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Probabilistic s_b-metric spaces

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Abstract
We propose an attractive development of metric spaces termed as a probabilistic s_b-metric spaces with some examples in this work. Also some of its properties were proved.

Keywords
Probabilistic metric spaces Probabilistic s_b-metric spaces, Strong neighbourhood, Continuous t-norm.

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47H09, 47H10, 54H25.

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Contents

1. Introduction

In 1942, K. Menger advanced the argument of Metric spaces, suggested a probabilistic conjecture of this theory. He proposed a probabilistic concept of length among the points p,q through a distribution function $F_{p,q}$. The intention of Menger was to use distribution functions instead of positive real number as values of the metric. The perception of a probabilistic metric space resembles to positions when we do not know truly the length between two points, but we know probabilities of possible values of the distance. A probabilistic rationalization of metric spaces shows to have fascinate in the survey of physical quantities and physiological thresholds.

2. Preliminaries

In this section, we recall some definitions and basic results which will be used throughout the paper.

Definition 2.1. A positive real function 'f' defined on $\mathfrak{R}^+ \cup \{\infty\}$ is said to be a Distance Distributive Function (d.d.f) [1] if it is increasing, left continuous on $(0,\infty)$ with $f(0)=0$ and $f(\infty)=1$. Here $\Delta^+ = \{d.d.f's\}$; and for all $f \in \Delta^+$ with $\lim_T(t \to \infty)[f(t)] = 1$ by D^+.

The following example, of distributive function is heavy side function in D^+,

$$H(S_1) = \begin{cases} 0 & \text{if } s_1 \le 0, \\ 1 & \text{if } s_1 > 0. \end{cases}$$

Definition 2.2. A commutative, associative and increasing mapping $T:[0,1]^2 \to [0,1]$ is said to be a t-norm [1] iff

(i) $T(b,1) = b$ for all $b \in [0,1]$

(ii) $T(0,0) = 0$.

Definition 2.3. A mapping T from $[0,1]^2$ into $[0,1]$ be a continuous t-norm [1] if T be:

(i) Commutative and Associative;

(ii) Continuous;

(iii) $T(p,1) = p$, for all $p \in [0,1]$;

(iv) $T(m,l) \le T(s,t)$ whenever $m \le s$ and $s \le t$ and $m,l,s,t \in [0,1]$.

Example 2.4. The examples of continuous t-norms are:
$T_p(q,e) = qe, T_M(q,e) = Min(q,e)$ and
$T_L(q,e) = Max\{q+e-1,0\}$.
If T is left-continuous then the operation, $\gamma_T : \Delta^+ \times \Delta^+ \to \Delta^+$ by, $\gamma_T(R,H)(y) = sup\{T(R(z),H(b)) : z+b=y\}$, is a triangle function

Definition 2.5. An s-Mertic [6] on E $(E \neq \phi)$ be a function $s: E^3 \to [o,\infty)$ that satisfies the coming settings, for all $i,e,s \in E$:

(i) $s(q,i,e) = 0$ iff $q = i = e$,

(ii) $s(q,i,e) \leq s(q,q,s) + s(i,i,s) + s(e,e,s)$

The pair (E,S) is termed as a s-Metric Space $(s-MS)$.

Definition 2.6. *A b-Metric [6] on $E(E \neq \emptyset)$ is a function $d : E^2 \to [o,\infty)$ if $\exists$ a real no. $k \geq 1$ s.t $q,i,e \in E$:*

(i) $d(q,i) = 0$ iff $q = i$,

(ii) $d(q,i) = d(i,q)$,

(iii) $d(q,e) \leq k[d(q,i) + d(i,e)]$.

Here (E,d) is a b-Metric space $(b-Ms)$.

Definition 2.7. *Take $E_1 \neq \emptyset$ and $k \leq 1$ be a given real no. A function $s_b : E_1^3 \to [o,\infty)$ is termed to be s_b-Me [19] iff for all $q,i,e,s \in E_1$, the following conditions hold:*

(i) $s_b(q,i,e) = 0$ iff $q = i = e$.

(ii) $s_b(q,q,i) = S_b(i,i,q)$ for all $q,i \in E_1$.

(iii) $s_b(q,i,e) \leq k[S_b(q,q,s) + S_b(i,i,s) + S_b(e,e,s)]$.

The pair (E,S_b) is labeled as a s_b-Metric Space $(s_b - MS)$.

Definition 2.8. *A real function D on $X \times X \times X$ is called a D-metric [7] on X if*

(i) $D(x,y,z) \geq 0$ for all $x,y,z \in X$(non-negative),

(ii) $D(x,y,z) = 0$ if and only if $x = y = z$(coincidence),

(iii) $D(x,y,z) = D(p(x,y,z))$ for every $x,y,z \in X$ and for any permutation $p(x,y,z)$ of x,y,z(symmetry),

(iv) $D(x,y,z) \leq D(x,y,u) + D(x,u,z) + D(u,y,z)$ for every $x,y,z,u \in X$ (tetrahedral inequality)

A D-metric space $(D-MS)$ be a pair (X,D) here D be a D-metric on X.

Definition 2.9. *A probabilistic generalized MS is a triple (Z,F,γ) where $Z \neq \emptyset$, $F : Z \times Z \to \Delta^+$. γ is a Δ^1 function, $\ni$: for all $a,t \in Z$ and $i,d \in Z$, $i,d \neq a,t$.*

(i) $F_{aa} = H$,

(ii) $F_{at} = H \Rightarrow a = t$,

(iii) $F_{at} = F_{ta}$,

(iv) $F_{at} \geq \gamma(F_{ai},\gamma(F_{id},F_{dt}))$ (Quadrilateral inequality)

If $\gamma = \gamma_T$ for some t-norm T, then (Z,F,γ_T) is labeled a generalized Menger space (GMS).

If T is a continuous t-norm, then (Z,F) satisfied (iv) under γ_T iff it satisfies,

(v) $F_{at}(q+e+f) \geq T(F_{ai}(q),T(F_{id}(t),F_{dt}(f)))$, for all $a,t \in Z$ and for all distinct points $i,d \in Z$, where $i,d \neq a,t$ about T.

Noted that a FMeS is a triple (Z,F,γ) hold (i)-(iii) and

(vi) $F_{ad} \geq \gamma(F_{at},F_{td})$ for all $a,t,d \in Z$.

$\therefore$ we'have $(vi)|rightarrow(iv)$. Then each PMS is a metric space $(PGMS)$.

3. Probabilistic s_b-Metric Spaces

Definition 3.1. *A Probabilistic s_b-Metric Spaces (Ps_bMs) is a quadruple (M,F,τ,s) here M is a nonempty set, F is a function from M^2 into Δ^+, τ is a triangle function, $S \geq 1$ is a real no. and the following conditions are satisfied; for all $x,y,z \in M$ and $p \geq 0$*

(i) $F_{xxx} = H_0$

(ii) $F_{xyz} = H \Rightarrow x = y = z$

(iii) $F_{xyz} = H_{xzy} = H_{yzx}$ for every x,y,z in M.

(iv) $F_{xyz}(sp) = \tau(F_{xyu},F_{xuz},F_{uyz})(p)$ for every x,y,z,u in M
If $\tau = \tau_T$ for a few t-norm T then (M,F,τ_T,s) is termed as s_b-MS.
If T is a continuous t-norm then (M,F) fulfils (iv) under τ_T iff it meets

(v) $F_{xyz}(s(p+q+r)) \geq T(F_{xyu}(p),F_{xuz}(q),F_{uyz}(r))$ for all $x,y,z \in M$ and for all $p,q,r > 0$ under T.

Remark 3.2. *Each PMS (PM-space) is a probabilistic s_b metric space with $s = 1$.*

Example 3.3. *Take $M = \{1,2,3,4\}$. Define $F : M^2 \to \Delta^+$ as follows:*

$$F_{(xyz)} = \begin{cases} H(t) & \text{if } x = y = z, \\ H(t-2) & \text{if } x,y,z \in \{1,2,3\}, x \neq y \neq z, \\ H(t-1) & \text{if } x,y,z \notin \{1,2,3\}, x \neq y \neq z. \end{cases}$$

Example 3.4. *Take $M = \{1,2,3,4\}$. Let $F : M \times M \to \Delta^+$ as follows:*

$$F_{(xyz)} = \begin{cases} H(t) & \text{if } x = y = z, \\ H(t-2) & \text{if } x,y,z \in \{1,2,3\}, x \neq y \neq z, \\ H(t-1) & \text{if } x,y,z \notin \{1,2,3\}, x \neq y \neq z. \end{cases}$$

Here (M,F,τ_{T_M}) is a probabilistic S_M metric space.

Example 3.5. *Let $M = [0,\infty)$. Consider $F : M^2 \to \Delta^+$ as follows:*

$F_{xyz}(t) = H(t - |x-y-z|^2)$. Here (M,F,τ_{T_M},s) is Ps_bMS, but (M,F,τ_{T_M}) is not a ideal PMS because it loss the triangle inequality.

Example 3.6. *Let $A = \{0,2\}$, $B = \{\frac{1}{n}, n \in N\}$, $C = \{1,2,3,4,..\}$, $M = A \cup B \cup C$. Consider $F : M \times M \to \Delta^+$ as*

$$F_{(xyz)} = \begin{cases} H(t) & \text{if } x = y = z, \\ H(t-1) & \text{if } x,y,z \in A \text{ or } x,y,z \in B \text{ or } x,y,z \in C \\ x \neq y \neq z, & \\ H(t-Z) & \text{if } x \in A, \ y \in B, \ z \in C. \end{cases}$$

Here (M,F,τ_{T_M},s) is Ps_bMS with τ_{T_M} continuous in which there does not exists $t > 0$ such that $N_0(t) \cap N_2(t) = \emptyset$. Hence (M,F,τ_{T_M}) enriched with the topology τ is not a Hausdroff topological space.

Definition 3.7. *Take (M,F) be a probabilistic semi -s_bMS (PSs_b-MS). Take any p in M and $t>0$ the strong neighbourhood p be the set $N_p(t)=\{q\in M : F_{pqr}(t)>1-t\}$.*

The strong neighbourhood system at p is the collection $P_p=\{N_p(t):t>0\}$ and the strong neighbourhood system for M is the union $P=\bigcup_{p\in M}P_p$.

Definition 3.8. *Let $\{x_n\}$ be a sequence in a probabilistic s_b semi metric space (M,F).*

(i) A sequence $\{x_n\}$ in M be called as Convergent to x in M if $\forall\ \varepsilon>0$ and $\delta\in(0,1)$ $\exists$ a positive integer $N(\varepsilon,\delta)$ such that $F_{x_n,x_m,x}(\varepsilon)>1-\delta$ whenever $n\geq N(\varepsilon,\delta)$.

(ii) A sequence $\{x_n\}$ in M is termed as Cauchy sequence if $\forall\ \varepsilon>0$ and $\delta\in(0,1)$ $\exists$ a positive integer $N(\varepsilon,\delta)$ such that $F_{x_n,x_m,x}(\varepsilon)>1-\delta$ whenever $m,p\geq N(\varepsilon,\delta)$. (M,F) is said to be Complete if every Cauchy sequence has a limit.

(iii) (M,F) is called as be Complete if for all CS has a limit.

Lemma 3.9. *Consider (M,F) as a Ps_bMS. Let $F:M^3\Delta^+$ by $F_{pqr}(t)=H(t-F(p,q,r))$, is a constant. Then*

(i) (M,F,τ_{T_M},s) is a Ps_bMS.

(ii) $((M,F,\tau_{T_M})$ is complete if and only if (M,d) is complete.

Proof. (i) It is fair that statements (i),(ii) and (iii) of probabilistic s_b-MS are fulfilled by F. For condition (v), let x,y,z in M, let t_1,t_2,t_3 in $[0,\infty]$.

If $Min(F_{xyu}(t_1),F_{xuz}(t_2),F_{uyz}(t_3))=0$ then

$F_{xyz}(s(t_1+t_2+t_3))\geq Min(F_{xyu}(t_1),F_{xuz}(t_2),F_{uyz}(t_3))$.

If $Min(F_{xyu}(t_1),F_{xuz}(t_2),F_{uyz}(t_3))=1$ then $t_1>F(x,y,u)$, $t_2>F(x,u,z)$ and $t_3>F(u,y,z)$.

Since (M,F) is a s_b-MS with constant 's', we've

$F(x,y,z)\leq s(F(x,y,u)+F(x,u,z)+F(u,y,z))<s(t_1+t_2+t_3)$

This implies $F_{xyz}(s(t_1+t_2+t_3))=1$.

Thus $F_{xyz}(s(t_1+t_2+t_3))\geq Min(F_{xyu}(t_1),F_{xuz}(t_2),F_{uyz}(t_3))$.

Hence condition (v) holds. Thus (M,F,τ_{T_M},s) is a probabilistic s_b metric space.

(ii) It is easy to check that $N_p(t)=\{q\in M:d(p,q)<t\}$. So (M,F,τ_{T_M}) is a complete Ps_bMS if and only if (M,F,t) is a complete s_b MS.

□

Example 3.10. *Consider (M,F,T) be a MS and $F'(x,y,z)=(F(x,y,z))^n$ where $n>1$ is a real number. Then F is a s_b-metric with $S=2n-1$. Clearly conditions (i) and (ii) of Ps_bMS are fulfilled.*

If $1<n<\infty$ then the convexity of the function $f(x)=x^n$ $(x>0)$ implies,

$$\begin{aligned}F'(x,y,z) &= (F(x,y,z))^n\\ &\leq [F(x,y,u)+F(x,u,z)+F(u,y,z)]^n\\ &\leq 2^{n-1}[F(x,y,u)+F(x,u,z)+F(u,y,z)]^n\\ &= 2^{n-1}[F'(x,y,u)+F'(x,u,z)+F'(u,y,z)]\end{aligned}$$

for each $x,y,z\in M$. Therefore condition (iii) of probabilistic s_b-MS is also satisfied and $(M,F,\tau_{T_M},2^{n-1})$ is a Ps_bMs with $F_{xyz}(t)=H(t-F'(x,y,z))$.

Remark 3.11. *In a PMS (M,F,τ) with τ being continuous M is enriched with the topology τ and M^2 with the corresponding product topology. Then the PM F is a continuous mapping from M^2 into Δ^+.*

However in a Ps_bMS (M,F,τ), the Ps_bMS F is not continuous in general even though τ is continuous, by the coming example

Example 3.12. *Let $M=N\cup\infty\ 0<b\leq 1$. Consider $F:M^2\to\Delta^+$ as follows:*

$$F^b_{(xyz)}(t)=\begin{cases}H(t) & \text{if } x=y=z,\\ H(t-7) & \text{if } x,y,z \text{ are odd and}\\ x\neq y\neq z, & \\ H(t-|\frac{b}{x}-\frac{b}{y}-\frac{b}{z}|) & \text{if } x,y,z \text{ are even and}\\ xyz=\infty, & \\ H(t-4) \ \ \text{Otherwise.} & \end{cases}$$

Here $(M,F^b,\tau_{T_M}),3)$ is a pS_bms with τ_{T_M} being continuous and take $a=1$.

Consider the sequence $x_n=3n,n\in N$. Then $F_{3n\infty}=H(t-\frac{1}{3n})$. Therefore $x_{n\to\infty}$ but $F_{3n1}(t)=H(t-4)\neq H(t-1)=F_{\infty 11}(t)$. Hence F is not continuous at ∞.

Theorem 3.13. *Take (X,F,T) be a probabilistic Menger s_b-MS under a continuous t-norm T such that $T\geq T_M$ and let d be the mapping from x^3 into R defined by $d_{s_b}(x,y,z)=sup\{\varepsilon\in[0,1):F_{x,y,z}(\varepsilon)\geq 1-\varepsilon\}$. Then we have*

(i) $d_{s_b}(x,y,z)<t$ if and only if $F_{x,y,z}(t)>1-t$

(ii) (X,d) is a D-metric space.

(iii) The Ps_b-MS F is convergent under the metric d.

Proof. (i) If $t>1$ then $d_{s_b}(x,y,z)\geq 1<t$ and also $F_{x,y,z}(t)\geq 0>1-t$. Suppose $d_{s_b}(x,y,z)<t\geq 1$ and choose δ such that $d_{s_b}(x,y,z)<\delta<t\geq 1$. Then $F_{x,y,z}(t)\geq F_{x,y,z}(\delta)>1-\delta>1-t$.

Conversely, suppose that $F_{x,y,z}(t)>1-t$ here $0<t\geq 1$. Then $\exists$ a δ_0 such that $d_{s_b}(x,y,z)<\delta_0<t$ where $0<\delta_0<t$.

This implies that $F_{x,y,z}(t)=\lim_T(\delta\to t^-)F_{x,y,z}(\delta)\geq\lim_T(\delta\to t^-)(1-\delta)=1-t$, $d_{s_b}(x,y,z)<\delta_0<t$.

(ii) If $x=y=z$ then $F_{x,y,z}=H_0$ and $d_{s_b}(x,y,z)=sup\{0\}=0$. Also if $d_{s_b}(x,y,z)=0$ then $F_{x,y,z}=H_0$ when $x=y=z$. So $d_{s_b}(x,y,z)=0 \Rightarrow x=y=z$.Symmetry is also true

Next we Show that d satisfies the tetrahedral inequality for a s_b metric space. It is enough to display that $d(x,y,z)<\varepsilon$ for that if $\delta_i, i=1,2,3$ such that $d(x,y,u)<\varepsilon_1, d(x,u,z)<\varepsilon_2, d(u,y,z)<\varepsilon_3 \Rightarrow d(x,y,z)<\varepsilon$ where $\varepsilon_1+\varepsilon_2+\varepsilon_3=\varepsilon$. We have

$$\begin{aligned} F_{x,y,z}(\varepsilon) \geq & \quad F_{x,y,z}(\delta_i) \\ \geq & \quad F_{x,y,z}(\delta_1+\delta_2+\delta_3) \\ \geq & \quad T(F_{x,y,u}(\delta_1), F_{x,y,u}(\delta_2), F_{dx,y,ud}(\delta_3)) \\ \geq & \quad T_m(1-\delta_1, 1-\delta_2, 1-\delta_3) \\ \geq & \quad 1-(\delta_1+\delta_2+\delta_3) \\ > & \quad 1-\varepsilon \end{aligned}$$

By (i) it follows that $d(x,y,z)<\varepsilon$. Thus the mapping d satisfies the Tetrahedral inequality.

(iii) This part follows from (i) and (ii). That is $d_{s_b}(x,y,z)<t$ if and only if $F_{x,y,z}(\varepsilon)>1-t$. Let $\{x_n\}$ be a sequence in X. Then given $\varepsilon>0 \ \exists$ a $t\in(0,1)$ such that $F_{x,y,z}(\varepsilon)>1-t$ by (i).

By (i), sequence $\{x_n\}$ converges to x in X, under the metric d.

□

Lemma 3.14. *Let (M,F,τ,s) be a Ps_bMS. If τ is continuous then the strong neighbourhood system $\mathscr{P}$ satisfies the following condition. If $x\neq y\neq z$ then there are $t_1,t_2,t_3>0$ such that $N_x(t_1)\cap N_y(t_2)\cap N_z(t_3)=\emptyset$.*

Proof. By the uniform continuity of τ, $\exists\ t>0$ such that $d_L(\tau(G_1,G_2,G_3),H)<\mathscr{P}$, whenever $d_L(G_1,H)<t_1, d_L(G_2,H)<t_2$ and $d_L(G_3,H)<t_3$.

Suppose that $N_x(t_1)\cap N_y(t_2)\cap N_z(t_3)=\emptyset$. So let $r\in N_x(t_1)\cap N_y(t_2)\cap N_z(t_3)$. Then $d_L(F_{xyu},H)<t_1$, $d_L(F_{xuz},H)<t_2$ and $d_L(F_{uyz},H)<t_3$ whence $d_L(F_{xyz},H)\leq d_L(\tau(F_{xyu},F_{xuz},F_{uyz},H))<\mathscr{P}=d_L(F_{xyz},H)$, which is a contradiction. Hence $N_x(t_1)\cap N_y(t_2)\cap N_z(t_3)=\emptyset$ and the proof is complete. □

Lemma 3.15. *Consider h be a distance distributive function in D^+. If $\exists\ \phi\in\varphi$ such that $h(t)\leq h(\phi(t))\ \forall\ t>0$ then $h=H$.*

Proof. Since h is monotonous, $h(t)\leq h(\varphi(t))$ implies that $h(t)=h(\varphi(t))$. Let $t>0$ and let us take $h(t)=h(\varphi(t))$. Then for each $n\geq 1$, $h(t)=g(\varphi^n(t))$. Now we shall show that $h(t)=1$.

Suppose to the inconsistent that $\exists$ a $t_0>0$ such that $h(t_0)<1$. Since $h\in D^+$ then $h(t)\to 1$ as $t\to\infty$, then $\exists$ a positive integer $n>1$ such that $\varphi^n(t_1)<t_0$. Since h is monotonous we have $h(t_0)\geq g(\varphi^n(t_1))$.

Thus $h(t_1)=h(\varphi^n(t_1))\leq h(t_0)$, which is a contradiction. Therefore $h(t)=1$, since $h\in D^+$. Hence $h=H$. □

4. Conclusion

The main result of this paper is the performance of proposing the image of Ps_b-MS as a conjecture of $Ps-MS$ and $b-MS$ an examine the topological properties of the same space.

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