f\big)(0)\big] = \Big(f^{-1} \circ \bar{B}_{2rs} \circ f\Big)(0).$$

This proves that

$$\bar{B}_r(0) \subset \big(f^{-1} \circ \bar{B}_{2rs} \circ f\big)(0).$$

Hence, by using Remark 3.6 and Theorems 3.1 and 4.7, we can already infer that

$$\bar{B}_r(x) = x + \bar{B}_r(0) \subset x + \Big(f^{-1} \circ \bar{B}_{2rs} \circ f\Big)(0) = \Big(f^{-1} \circ \bar{B}_{2rs} \circ f\Big)(x) = f^{-1}\big[\big(\bar{B}_{2rs} \circ f\big)(x)\big],$$

and thus

$$\big(f \circ \bar{B}_r\big)(x) = f\big[\bar{B}_r(x)\big] \subset f\Big[f^{-1}\big[\big(\bar{B}_{2rs} \circ f\big)(x)\big]\Big] \subset \big(\bar{B}_{2rs} \circ f\big)(x)$$

for all $x \in X$. Therefore, $f \circ \bar{B}_r \subset \bar{B}_{2rs} \circ f$, and thus the required inclusion is also true. □

Remark 5.15. The above proof shows that condition (5.3) can be weakened by requiring only that

$$\Big(\big(\bar{B}_r \setminus B_r\big) \cap \varphi^{-1}\Big)(0) \subset \Big(\psi^{-1} \circ \bar{B}_{rs}\Big)(0)$$

The forthcoming Proposition 6.1 will shows that the latter inclusion is actually equivalent to condition (5.3).

6 Equivalent reformulations of condition (5.3)

Condition (5.3) can also be naturally weakened with the help of the following

Proposition 6.1. *If φ and ψ are continuous and homogeneous functions of one normed space X to another Y, then for any $r > 0$ and $s > 0$ the following inclusions are equivalent:*

(a) $\big(B_r \cap \varphi^{-1}\big)(0) \subset \big(\psi^{-1} \circ \bar{B}_s\big)(0)$;

(b) $\big(\bar{B}_r \cap \varphi^{-1}\big)(0) \subset \big(\psi^{-1} \circ \bar{B}_s\big)(0)$;

(c) $\big(\big(\bar{B}_r \setminus B_r\big) \cap \varphi^{-1}\big)(0) \subset \big(\psi^{-1} \circ \bar{B}_s\big)(0)$.

Proof. Since $\bar{B}_r = B_r \cup \big(\bar{B}_r \setminus B_r\big)$, it is clear that (b) implies both (a) and (c) even if φ and ψ are arbitrary relations. Therefore, we need only show that both (a) and (c) imply (b).

If $x \in \big(\bar{B}_r \cap \varphi^{-1}\big)(0)$, then $x \in \bar{B}_r(0) \cap \varphi^{-1}(0)$, and thus

$$\|x\| \leq r \qquad \text{and} \qquad \varphi(x) = 0.$$

Hence, if $x \neq 0$, then by taking

$$u = r\,\|x\|^{-1} x,$$

we can infer that

$$\|u\| = r\,\|x\|^{-1}\,\|x\| = r \qquad \text{and} \qquad \varphi(u) = r\,\|x\|^{-1}\varphi(x) = 0.$$

This implies that

$$u \in \big(\bar{B}_r(0) \setminus B_r(0)\big) \cap \varphi^{-1}(0) = \big(\bar{B}_r \setminus B_r\big)(0) \cap \varphi^{-1}(0) = \Big(\big(\bar{B}_r \setminus B_r\big) \cap \varphi^{-1}\Big)(0).$$

Therefore, if (c) holds, then we also have

$$u \in \Big(\psi^{-1} \circ \bar{B}_s\Big)(0) = \psi^{-1}\big[\bar{B}_s(0)\big].$$

This implies that $\psi(u) \in \bar{B}_s(0)$, and thus $\|\psi(u)\| \leq s$. Hence, we can infer that

$$\|\psi(x)\| = \left\|\psi\left(r^{-1}\|x\|u\right)\right\| = r^{-1}\|x\|\,\|\psi(u)\| \leq s,$$

and thus $\psi(x) \in \bar{B}_s(0)$. Moreover, we can note that $\psi(0) = 0 \in \bar{B}_s(0)$ also holds. Therefore,

$$x \in \psi^{-1}\left[\bar{B}_s(0)\right] = \left(\psi^{-1} \circ \bar{B}_s\right)(0)$$

even if $x = 0$. This shows that (c) implies (b) even if φ and ψ are only assumed to be homogeneous.

On the other hand, if $x \in \left(\bar{B}_r \cap \varphi^{-1}\right)(0)$, and thus $\|x\| \leq r$ and $\varphi(x) = 0$, then by taking

$$x_n = n\,(n+1)^{-1}x$$

for each $n \in \mathbb{N}$, we can see that

$$\|x_n\| = n\,(n+1)^{-1}\|x\| < r \qquad \text{and} \qquad \varphi(x_n) = n\,(n+1)^{-1}\varphi(x) = 0.$$

This implies that

$$x_n = B_r(0) \cap \varphi^{-1}(0) = \left(B_r \cap \varphi^{-1}\right)(0).$$

Therefore, if (a) holds, then we also have

$$x_n \in \left(\psi^{-1} \circ \bar{B}_{rs}\right)(0) = \psi^{-1}\left[\bar{B}_s(0)\right],$$

and thus $\psi(x_n) \in \bar{B}_s(0)$. This implies that $\|\psi(x_n\| \leq s$. Hence, by using that

$$\lim_{n\to\infty} x_n = x, \qquad \text{and thus} \qquad \lim_{n\to\infty} \psi(x_n) = \psi(x),$$

we can infer already that $\|\psi(x)\| \leq s$. Therefore, $\psi(x) \in \bar{B}_s(0)$, and thus $x \in \left(\psi^{-1} \circ \bar{B}_s\right)(0)$ also holds. This shows that (a) implies (b) even if φ and ψ are only assumed to be homogeneous and continuous, respectively. □

In this respect, it is also worth mentioning that we also have the following

Proposition 6.2. *If φ is a continuous homogeneous functions of one normed space X to another Y, then for any $r > 0$ we have*

$$\left(\bar{B}_r \cap \varphi^{-1}\right)(0) = \overline{\left(B_r \cap \varphi^{-1}\right)(0)}.$$

Proof. From the proof of the implication (a) $\Longrightarrow$ (b) we can see that

$$\left(\bar{B}_r \cap \varphi^{-1}\right)(0) \subset \overline{\left(B_r \cap \varphi^{-1}\right)(0)}$$

even if φ is only assumed to be homogeneous.

Moreover, we can note that $\varphi(0) = \varphi^{-1}[\{0\}]$ is a closed subset of X even if φ is only assumed to be continuous. Thus,

$$\left(\bar{B}_r \cap \varphi^{-1}\right)(0) = \bar{B}_r(0) \cap \varphi^{-1}(0)$$

is also a closed subset of X. Hence, it is clear that

$$\overline{\left(B_r \cap \varphi^{-1}\right)(0)} \subset \overline{\left(\bar{B}_r \cap \varphi^{-1}\right)(0)} = \left(\bar{B}_r \cap \varphi^{-1}\right)(0),$$

and thus the required equality is also true. □

Remark 6.16. The latter proposition allows of a shorter proof of the implication (a) $\Longrightarrow$ (b) in Proposition 6.1.

Namely, if (a) holds, then by noticing that

$$\left(\psi^{-1} \circ \bar{B}_s\right)(0) = \psi^{-1}\left[\bar{B}_s(0)\right]$$

is also a closed subset of X, we can at once see that

$$\left(\bar{B}_r \cap \varphi^{-1}\right)(0) = \overline{\left(B_r \cap \varphi^{-1}\right)(0)} \subset \overline{\left(\psi^{-1} \circ \bar{B}_s\right)(0)} = \left(\psi^{-1} \circ \bar{B}_s\right)(0),$$

and thus (b) also holds.

7 Equivalent reformulations of inclusions (5.3) and (5.4)

Inclusions (5.3) and (5.4) can also be reformulated by using the following basic proposition whose proof is included here only for the reader's convenience.

Proposition 7.3. *If Ψ is a relation on one set X to another Y, then*

(a) $\Delta_X \subset \Psi^{-1} \circ \Psi$ *if* Ψ *is total ;*

(b) $\Psi \circ \Psi^{-1} \subset \Delta_Y$ *if* Ψ *is a function .*

Proof. If Ψ is total, then for each $x \in X$ there exists $y \in Y$ such that $y \in \Psi(x)$. Hence, it is clear that $x \in \Psi^{-1}(y)$, and thus

$$x \in \Psi\left[\Psi^{-1}(x)\right] = \left(\Psi \circ \Psi^{-1}\right)(x).$$

Therefore, $(x, x) \in \Psi \circ \Psi^{-1}$. This shows that (a) is true.

On the other hand, if $(y, z) \in \Psi \circ \Psi^{-1}$, then we can note that

$$z \in \left(\Psi \circ \Psi^{-1}\right)(y) = \Psi\left[\Psi^{-1}(y)\right].$$

Therefore, there exists $x \in \Psi^{-1}(y)$ such that $z \in \Psi(x)$. Hence, if Ψ is a function, we can already infer that $y = \Psi(x) = z$. Therefore, (b) is also true. □

By this proposition, it is clear that in particular we also have the following

Proposition 7.4. *If Ψ is a relation on one set X to another Y, then for any $A \subset X$ and $B \subset Y$*

(a) $\Psi[A] \subset B$ *implies* $A \subset \Psi^{-1}[B]$ *if* Ψ *is total ;*

(b) $A \subset \Psi^{-1}[B]$ *implies* $\Psi[A] \subset B$ *if* Ψ *is a function .*

Proof. If for instance $\Psi[A] \subset B$ and Ψ is total, then Proposition 7.3 we have

$$A = \Delta_X[A] \subset \left(\Psi^{-1} \circ \Psi\right)[A] = \Psi^{-1}\left[\Psi[A]\right] \subset \Psi^{-1}[B].$$

□

A simple application of this proposition gives the following

Proposition 7.5. *If Ψ is a relation on one set X to another Y, and R and S are relations on X and Y, respectively, then for any $A \subset X$*

(1) $(\Psi \circ R)[A] \subset S[A]$ *implies* $R[A] \subset \left(\Psi^{-1} \circ S\right)[A]$ *if* Ψ *is total ;*

(2) $R[A] \subset \left(\Psi^{-1} \circ S\right)[A]$ *implies* $(\Psi \circ R)[A] \subset S[A]$ *if* Ψ *is a function .*

Proof. If for instance $(\Psi \circ R)[A] \subset S[A]$ holds, then we also have $\Psi\left[R[A]\right] \subset S[A]$. Hence, if Ψ is total, then by using Proposition 7.4 we can infer that

$$R[A] \subset \Psi^{-1}\left[S[A]\right] = \left(\Psi^{-1} \circ S\right)[A].$$

□

Remark 7.17. By this proposition, it is clear that condition (5.3) of Lemma 5.5 is equivalent to the inclusion

$$\Big(\psi \circ \left(\bar{B}_r \cap \varphi^{-1}\right)\Big)(0) \subset \bar{B}_{rs}(0).$$

Moreover, by using Proposition 7.4, we can also easily establish the following

Proposition 7.6. *If F is a relation on one set X to another Y, and R and S are relations on X and Y, respectively, then for any $A \subset X$*

(1) $(F \circ R)[A] \subset (S \circ F)[A]$ *implies* $R[A] \subset \left(F^{-1} \circ S \circ R\right)[A]$ *if* F *is total ;*

(2) $R[A] \subset \left(F^{-1} \circ S \circ R\right)[A]$ *implies* $(F \circ R)[A] \subset (S \circ F)[A]$ *if* F *is a function .*

Proof. If for instance $(F \circ R)[A] \subset (S \circ F)[A]$ holds, then we also have $F[R[A]] \subset (S \circ F)[A]$. Hence, if F is total, then by using Proposition 7.5 we can infer that

$$R[A] \subset F^{-1}\Big[(F^{-1} \circ S \circ R)[A]\Big] = (F^{-1} \circ S \circ R)[A].$$

□

Remark 7.18. By this proposition, it is clear that conclusion (5.4) of Lemma 5.5 is equivalent to the inclusion

$$\bar{B}_r \subset (\varphi - \alpha\,\psi)^{-1} \circ \bar{B}_{2rs} \circ (\varphi - \alpha\,\psi).$$

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