Malaya Journal of Matematik, Vol. 8, No. 3, 939-944, 2020
https://doi.org/10.26637/MJM0803/0035

Cassini determinant involving the (a,b)-hyper-Fibonacci numbers

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Abstract
In the present paper, we establish some combinatorial properties of the (a,b)-hyper-Fibonacci numbers in order to extend the Cassini determinant.

Keywords
Generalized Fibonacci numbers; Generalized hyper-Fibonacci numbers; Cassini determinant.

AMS Subject Classification
11B39, 11B37, 15B36.

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Article History: Received **19** October **2019**; Accepted **22** June **2020**

Contents

1. Introduction

Let $(b_k)_{k\geq 0}$ and $(c_k)_{k\geq 0}$ be two sequences satisfying the following recurrence relation $a_{n+2} = \alpha a_{n+1} + \beta a_n$, where α and β are integers. According to [1], we have the identity

$$b_n c_{n-1} - b_{n-1} c_n = (-\beta)^{n-1}(b_1 c_0 - b_0 c_1). \tag{1.1}$$

If we take $b_n = F_{n+2}$ and $c_n = F_{n+1}$, then identity (1.1) reduces to

$$F_n F_{n+2} - F_{n+1}^2 = (-1)^{n+1}, \tag{1.2}$$

where (F_n) denote the well known Fibonacci numbers. Identity (1.2) is called the Cassini identity [2–4], we can write it as a 2×2 determinant

$$\begin{vmatrix} F_n & F_{n+1} \\ F_{n+1} & F_{n+2} \end{vmatrix} = (-1)^{n+1}. \tag{1.3}$$

Martinjak and Urbiha [5] extend the Cassini determinant (1.3) to the hyper-Fibonacci numbers defined by

$$F_n^{(r+1)} = \sum_{k=0}^{n} F_k^{(r)}, \quad F_n^{(0)} = F_n, \quad F_0^{(r)} = 0, \quad F_1^{(r)} = 1, \tag{1.4}$$

where r is a nonnegative integer. The number $F_n^{(r)}$ is called the nth hyper-Fibonacci number of the rth generation. Hyper-Fibonacci numbers were introduced by Dil and Mező [6], they satisfy many interesting number-theoretical and combinatorial properties, e.g. [7]. Martinjak and Urbiha [5] define the matrix

$$A_{r,n} = \begin{pmatrix} F_n^{(r)} & F_{n+1}^{(r)} & \cdots & F_{n+r+1}^{(r)} \\ F_{n+1}^{(r)} & F_{n+2}^{(r)} & \cdots & F_{n+r+2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n+r+1}^{(r)} & F_{n+r+2}^{(r)} & \cdots & F_{n+2r+2}^{(r)} \end{pmatrix}$$

and prove that $\det(A_{r,n}) = (-1)^{n+\lfloor (r+3)/2 \rfloor}$, where $n \geq 0$ and $r \geq 0$ are integers. It is clear that for $r = 0$ we find (1.3).

In this paper we consider the (a,b)-Fibonacci numbers $(G_n)_{n\geq 0}$ defined by

$$\begin{cases} G_0 = a, \ G_1 = b, \\ G_{n+2} = G_{n+1} + G_n, \qquad (n \geq 0) \end{cases} \tag{1.5}$$

where a and b are any integers. If we take $b_n = G_{n+2}$ and $c_n = G_{n+1}$, then identity (1.1) reduces to

$$G_n G_{n+2} - G_{n+1}^2 = (-1)^{n-1}(b^2 - ab - a^2). \tag{1.6}$$

In Section 2 we define the (a,b)-hyper-Fibonacci numbers associated to the sequence $(G_n)_{n\geq 0}$ and we give some properties. In Section 3 we extend identity (1.6) to these generalized hyper-Fibonacci numbers.

Throughout this paper we denote by C_n^k the binomial coefficient which is defined for a nonnegative integer n and an integer k by

$$C_n^k = \begin{cases} \dfrac{n!}{k!(n-k)!} & \text{if } 0 \leq k \leq n \\ 0 & \text{othewise} \end{cases} \tag{1.7}$$

For a negative integer n and an integer k we have

$$C_n^k = \begin{cases} (-1)^k C_{-n+k-1}^k & \text{if } k \geq 0 \\ (-1)^{n-k} C_{-k-1}^{n-k} & \text{if } k \leq n \\ 0 & \text{othewise} \end{cases} \tag{1.8}$$

2. The (a,b)-hyper-Fibonacci numbers

The (a,b)-hyper-Fibonacci numbers associated to the sequence $(G_n)_{n\geq 0}$ are defined by

$$G_n^{(r+1)} = \sum_{k=0}^{n} G_k^{(r)}, \quad G_n^{(0)} = G_n, \quad G_0^{(r)} = a, \quad G_1^{(r)} = ar+b, \tag{2.1}$$

where r be a nonnegative integer. The number $G_n^{(r)}$ is called the nth (a,b)-hyper-Fibonacci number of the rth generation.

In this section we give some properties satisfied by the (a,b)-hyper-Fibonacci numbers.

Lemma 2.1. *Let $n \geq 0$ be an integer, then*

$$G_n^{(1)} = G_{n+2} - b. \tag{2.2}$$

Proof. By induction on n. For $n = 0$, identity (2.2) is trivially checked. Now assume that (2.2) is true for an integer $n \geq 0$, then

$$\begin{aligned} G_{n+1}^{(1)} &= \sum_{k=0}^{n+1} G_k \\ &= \sum_{k=0}^{n} G_k + G_{n+1} \\ &= G_n^{(1)} + G_{n+1} \\ &= G_{n+2} - b + G_{n+1} \\ &= G_{n+3} - b. \end{aligned}$$

We conclude that (2.2) is true for all $n \geq 0$. □

The following proposition expresses an (a,b)-hyper-Fibonacci number of any generation $r \geq 1$ in terms of (a,b)-Fibonacci numbers.

Proposition 2.2. *Let $r \geq 1$ be an integer, then*

$$G_n^{(r)} = G_{n+2r} - \sum_{l=0}^{r-1} C_{n+r-l-1}^{r-l-1} G_{2l+1}, \quad n \geq 0. \tag{2.3}$$

Proof. We deduce from Lemma 2.1 that (2.3) is true for $r = 1$. Now assume that (2.3) is true for an integer $r \geq 1$, then

$$\begin{aligned} G_n^{(r+1)} &= \sum_{k=0}^{n} G_k^{(r)} \\ &= \sum_{k=0}^{n} \left(G_{k+2r} - \sum_{l=0}^{r-1} C_{k+r-l-1}^{r-l-1} G_{2l+1} \right) \\ &= \sum_{k=0}^{n} G_{k+2r} - \sum_{k=0}^{n} \sum_{l=0}^{r-1} C_{k+r-l-1}^{r-l-1} G_{2l+1} \\ &= \sum_{l=0}^{n+2r} G_l - \sum_{l=0}^{2r-1} G_l - \sum_{l=0}^{r-1} G_{2l+1} \sum_{k=0}^{n} C_{k+r-l-1}^{r-l-1} \\ &= G_{n+2r}^{(1)} - G_{2r-1}^{(1)} - \sum_{l=0}^{r-1} C_{n+r-l}^{r-l} G_{2l+1} \\ &= G_{n+2r+2} - G_{2r+1} - \sum_{l=0}^{r-1} C_{n+r-l}^{r-l} G_{2l+1} \\ &= G_{n+2r+2} - \sum_{l=0}^{r} C_{n+r-l}^{r-l} G_{2l+1}. \end{aligned}$$

We deduce that (2.3) is true for all $r \geq 1$.

□

The next proposition expresses an (a,b)-hyper-Fibonacci number of any positive generation in terms of an (a,b)-hyper-Fibonacci number of the preceding generation.

Proposition 2.3. *Let $r \geq 0$ be an integer, then*

$$G_n^{(r+1)} = G_{n+2}^{(r)} - aC_{n+r+1}^{r-1} - bC_{n+r+1}^{r}, \quad n \geq 0. \tag{2.4}$$

Proof. We deduce from Lemma 2.1 that (2.4) is true for $r = 0$. Now assume that (2.4) is true for an integer $r \geq 0$, then

$$\begin{aligned} G_n^{(r+2)} &= \sum_{k=0}^{n} G_k^{(r+1)} \\ &= \sum_{k=0}^{n} \left(G_{k+2}^{(r)} - aC_{k+r+1}^{r-1} - bC_{k+r+1}^{r} \right) \\ &= \sum_{k=0}^{n} G_{k+2}^{(r)} - a\sum_{k=0}^{n} bC_{k+r+1}^{r-1} - b\sum_{k=0}^{n} bC_{k+r+1}^{r} \\ &= \sum_{l=2}^{n+2} G_l^{(r)} - a\sum_{l=2}^{n+2} C_{l+r-1}^{r-1} - b\sum_{l=2}^{n+2} C_{l+r-1}^{r} \\ &= \sum_{l=0}^{n+2} G_l^{(r)} - a - ar - b - a\sum_{l=2}^{n+2} C_{l+r-1}^{r-1} - b\sum_{l=2}^{n+2} C_{l+r-1}^{r} \\ &= \sum_{l=0}^{n+2} G_l^{(r)} - a\sum_{l=0}^{n+2} C_{l+r-1}^{r-1} - b\sum_{l=1}^{n+2} C_{l+r-1}^{r} \\ &= G_{n+2}^{(r+1)} - aC_{n+r+2}^{r} - bC_{n+r+2}^{r+1}. \end{aligned}$$

We deduce that (2.4) is true for all $r \geq 0$. □

We get the following corollary as a simple and immediate consequence, it allows us to define the (a,b)-hyper-Fibonacci numbers of negative subscripts.

Corollary 2.4. *Let $r \geq 0$ and $n \geq 0$ be integers, then*

$$G_n^{(r+1)} = G_{n+2}^{(r+1)} - G_{n+1}^{(r+1)} - aC_{n+r+1}^{r-1} - bC_{n+r+1}^{r}. \tag{2.5}$$

Proof. According to definition (2.1), we have

$$G_{n+2}^{(r)} = G_{n+2}^{(r+1)} - G_{n+1}^{(r+1)}. \quad (2.6)$$

Replacing in (2.4) we obtain

$$G_n^{(r+1)} = G_{n+2}^{(r+1)} - G_{n+1}^{(r+1)} - aC_{n+r+1}^{r-1} - bC_{n+r+1}^{r}.$$

□

Remark 2.5. *For $r \geq 1$, the the (a,b)-hyper-Fibonacci numbers for negative subscripts are defined as*

$$G_{-n}^{(r+1)} = G_{-n+2}^{(r+1)} - G_{-n+1}^{(r+1)} - aC_{-n+r+1}^{r-1} - bC_{-n+r+1}^{r}, \quad n > 0.$$

It is easy to see that

$$G_{-n}^{(r)} = 0 \quad \text{for } 1 \leq n \leq r \quad \text{and} \quad G_{-r-1}^{(r)} = (-1)^{r-1}(a-b).$$

The following proposition is the key assertion behind the computation of Cassini determinant.

Proposition 2.6. *Let $r \geq 0$ be an integer, then*

$$G_{n+r+2}^{(r)} = \sum_{k=0}^{r+1} (-1)^{r-k} \left(C_r^k - C_{r+1}^{k-1} \right) G_{n+k}^{(r)}, \qquad n \geq -r. \quad (2.7)$$

Proof. Let us show identity (2.7) by induction on $r \geq 0$. For $r = 0$ we get $G_{n+2} = G_{n+1} + G_n$ for $n \geq 0$ which is true by definition of the sequence $(G_n)_n$. Now assume that (2.7) is true for an integer $r \geq 0$, since $n+1 \geq n \geq -r$, we have

$$G_{n+r+3}^{(r)} = \sum_{k=0}^{r+1} (-1)^{r-k} \left(C_r^k - C_{r+1}^{k-1} \right) G_{n+k+1}^{(r)}. \quad (2.8)$$

Since $n+r+3 \geq n+r+2 \geq 0$, we have $G_{n+r+3}^{(r)} = G_{n+r+3}^{(r+1)} - G_{n+r+2}^{(r+1)}$. For $k = 0, 1 \ldots, r+1$, we have $G_{n+k+1}^{(r)} = G_{n+k+1}^{(r+1)} - G_{n+k}^{(r+1)}$ because

- If $n+k+1 < 0$ then we obtain $0 = 0 - 0$.
- If $n+k+1 \geq 0$ and $n+k < 0$ then $n+k = -1$, we obtain $a = a - 0$.
- If $n+k \geq 0$ then $n+k+1 > 0$ and we obtain $G_{n+k+1}^{(r)} = G_{n+k+1}^{(r+1)} - G_{n+k}^{(r+1)}$.

Thus, we get from (2.8) that

$$G_{n+r+3}^{(r+1)} - G_{n+r+2}^{(r+1)} = \sum_{k=0}^{r+1} (-1)^{r-k} \left(C_r^k - C_{r+1}^{k-1} \right) \left(G_{n+k+1}^{(r+1)} - G_{n+k}^{(r+1)} \right).$$

We deduce that

$$\begin{aligned}
& G_{n+r+3}^{(r+1)} \\
= \; & G_{n+r+2}^{(r+1)} + \sum_{k=0}^{r+1} (-1)^{r-k} \left(C_r^k - C_{r+1}^{k-1} \right) G_{n+k+1}^{(r+1)} \\
& + \sum_{k=0}^{r+1} (-1)^{r+1-k} \left(C_r^k - C_{r+1}^{k-1} \right) G_{n+k}^{(r+1)} \\
= \; & (r+2) G_{n+r+2}^{(r+1)} + \sum_{k=0}^{r} (-1)^{r-k} \left(C_r^k - C_{r+1}^{k-1} \right) G_{n+k+1}^{(r+1)} \\
& + (-1)^{r+1} G_n^{(r+1)} + \sum_{k=1}^{r+1} (-1)^{r+1-k} \left(C_r^k - C_{r+1}^{k-1} \right) G_{n+k}^{(r+1)} \\
= \; & (r+2) G_{n+r+2}^{(r+1)} + \sum_{l=1}^{r+1} (-1)^{r+1-l} \left(C_r^{l-1} - C_{r+1}^{l-2} \right) G_{n+l}^{(r+1)} \\
& + (-1)^{r+1} G_n^{(r+1)} + \sum_{k=1}^{r+1} (-1)^{r+1-k} \left(C_r^k - C_{r+1}^{k-1} \right) G_{n+k}^{(r+1)} \\
= \; & \sum_{k=0}^{r+2} (-1)^{r+1-k} \left(C_r^{k-1} + C_r^k - C_{r+1}^{k-2} - C_{r+1}^{k-1} \right) G_{n+k}^{(r+1)} \\
= \; & \sum_{k=0}^{r+2} (-1)^{r+1-k} \left(C_{r+1}^k - C_{r+2}^{k-1} \right) G_{n+k}^{(r+1)}.
\end{aligned}$$

We conclude that (2.7) is true for all $r \geq 0$. □

3. Cassini determinant for (a,b)-hyper-Fibonacci numbers

Cassini identity (1.6) can be expressed as a determinant in the following way

$$\begin{vmatrix} G_n & G_{n+1} \\ G_{n+1} & G_{n+2} \end{vmatrix} = (-1)^{n-1}(b^2 - ab - a^2).$$

For $n, r \in \mathbb{Z}$ such that $n \geq 0$ and $r \geq 0$, let's define the $(r+2) \times (r+2)$ matrix

$$C_{r,n} = \begin{pmatrix} G_n^{(r)} & G_{n+1}^{(r)} & \cdots & G_{n+r+1}^{(r)} \\ G_{n+1}^{(r)} & G_{n+2}^{(r)} & \cdots & G_{n+r+2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ G_{n+r+1}^{(r)} & G_{n+r+2}^{(r)} & \cdots & G_{n+2r+2}^{(r)} \end{pmatrix}.$$

Note that

$$C_{0,n} = \begin{pmatrix} G_n & G_{n+1} \\ G_{n+1} & G_{n+2} \end{pmatrix}.$$

Our aim is to evaluate the determinant of the matrix $C_{r,n}$.

From Proposition 2.6, we can write

$$G_{n+r+2}^{(r)} = \sum_{k=0}^{r+1} q_k G_{n+k}^{(r)}, \qquad n \geq -r,$$

where

$$q_k = (-1)^{r-k} \left(C_r^k - C_{r+1}^{k-1} \right), \quad 0 \leq k \leq r+1.$$

Let

$$V_{r+2}=\begin{pmatrix}0&1&0&\cdots&0&0\\0&0&1&\cdots&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots\\0&0&0&\cdots&1&0\\0&0&0&\cdots&0&1\\q_0&q_1&q_2&\cdots&q_r&q_{r+1}\end{pmatrix}.$$

Thus, we deduce from Proposition 2.6 that the (a,b)-hyper-Fibonacci numbers $\left(G_n^{(r)}\right)_n$ can be defined by the vector recurrence relation

$$\begin{pmatrix}G_{n+1}^{(r)}\\G_{n+2}^{(r)}\\\vdots\\G_{n+r+2}^{(r)}\end{pmatrix}=V_{r+2}\begin{pmatrix}G_{n}^{(r)}\\G_{n+1}^{(r)}\\\vdots\\G_{n+r+1}^{(r)}\end{pmatrix},\tag{3.1}$$

where $n+r\geq 0$.

Lemma 3.1. *Let n and r be nonnegative integers, then*

$$C_{r,n}=V_{r+2}^{n}C_{r,0}.$$

Proof. From relation (3.1) we can write $C_{r,n}=V_{r+2}C_{r,n-1}$. It follows that

$$C_{r,n}=V_{r+2}C_{r,n-1}=V_{r+2}^{2}C_{r,n-2}=\cdots=V_{r+2}^{n}C_{r,0}.$$

□

Lemma 3.2. *Let r be a nonnegative integer, then*

$$\det(V_{r+2})=-1.$$

Proof. It is clear that

$$\det(V_{r+2})=(-1)^{r+3}q_0=(-1)^{r+3}(-1)^{r+2}=-1.$$

□

Theorem 3.3. *Let n and r be nonnegative integers, then*

$$\det(C_{r,n})=(-1)^{n+\lfloor (r+3)/2\rfloor}(b^2-ab-a^2)b^r.\tag{3.2}$$

Proof. For $r=0$ the result follows from identity (1.6). Thus, assume that $r\geq 1$. We deduce from (3.1) that multiplication by V_{r+2}^{-1} decreases by 1 the subscript of each component, i.e.,

$$V_{r+2}^{-1}C_{r,0}=\begin{pmatrix}G_{-1}^{(r)}&G_{0}^{(r)}&\cdots&G_{r}^{(r)}\\G_{0}^{(r)}&G_{1}^{(r)}&\cdots&G_{r+1}^{(r)}\\\vdots&\vdots&\ddots&\vdots\\G_{r}^{(r)}&G_{r+1}^{(r)}&\cdots&G_{2r+1}^{(r)}\end{pmatrix}.$$

Thus,

$$V_{r+2}^{-r}C_{r,0}=\begin{pmatrix}G_{-r}^{(r)}&G_{1-r}^{(r)}&\cdots&G_{1}^{(r)}\\G_{1-r}^{(r)}&G_{2-r}^{(r)}&\cdots&G_{2}^{(r)}\\\vdots&\vdots&\ddots&\vdots\\G_{1}^{(r)}&G_{2}^{(r)}&\cdots&G_{r+2}^{(r)}\end{pmatrix}.$$

Since $G_{-n}^{(r)}=0$ for $1\leq n\leq r$, then

$$V_{r+2}^{-r}C_{r,0}=\begin{pmatrix}0&\cdots&G_{0}^{(r)}&G_{1}^{(r)}\\\vdots&\vdots&⋰&\vdots\\G_{0}^{(r)}&G_{1}^{(r)}&\cdots&G_{r+1}^{(r)}\\G_{1}^{(r)}&G_{2}^{(r)}&\cdots&G_{r+2}^{(r)}\end{pmatrix}.$$

Thus,

$$\det(C_{r,0})=\det(V_{r+2})^r\cdot\Delta,\tag{3.3}$$

where

$$\Delta=\begin{vmatrix}0&\cdots&G_{0}^{(r)}&G_{1}^{(r)}\\\vdots&\vdots&⋰&\vdots\\G_{0}^{(r)}&G_{1}^{(r)}&\cdots&G_{r+1}^{(r)}\\G_{1}^{(r)}&G_{2}^{(r)}&\cdots&G_{r+2}^{(r)}\end{vmatrix}.$$

Let L_j denotes the jth Line of Δ where $j=1,2,\ldots,r+2$. First, we replace L_{i+1} by $L_{i+1}-L_i$ for $i=r+1,r,\ldots,1$. Since $G_{i+1}^{(r-1)}=G_{i+1}^{(r)}-G_{i}^{(r)}$, we get

$$\Delta=\begin{vmatrix}0&0&\cdots&G_{0}^{(r-1)}&G_{1}^{(r-1)}+a\\0&0&\cdots&G_{1}^{(r-1)}&G_{2}^{(r-1)}\\\vdots&\vdots&⋰&\vdots&\vdots\\G_{0}^{(r-1)}&G_{1}^{(r-1)}&\cdots&G_{r}^{(r-1)}&G_{r+1}^{(r-1)}\\G_{1}^{(r-1)}&G_{2}^{(r-1)}&\cdots&G_{r+1}^{(r-1)}&G_{r+2}^{(r-1)}\end{vmatrix}.$$

Using the same method $(r-1)$ times again, we obtain

$$\Delta=\begin{vmatrix}0&0&\cdots&0&G_0&G_1+d_1\\0&0&\cdots&G_0&G_1&G_2+d_2\\\vdots&\vdots&\vdots&⋰&\vdots&\vdots\\0&G_0&\cdots&G_{r-2}&G_{r-1}&G_r+d_r\\G_0&G_1&\cdots&G_{r-1}&G_r&G_{r+1}\\G_1&G_2&\cdots&G_r&G_{r+1}&G_{r+2}\end{vmatrix},$$

where $d_i=a(-1)^{i-1}C_r^i$ for $1\leq i\leq r$. Now let C_j denotes the jth column of this last determinant, where $j=1,\ldots,r+2$. Replacing the column C_i by $C_i-C_{i-1}-C_{i-2}$ for $i=r+2,r+1,\ldots,3$ and using the fact that $G_i=G_{i-1}+G_{i-2}$ gives

$$\Delta=\begin{vmatrix}0&0&0&\cdots&0&G_0&G_1-G_0+d_1\\0&0&0&\cdots&G_0&G_1-G_0&d_2\\0&0&0&\cdots&G_1-G_0&0&d_3\\\vdots&\vdots&\vdots&⋰&\vdots&\vdots&\vdots\\0&G_0&G_1-G_0&\cdots&0&0&d_r\\G_0&G_1&0&\cdots&0&0&0\\G_1&G_2&0&\cdots&0&0&0\end{vmatrix}.$$

Now we permute the column C_i with column C_{r+3-i} for $1\leq i\leq\lfloor (r+2)/2\rfloor$, we obtain

$$\Delta=(-1)^{\lfloor\frac{r+2}{2}\rfloor}\begin{vmatrix}G_1-G_0+d_1&G_0&0&\cdots&0&0&0\\G_2&G_1-G_0&G_0&\cdots&0&0&0\\G_3&0&G_1-G_0&\cdots&0&0&0\\\vdots&\vdots&\vdots&\ddots&\vdots&\vdots&\vdots\\d_r&0&0&\cdots&G_1-G_0&G_0&0\\0&0&0&\cdots&0&G_1&G_0\\0&0&0&\cdots&0&G_2&G_1\end{vmatrix}.$$

We deduce that

$$\Delta = (-1)^{\lfloor \frac{r+2}{2} \rfloor}\Delta' \begin{vmatrix} G_1 & G_0 \\ G_2 & G_1 \end{vmatrix}, \tag{3.4}$$

where

$$\Delta' = \begin{vmatrix} d_1+b-a & a & 0 & \cdots & 0 & 0 \\ d_2 & b-a & a & \cdots & 0 & 0 \\ d_3 & 0 & b-a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{r-1} & 0 & 0 & \cdots & b-a & a \\ d_r & 0 & 0 & \cdots & 0 & b-a \end{vmatrix}.$$

We distinguish two cases for the compute of Δ'.

- If $a = b \neq 0$, let C_j denotes the jth column of Δ' where $j = 1, \ldots, r$. Replacing C_1 by $C_1 - \sum_{k=2}^{r} \frac{d_{k-1}}{a} C_k$ gives

$$\Delta' = \begin{vmatrix} 0 & a & 0 & \cdots & 0 & 0 \\ 0 & 0 & a & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a \\ d_r & 0 & 0 & \cdots & 0 & 0 \end{vmatrix} = (-1)^{r+1} d_r a^{r-1} = a^r.$$

- If $a \neq b$, let L_j denotes the jth line of Δ' where $j = 1, \ldots, r$. Replacing the line L_i by $L_i + \frac{a}{a-b} L_{i+1}$ for $i = r-1, \ldots, 1$ gives

$$\Delta' = \begin{vmatrix} (b-a) + \sum_{i=1}^{r} \left(\frac{a}{a-b}\right)^{i-1} d_i & 0 & \cdots & 0 & 0 \\ & b-a & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{r-1} + \frac{a}{a-b} d_r & 0 & \cdots & b-a & 0 \\ d_r & 0 & \cdots & 0 & b-a \end{vmatrix}.$$

We deduce that

$$\begin{aligned} \Delta' &= (b-a)^{r-1}\left[(b-a) + \sum_{i=1}^{r}\left(\frac{a}{a-b}\right)^{i-1} d_i\right] \\ &= (b-a)^{r-1}\left[(b-a) + a\sum_{i=1}^{r}\left(\frac{a}{b-a}\right)^{i-1} C_r^i\right] \\ &= (b-a)^{r-1}\left[(b-a) + (b-a)\sum_{i=1}^{r}\left(\frac{a}{b-a}\right)^{i} C_r^i\right] \\ &= (b-a)^{r-1}\left[(b-a)\sum_{i=0}^{r}\left(\frac{a}{b-a}\right)^{i} C_r^i\right] \\ &= (b-a)^{r}\left(\frac{a}{b-a}+1\right)^{r} \\ &= b^r, \end{aligned}$$

which coincides with the case $a = b \neq 0$. Since $\begin{vmatrix} G_1 & G_0 \\ G_2 & G_1 \end{vmatrix} = b^2 - ab - a^2$, we deduce from Identities (3.3), (3.4) and Lemmas 3.1, 3.2 that

$$\det(C_{r,n}) = (-1)^{n+\lfloor (3r+2)/2 \rfloor}(b^2 - ab - a^2) b^r.$$

It is easy to see that $(-1)^{\lfloor (3r+2)/2 \rfloor} = (-1)^{\lfloor (r+3)/2 \rfloor}$, thus

$$\det(C_{r,n}) = (-1)^{n+\lfloor (r+3)/2 \rfloor}(b^2 - ab - a^2) b^r.$$

□

Corollary 3.4. *Let n and r be nonnegative integers, then*

$$\begin{vmatrix} F_n^{(r)} & F_{n+1}^{(r)} & \cdots & F_{n+r+1}^{(r)} \\ F_{n+1}^{(r)} & F_{n+2}^{(r)} & \cdots & F_{n+r+2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ F_{n+r+1}^{(r)} & F_{n+r+2}^{(r)} & \cdots & F_{n+2r+2}^{(r)} \end{vmatrix} = (-1)^{n+\lfloor (r+3)/2 \rfloor}.$$

Proof. Follows from identity (3.2) for $a = 0$ and $b = 1$. □

The hyper-Lucas numbers associated to the well-known Lucas numbers $(L_n)_n$ are given by [6]

$$L_n^{(r+1)} = \sum_{k=0}^{n} L_k^{(r)}, \quad L_n^{(0)} = L_n, \quad L_0^{(r)} = 2, \quad L_1^{(r)} = 2r+1, \tag{3.5}$$

where r is a nonnegative integer. The following corollary extends the Cassini identity [1]

$$L_n L_{n+2} - L_{n+1}^2 = 5(-1)^n. \tag{3.6}$$

Corollary 3.5. *Let n and r be nonnegative integers, then*

$$\begin{vmatrix} L_n^{(r)} & L_{n+1}^{(r)} & \cdots & L_{n+r+1}^{(r)} \\ L_{n+1}^{(r)} & L_{n+2}^{(r)} & \cdots & L_{n+r+2}^{(r)} \\ \vdots & \vdots & \ddots & \vdots \\ L_{n+r+1}^{(r)} & L_{n+r+2}^{(r)} & \cdots & L_{n+2r+2}^{(r)} \end{vmatrix} = 5(-1)^{n+\lfloor (r+1)/2 \rfloor}.$$

Proof. Follows from identity (3.2) for $a = 2$ and $b = 1$.

□

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ISSN(P):2319 – 3786
Malaya Journal of Matematik
ISSN(O):2321 – 5666
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