Malaya Journal of Matematik, Vol. 8, No. 4, 1370-1374, 2020

https://doi.org/10.26637/MJM0804/0005

# On $g^*\beta$ -compactness and $g^*\beta$ -connectedness in topological spaces

A. Punitha Tharani[1] and H. Sujitha[2]*

**Abstract**

We introduce the concept of $g^*\beta-$compact and $g^*\beta-$ connected spaces in topological spaces and investigate their basic properties and are obtained using $g^*\beta$ -closed sets.

**Keywords**

$g^*\beta$ -closed set, $g^*\beta$ -connected space, $g^*\beta$-separated, $g^*\beta$ -compact space.

**AMS Subject Classification**

54B10, 54C10, 54D05, 54D30.

[1] *Department of Mathematics, St. Mary's College (Autonomous), Thoothukudi-628001, Tamil Nadu, India.*

[2] *Research Scholar (Part-time Internal) (Register Number: 19122212092003), Department of Mathematics, St. Mary's College (Autonomous), Thoothukudi-628001, Tamil Nadu, India.*

[1,2] *Affiliated to Manonmaniam Sundaranar University, Abishekapatti, Tirunelveli-627012, Tamil Nadu, India.*

*__Corresponding author__: [1] punitha_tharani@yahoo.co.in; [2]suji.sujitha16@gmail.com

**Article History**: Received **01** June **2020**; Accepted **21** August **2020**



## Contents



## 1. Introduction

Topological spaces are mathematical structures that allow the formal definitions of concepts such as connectedness, compactness, interior and closure. Compactness is one of the most important, useful and fundamental concepts in topology. Compactness is the generalization to topological spaces of the property of closed and bounded subsets of the real line. In 1970 , Levine [4] introduced the notion of generalized closed (briefly g-closed) sets in topological spaces. In 1981 , Dorsett [2] introduced and studied the concept of Semi compact spaces. Punitha Tharani. A and Sujitha. H [8] introduced the concept of $g^*\beta$-closed sets in topological spaces.

In this paper, we introduce the concept of $g^*\beta$ compactness and $g^*\beta$ -connectedness in topological spaces and also discuss some of their properties. For the concept of compact space and connected space, we refer [1,7,9].

## 2. $g^*\beta$- Compactness

**Definition 2.1.** *A subset $\mathscr{H}$ of a topological space $X$ is $g^*\beta$ - compact relative to $X$ if for every collection $\{S_i : i \in \Omega\}$ of $g^*\beta$ -open susbsets of $X$ such that $\mathscr{H} \subseteq \cup_{i\in\Omega} S_i$ there exists a finite subset $\Omega_0$ of $\Omega$ such that $\mathscr{H} \subseteq \cup_{i\in\Omega_0} S_i$ If $\mathscr{H} = X$ and if $\mathscr{H}$ is $g^*\beta$ - compact relative to $X$ then $X$ is $g^*\beta$ - compact.*

**Definition 2.2.** *A subset $\mathscr{H}$ of a topological space $X$ is said to be $g^*\beta$ -compact if $\mathscr{H}$ is $g^*\beta$ - compact as a subspaces of $X$.*

**Theorem 2.3.** *A $g^*\beta$ -closed subset $\mathscr{H}$ of as $g^*\beta$ compact space $X$ is $g^*\beta$ - compact relative to $X$.*

*Proof.* Let $\mathscr{H}$ be a $g^*\beta$ -closed subset of a $g^*\beta$ - compact space $X$. Then $X\backslash S$ is $g^*\beta$ -open. Let $\xi$ be a cover for $\mathscr{H}$ by $g^*\beta$ -open subsets of $X$. Since $X$ is $g^*\beta$ - compact it has finite subcover say $\{\mathscr{T}_1, \mathscr{T}_2, \mathscr{T}_3 \ldots\ldots, \mathscr{T}_n\}$. Then $\{\mathscr{J}_1, \mathscr{T}_2, \mathscr{T}_3 \ldots, \mathscr{T}_n\} \cup \{X\backslash\mathscr{H}\}$ is a finite subcover of $\xi$ for $\mathscr{H}$. Thus $\mathscr{H}$ is $g^*\beta$ - compact relative to $X$. ☐

**Theorem 2.4.** *Let $(X,\tau)$ and $(Y,\sigma)$ be topological spaces and $f:(X,\tau)\to(Y,\sigma)$ be a function. Then*

a) *$f$ is $g^*\beta$ -irresolute and $S$ is $g^*\beta$ -compact subset of $X \Rightarrow f(A)$ is $g^*\beta$ -compact subset of $Y$.*

b) *$f$ is one-one, $g^*\beta$ -resolute map and $\mathscr{T}$ is a $g^*\beta$ -compact subset $\Rightarrow f^{-1}(\mathscr{J})$ is $g^*\beta$ - compact subset of $X$.*

*Proof.* Proof follows from definition. □

**Theorem 2.5.** *Let $(X,\tau)$ and $(Y,\sigma)$ be topological spaces and $f:(X,\tau)\rightarrow(Y,\sigma)$ be a function.*

a) *$f$ is onto, $g^*\beta$ -irresolute and $X$ is $g^*\beta$ -compact which implies $Y$ is $g^*\beta$ - compact.*

b) *$f$ is bijection and $g^*\beta$ -resolute then $Y$ is $g^*\beta$ -compact which implies $X$ is $g^*\beta$ -compact.*

*Proof.* a) Let $\{G_i : i\in\mathscr{J}\}$ be a $g^*\beta$ -open cover for $Y$. Then $\left\{f^{-1}(G_i) : i\in\mathscr{J}\right\}$ is a $g^*\beta$ -open cover for $X$. since $X$ is $g^*\beta$ - compact, there exists $G_{i_1},G_{i_2},G_{i_3},\ldots\ldots,G_{i_n}$ such that $f^{-1}(G_{i_1})\cup f^{-1}(G_{i_2})\cup f^{-1}(G_{i_2})\ldots\ldots\cup f^{-1}(G_{i_n})=X$. Now $f\left(f^{-1}(G_{i_1})\cup f^{-1}(G_{i_2})\cup f^{-1}(G_{i_2})\ldots\ldots\ldots\cup\ f^{-1}(G_{i_n})\right)=f(X)$. That is $f(X)=ff^{-1}(G_{i_1})\cup ff^{-1}(G_{i_2})\cup ff^{-1}\left(G_{i_3}\right)\ldots\ldots ff^{-1}(G_{i_n})\subseteq G_{i_1}\cup G_{i_2}\cup\ G_{i_3}\cdots\cdots\cdots\cup G_{i_n}$ . Since $f$ is onto, $f(X)=Y$. Hence $Y$ is $g^*\beta$ -compact.
b) Proof is similar to that of (a). □

**Theorem 2.6.** *A topological space $(X,\tau)$ is $g^*\beta$ - compact iff any family of $g^*\beta - Cl(X)$ having finite intersection property have a nonempty intersection.*

*Proof.* Assume $X$ is $g^*\beta$ - compact. Let $\{S_i\}$ be a family of $g^*\beta - Cl(X)$ with finite intersection property. To prove that $\cap_i S_i\neq\Phi$. Suppose $\cap_i S_i=\Phi$. Then $X\backslash(\cap_i S_i)=X$ Hence $\cup_i(X\backslash S_i)=X$. Also since each $S_i$ is $g^*\beta-$ Closed $X\backslash S_i$ is $g^*\beta$ -open. Therefore $\{X\backslash S_i\}$ is a cover for $X$ by $g^*\beta$ -open sets of $X$. since $X$ is $g^*\beta$ - compact, this cover has a finite subcover say $\{X\backslash S_1, X\backslash S_2,\ldots\ldots X\backslash S_n\}$. Therefore $\cup_i(X\backslash S_i)=X$. It follows that $X\backslash(\cap_i S_i)=X$ which implies $\cap_{i=1}^n S_i=\Phi$ which is a contradiction to the finite intersection property. Hence $\cap_i S_i\neq\Phi$.

Conversely, suppose that each family of $g^*\beta$ closed sets with finite intersection property has nonempty intersection. To prove that $X$ is $g^*\beta$ - compact. Let $\{S_i : i\in\Omega\}$ be a cover of $X$ by $g^*\beta$ -open sets. Then $\cup_{i\in\Omega}S_i=X$ that implies $X\backslash\cup_{i\in\Omega}S_i=\Phi$. Hence $\cap_{i\in\Omega}(X\backslash S_i)=\Phi$. Since each $S_i$ is $g^*\beta$ -open $X\backslash S_i$ is $g^*\beta-$ closed for each $i$. Hence $(X\backslash S_i : i\varepsilon\Omega)$ is a family of $g^*\beta-$ closed sets whose intersection is empty. By hypothesis there exists a finite subcollection of $g^*\beta$ closed subsets of $X$ say $\{X\backslash S_1, X\backslash S_2,\ldots\ldots X\backslash S_n\}$ such that $\cap_{i=1}^n S_i=\Phi$. Which implies $\cup_{i=1}^n S_i=$ X. This proves that X is compact. □

## 3. $g^*\beta$-Connected spaces

**Definition 3.1.** *Let $(X,\tau)$ be a topological space. Two nonempty subsets $S$ and $\mathscr{T}$ of $X$ are called $g^*\beta$ -separated if $S\cap g^*\beta cl(\mathscr{T})=\Phi=g^*\beta cl(S)\cap\mathscr{T}$.*

**Example 3.2.** *Consider $\tau=\{\Phi,X,\{e\},\{f\},\{e,f\},\{f,g\},\{e,f,g\}\}$ on $X=\{e,f,g,h\}$. Let $S=\{f\}$, $\mathscr{T}=\{e,g\}$, $\mathscr{P}=\{e,f,h\}$ be the subsets of $X$. Then $g^*\beta cl(S)=\{f,h\}$, $g^*\beta cl(\mathscr{T})=\{e,g,h\}$ and $g^*\beta cl(\mathscr{P})=\{e,f,h\}$. $S\cap g^*\beta cl(\mathscr{T})=\Phi$ and $g^*\beta cl(S)\cap\mathscr{T}=\Phi$. This shows that $S$ and $\mathscr{T}$ are $g^*\beta$ -separated sets. But $\mathscr{T}\cap g^*\beta cl(\mathscr{P})=\{e\}$ and $g^*\beta cl(\mathscr{T})\cap\mathscr{P}=\{e,h\}$. Hence $\mathscr{T}$ and $\mathscr{P}$ are not $g^*\beta$ -separated sets.*

**Theorem 3.3.** *If $S$ and $\mathscr{T}$ are $g^*\beta$ -separation then they are disjoint.*

*Proof.* $S\cap\mathscr{T}\subseteq g^*\beta cl(S)\cap\mathscr{T}=\Phi$. □

**Theorem 3.4.** *If $S$ and $\mathscr{T}$ are $g^*\beta$ -separated subsets of a space $X$ and $\mathscr{P}\subseteq S$ and $Q\subseteq\mathscr{T}$ then $\mathscr{P}$ and $Q$ are also $g^*\beta$ -separated.*

*Proof.* Suppose $S$ and $\mathscr{T}$ are $g^*\beta$ -separated subsets of a space $X$, by Definition 3.1, $S\cap g^*\beta cl(\mathscr{T})=\Phi$ and $g^*\beta cl(S)\cap\mathscr{T}=\Phi$. Since $\mathscr{P}\subseteq S$, we have $g^*\beta\,\mathrm{cl}(\mathscr{P})\subseteq g^*\beta cl(S)$ and since $Q\subseteq\mathscr{T}, g^*\beta cl(Q)\subseteq g^*\beta cl(\mathscr{T})$. Hence $\mathscr{P}\cap g^*\beta cl(Q)=S\cap g^*\beta cl(\mathscr{T})=\Phi$ and $g^*\beta cl(\mathscr{P})\cap Q\subseteq g^*\beta cl(S)\cap\mathscr{T}=\Phi$. Therefore $\mathscr{P}$ and $Q$ are also $g^*\beta$ -separated sets. □

**Theorem 3.5.** *If $S$ is $g^*\beta$ -open and $S\cap\mathscr{T}=\Phi$ then if $S\cap g^*\beta cl(\mathscr{T})=\Phi$.*

*Proof.* Suppose if $S\cap g^*\beta cl(\mathscr{T})\neq\Phi$. Choose if $a\in S\cap g^*\beta cl(\mathscr{T})$. Then $a\in S$ and $a\in g^*\beta cl(\mathscr{T})$. It follows that $a\in\mathscr{T}'$ where $\mathscr{T}'$ is a $g^*\beta$ -closed superset of $\mathscr{T}$. In particular $a\in X\backslash S$ that is $a\notin S$ which is a contradiction to $a\in S$. Therefore $S\cap g^*\beta cl(\mathscr{T})=\Phi$. □

**Definition 3.6.** *A topological space $(X,\tau)$ is said to be $g^*\beta$ -connected if $X$ is not expressed as a union of two nonempty $g^*\beta$ -separated subsets of $X$. Otherwise $X$ is said to be $g^*\beta$ -disconnected.*

**Example 3.7.** *a) Let $X=\{e,f,g,h\}$. Consider the topology $\tau=\{\Phi,X,\{e\},\{e,f\},\{e,h\},\{e,f,g\},\{h\},\{e,f,h\}\}$ Here $X$ is not $g^*\beta$ - connected and not connected. b) Consider the Sierpinski topology $\tau=\{\Phi,X,\{e\}\}$ Then $g^*\beta-C(X,\tau)=\{\Phi,X,\{f\}\}$. Here $X$ is both $g^*\beta$ connected and connected. c) Consider the finite excluded point topology $\tau=\{\Phi,X,\{g\},\{e,g,h\}\}$ on $X=\{e,f,g,h\}$. Then $g^*\beta C(X,\tau)=\left\{\begin{array}{c}\Phi,X,\{e\},\{f\},\{h\},\{e,f\},\{e,h\},\{f,g\}\\ \{f,h\},\{e,f,g\},\{f,g,h\},\{e,f,h\}\end{array}\right\}$.*

*Here $X$ is connected but not $g^*\beta$ -connected.*

**Theorem 3.8.** *For a topological space $X$ the following are equivalent.*

a) *$X$ is $g^*\beta$ -connected.*

b) *$X$ cannot be expressed as the union of two nonempty disjoint $g^*\beta$ -open sets.*

c) *The only subsets of $X$ which are both $g^*\beta$ -open and $g^*\beta-$ closed are the empty set $\Phi$ and $X$.*

d) *Each $g^*\beta$ -continuous function of $X$ into discrete space $Y$ with atleast two points is a constant map.*

*Proof.* Suppose (a) holds. Let $X = S \cup \mathscr{T}$ where $S$ and $\mathscr{T}$ be the nonempty disjoint $g^*\beta$ -open sets. By theorem 3.5 $S \cap g^*\beta cl(\mathscr{J}) = \Phi$ and $\mathscr{T} \cap g^*\beta cl(S) = \Phi$. By definition 3.1,$S$ and $T$ be are $g^*\beta$ -separated sets of $X$. Therefore $X$ is not $g^*\beta$ -connected which is contradiction. This proves (a) $\Rightarrow (b)$.

Now to prove (b) $\Rightarrow$ (a). Suppose (b) holds. If $X$ is not $g^*\beta$ -connected then by definition 3.6,$X$ can be expressed as a union of two nonempty disjoint $g^*\beta$ -separated sets. This proves (b) $\Rightarrow$ (a).

Now to prove (b) $\Rightarrow$ (c). Suppose (b) holds. Let $S$ be a subset of X which is both $g^*\beta$ -open and $g^*\beta$ -closed. Then $X \backslash S$ is both $g^*\beta$ -open and $g^*\beta$ -closed. Suppose $S \neq \Phi$ and $S \neq$ X. Since $S \neq \Phi$ we have $X \backslash S \neq X$. And since $S \neq$ X, we have $X \backslash S \neq \Phi$. Therefore X $=$ S $\cup$ (X$\backslash$S) is a disjoint union of nonempty $g^*\beta$ -open sets. This contradicts to (b).

Hence $S = \Phi$ or $X$. This proves (b) $\Rightarrow$ (c). Suppose (c) holds. Let $f : X \to Y$ is a $g^*\beta$ -continuous function where $Y$ is a discrete space with atleast two points.

Fix $b_0 \in Y$ such that $f(a_0) = b_0$ for some $a_0 \in X$. since $Y$ is discrete, $\{b_0\}$ is both closed and open in $Y$. since $f$ is $g^*\beta$ -continuous, $f^{-1}(\{b_0\})$ is both or $g^*\beta$ -closed and $g^*\beta$ -open by (c) $f^{-1}(\{b_0\}) = \Phi$ or $X$. since $f^{-1}(\{b_0\}) \neq \Phi$, therefore $f^{-1}(\{b_0\}) =$ X. That is $f(x) = b_0$ for $x \in X$ This implies that $f$ is a constant map. This implies (c) $\Rightarrow$ (d). Suppose (d) holds. Suppose $X$ is not $g^*\beta$ -connected. Let X $= S \cup \mathscr{T}$ where $S \neq \Phi, \mathscr{T} \neq \Phi$ $S \cap \mathscr{J} \neq \Phi$,$S$ and $\mathscr{T}$ are $g^*\beta$ -open set. Let $Y$ be discrete space and $|y| > 1$. Fix $b_0$ and $b_1$ in $Y$ and $b_0 \neq b_1$. Define $f : X \to Y$ with $f(x) = \begin{cases} b_0 \text{ if } x \in A \\ b_1 \text{ if } x \in B \end{cases}$. Let $G$ be an open set in $Y$.

Then $f^{-1}(G) = \begin{cases} \Phi \text{ if } b_0 \notin G, b_1 \notin G \\ X \text{ if } b_0 \in G, b_1 \in G \\ S \text{ if } b_0 \in G, b_1 \notin G \\ \mathscr{T} \text{ if } b_0 \notin G, b_1 \in G \end{cases}$ Therefore $f$ is $g^*\beta$ -continuous but $f$ is not a constant map. Which is a contradiction to (d). Hence proved. □

**Definition 3.9.** *A subset $S$ of $X$ is $g^*\beta$ -connected if $S$ cannot be written as the union of two nonempty disjoint $g^*\beta$ -open subsets of $X$.*

**Theorem 3.10.** *(a) If a space $X$ is $g^*\beta$ -connected then it is connected. (b) If a space $X$ is $g^*\beta$ -connected then it is $g^{**}$ connected.*

*Proof.* (a) Let $X$ be $g^*\beta$ -connected. Suppose $X$ is not connected there exists disjoint nonempty open sets $S$ and $\mathscr{T}$ such that X $= S \cup \mathscr{T}$. Therefore $S$ and $\mathscr{T}$ are $g^*\beta$ -open sets, which is contradiction to $X$ be $g^*\beta$ -connected. Hence the condition (a) is proved.

(b) Let $X$ be $g^*\beta$ -connected. Suppose $X$ is not $g^{**}$ -connected there exists disjoint nonempty open sets $S$ and $\mathscr{T}$ such that $X = S \cup \mathscr{T}$. Therefore $S$ and $\mathscr{T}$ are $g^*\beta$ open sets, which is contradiction to $X$ be $g^*\beta$ -connected. Hence the condition (b) is proved. The converse of the above theorem 3.10 need not be true. □

**Theorem 3.11.** *If $f : X \to Y$ is $g^*\beta$ -continuous surjection and $X$ is $g^*\beta$ -connected then $Y$ is connected.*

*Proof.* Suppose that $Y$ is not connected. Let $Y = S \cup \mathscr{T}$ where $S$ and $\mathscr{T}$ are disjoint nonempty open sets in $Y$. Since $f$ is $g^*\beta$ -continuous and onto, $X = f^{-1}(S) \cup f^{-1}(\mathscr{T})$ where $f^{-1}(S)$ and $f^{-1}(\mathscr{T})$ are disjoint nonempty $g^*\beta$ -open sets in $X$, which is contradiction to $X$ is $g^*\beta-$ connected. Therefore $Y$ is connected. □

**Theorem 3.12.** *If $f : X \to Y$ is $g^*\beta$ -irresolute surjection and $X$ is $g^*\beta$ -connected then $Y$ is $g^*\beta$ -connected.*

*Proof.* Suppose that $Y$ is not $g^*\beta$ -connected. Let Y $= S \cup \mathscr{T}$ where $S$ and $\mathscr{T}$ are disjoint nonempty $g^*\beta-$ open sets in $Y$. since $f$ is $g^*\beta$ -irresolute and onto, $X = f^{-1}(S) \cup f^{-1}(\mathscr{T})$ where $f^{-1}(S)$ and $f^{-1}(\mathscr{T})$ are disjoint nonempty $g^*\beta$ -open sets in $X$, which is contradiction to $X$ is $g^*\beta$ -connected. Therefore $Y$ is $g^*\beta$ -connected. □

**Theorem 3.13.** *Suppose that $X$ is ${}_{\beta}T^{**}_{1/2}$ then $X$ is connected if and only if it is $g^*\beta$ -connected.*

*Proof.* Suppose that $X$ is connected. Then $X$ cannot be expressed as disjoint union of two nonempty proper subsets of $X$. Suppose $X$ is not a $g^*\beta$ -connected space. Let $S \cap \mathscr{T}$ and any two $g^*\beta$ -open subsets of $X$ such that X $= S \cup \mathscr{T}$ where Sn$\mathscr{T} = \Phi$ and $S \subset$ X,$\mathscr{T} \subset$ X. Since $X$ is ${}_{\beta}T^{**}_{1/2}$ space and $S, \mathscr{T}$ are $g^*\beta$ - open. $S, \mathscr{T}$ are open subsets of $X$. Which is contradiction to $X$ is $g^*\beta$ connected. Conversely every open set is $g^*\beta$ -open. Therefore every $g^*\beta$ -connected space is connected. □

**Definition 3.14.** *A subset $S$ of $X$ is $g^*\beta$ -connected if $S$ cannot be written as the union of two nonempty disjoint $g^*\beta$ -open subsets of $X$.*

**Theorem 3.15.** *Let $(X, \tau)$ be a topological space and let $\mathscr{M} \subseteq S \cup \mathscr{T}$ be $g^*\beta$ -connected where $S$ and $\mathscr{T}$ are $g^*\beta$ -separated sets in $X$. Then $\mathscr{M} \subseteq S$ or $\mathscr{M} \subseteq \mathscr{T}$ that is $\mathscr{M}$ cannot intersect both $S$ and $\mathscr{T}$.*

*Proof.* Since $S$ and $\mathscr{T}$ are $g^*\beta$ -separated sets by definition 3.1,$S \cap g^*\beta cl(\mathscr{T}) = \Phi$ and $g^*\beta cl(S) \cap \mathscr{T} = \Phi$. Now $\mathscr{M} \subseteq S \cup \mathscr{T}$ that implies $\mathscr{M} = \mathscr{M} \cap (S \cup \mathscr{T}) = (\mathscr{M} \cap S) \cup (\mathscr{M} \cap \mathscr{T})$. We would like to prove that one of the sets $\mathscr{M} \cap S$ and $\mathscr{M} \cap \mathscr{T}$ is empty. Suppose none of these sets is empty that is $\mathscr{M} \cap S \neq \Phi$ and $\mathscr{M} \cap \mathscr{T} \neq \Phi$. Then $(\mathscr{M} \cap S) \cap g^*\beta cl(\mathscr{M} \cap \mathscr{T}) \subseteq (\mathscr{M} \cap S) \cap (g^*\beta cl(\mathscr{M}) \cap g^*\beta cl(\mathscr{J}))$ $= (\mathscr{M} \cap g^*\beta cl(\mathscr{M})) \cap (\mathscr{M} \cap g^*\beta cl(\mathscr{T})) = (\mathscr{M} \cap g^*\beta cl(\mathscr{M}))$ $\cap \Phi = \Phi$. Similarly $g^*\beta\,\mathrm{cl}(\mathscr{M} \cap S) \cap (\mathscr{M} \cap \mathscr{T}) = \Phi$. Hence $\mathscr{M} \cap S$ and $\mathscr{M} \cap \mathscr{T}$ are $g^*\beta$ -separated sets. Thus $\mathscr{M}$ has been expressed as the union of two nonempty $g^*\beta$ -separated sets. Thus $\mathscr{M}$ is $g^*\beta$ -disconnected subset of $X$. But this is a contradiction to our assumption. Hence one of the sets $\mathscr{M} \cap S$ and

$\mathscr{M} \cap \mathscr{T}$ is empty. If $\mathscr{M} \cap S = \Phi$ then $\mathscr{M} = \mathscr{M} \cap \mathscr{T}$ which implies that $\mathscr{M} \subseteq \mathscr{T}$. similarly if $\mathscr{M} \cap \mathscr{T} = \Phi$ then $\mathscr{M} \subseteq S$. Therefore $\mathscr{M} \subseteq S$ or $\mathscr{M} \subseteq \mathscr{T}$. □

**Theorem 3.16.** *If $Y = \cup S_i$ where each $S_i$ is a $g^*\beta$ -connected subsets of $X$ and $\cap S_i \neq \Phi$ then $Y$ is $g^*\beta$ -connected.*

*Proof.* Let $\mathscr{K}$ be a point of $\cap S_i$. Suppose $Y = S \cup \mathscr{T}$ where $S$ and $T$ are $g^*\beta$ -separated by $g^*\beta$ -open sets in $X$. The point $\mathscr{K}$ is in one of the sets $S$ or $\mathscr{T}$. Suppose $k \in S$. Since each $S_i$ is $g^*\beta$ -connected by theorem 3.15,$S_i \subseteq S$ or $S_i \subseteq \mathscr{T}$. It cannot lie in $\mathscr{T}$. Because it contains the point $k$ of $S$. Hence $S_i \subset S$ for all $i$. It follows that $\cup S_i \subseteq S \Rightarrow \mathscr{T} = \Phi$. Which is a contradiction. Therefore $Y$ is $g^*\beta$ -connected. □

**Theorem 3.17.** *Let $S$ be a $g^*\beta$ -connected subsets of $X$. If $\mathscr{T}$ is a subset of $X$ such that $S \subseteq \mathscr{T} \subseteq g^*\beta cl(S)$ then $\mathscr{T}$ is $g^*\beta$ -connected. In particular $g^*\beta cl(S)$ is $g^*\beta$ -connected set of $X$ provided $S$ is $g^*\beta$ -connected.*

*Proof.* Suppose $\mathscr{T}$ is not a $g^*\beta$ -connected set of $X$. Then there exists a nonempty $g^*\beta$ -open sets $\mathscr{P}$ and $Q$ in $X$ such that $\mathscr{P} \cap g^*\beta cl(Q) = \Phi, g^*\beta cl(\mathscr{P}) \cap Q = \Phi$ and $\mathscr{P} \cup Q = \mathscr{T}$. since $S \subseteq \mathscr{T} = \mathscr{P} \cup Q$ by theorem 3.15 we have $S \subseteq \mathscr{P}$ or $S \subseteq Q$. Let $S \subseteq \mathscr{P}$ which implies that $g^*\beta cl(S) \subseteq g^*\beta cl(\mathscr{P})$ that implies $g^*\beta cl(S) \cap Q \subseteq g^*\beta cl(\mathscr{P}) \cap Q = \Phi$. Therefore $g^*\beta cl(S) \cap Q = \Phi$. Also $\mathscr{P} \cup Q = \mathscr{T} \subseteq g^*\beta cl(S)$. That is $Q \subseteq \mathscr{T} \subseteq g^*\beta cl(S)$ which implies $g^*\beta cl(S) \cap Q = Q$ Hence $Q = \Phi$ Hence $\mathscr{T}$ must be $g^*\beta$ -connected. Again since $S \subseteq \mathscr{T} \subseteq g^*\beta cl(S)$ we have $g^*\beta cl(S)$ is $g^*\beta$ -connected. □

**Theorem 3.18.** *If every two points of a subset $\mathscr{M}$ of a topological space $X$ are contained in some $g^*\beta$ -connected set of $\mathscr{M}$, then $\mathscr{M}$ is $g^*\beta$ -connected.*

*Proof.* Suppose $\mathscr{M}$ is not $g^*\beta$ -connected. By definition 3.1, there exists two nonempty open sets $S$ and $\mathscr{T}$ of $X$ such that $S \cap g^*\beta cl(\mathscr{T}) = \Phi, g^*\beta cl(S) \cap \mathscr{T} = \Phi$ and $\mathscr{M} = S \cup \mathscr{T}$ since $S$ and $\mathscr{T}$ are nonempty there exists $s,t$ with $s \in S$ and $t \in \mathscr{T}$ since $\mathscr{M} \subseteq S \cup \mathscr{T}$ by theorem 3.15,$\mathscr{M} \subseteq S$ or $\mathscr{M} \subseteq \mathscr{T}$ Therefore $s,t$ are both in $S$ or both in $\mathscr{T}$, which is a contradiction. Since $S, \mathscr{T}$ are disjoint sets. Hence $\mathscr{M}$ must be $g^*\beta$ - connected. □

**Definition 3.19.** *Let $(X, \tau)$ be a topological space. A maximal $g^*\beta$ -connected subset of $X$ is called a $g^*\beta$ component in $X$.*

**Example 3.20.** *Let $X = \{e, f, g, h\}$, $\tau = \{\Phi, X, \{f\}, \{g\}, \{f, g\}, \{e, f\}, \{e, f, g\}, \{f, g, h\}\}$. Then $g^*\beta O(X, \tau) = \{\Phi, X, \{f\}, \{g\}, \{f, g\}, \{e, f\}, \{e, f, g\}, \{f, g, h\}\}$. Then $g^*\beta$ -components are $\{e, f, g\}, \{f, g, h\}$.*

**Proposition 3.21.** *Let $S$ be a $g^*\beta$ -component set in a topological space $(X, \tau)$. Then $S = g^*\beta cl(S)$.*

*Proof.* Let $S$ be a $g^*\beta$ -component set of $(X, \tau)$. Since $A$ is $g^*\beta$ -connected by theorem 3.17, $g^*\beta cl(S)$ is also $g^*\beta$ connected. Further since $S$ is $g^*\beta$ -component, by definition 3.19,$S$ is a maximal $g^*\beta$ -connected set. Hence $g^*\beta cl(S) \subseteq S$ But $S \subseteq g^*\beta cl(S)$. Therefore $S = g^*\beta cl(S)$. □

**Theorem 3.22.** *Let $(X, \tau)$ be a topological space. Then*

*(a) Each point in $X$ is contained in exactly one $g^*\beta$ -component of $X$*

*(b) The $g^*$ -component of $X$ form a partition of $X$ that is any two $g^*\beta$ -components are either disjoint or identical and the union of all $g^*\beta$ components in $X$.*

*(c) Each $g^*\beta$ -connected subset of $X$ is contained in a $g^*\beta$ -component of $X$.*

*(d) Each $g^*\beta$ -connected subset of $X$ which is both $g^*\beta$-open and $g^*\beta$-closed is a $g^*\beta$ component of $X$.*

*Proof.* Let $a$ be any arbitrary point of $X$. Let $\{S_i\}$ be a collection of all $g^*\beta$ -connected subsets of $X$ which contain $a$. This collection is nonempty. Since $\{a\}$ is $g^*\beta$ connected. Further $\cap S_i \neq \Phi$ since $a$ is a point of each $S_i$. Hence $S_a = \cup S_i$ is a $g^*\beta$ -connected by theorem 3.15 Also $S_a$ is maximal and contains $a$. For let $\mathscr{T}$ be any $g^*\beta$ connected set such that $S_a \subseteq \mathscr{T}$. Then $a \in \mathscr{T}$. Therefore $\mathscr{T}$ is one of the members of the collection $\{S_i\}$. Hence $\mathscr{T} \subseteq S_a$. Thus $S_a = \mathscr{T}$. Hence $S_a$ is a $g^*\beta$ -component of $X$ containing $a$. Let $a$ be any other $g^*\beta$ -component of $X$ containing $a$. Then $S_a$ is one of the members of $S'_a s$ and so $S^*_a \subseteq S_a$. But since $S_a$ is maximal as a $g^*\beta$-connected subset of ,$S^*_a \subseteq S_a$. Hence Proved (a).

Let $\mathscr{P} = \{S_a : a \in X\}$ where $S_a$ is defined as in (a). We claim that $\mathscr{P}$ contains all the $g^*\beta$ -component of $X$. By (a) each $S_a \in \mathscr{P}$ is a $g^*\beta$ -component. And if $\mathscr{M}$ is any other $g^*\beta$ -component, then $\mathscr{M}$ being nonempty contains some point $a_0 \in X$ and by (a), we have $\mathscr{M} = S_{a_n} \in \mathscr{P}$ Our objective is to prove that $\mathscr{P}$ forms a partition of $X$. Let $Y = \cup\{S_a : a \in X\}$. Let $S_u$ and $S_v$ be any two $g^*\beta-$ components such that $S_u \cap S_v \neq \Phi$. We need to prove wre $S_v$. Since $S_u$ and $S_v$ are $g^*\beta$ -connected subsets containing $u$ and $S_w$ is a $g^*\beta$ -component containing $u$. We have $S_u \subseteq S_w$ and $S_v \subseteq S_{wr}$. But since $S_u$ and $S_v$ are $g^*\beta$ -components we have $S_u = S_w = S_{v^*}$. Hence $g^*\beta$ -component of $X$ form a partition of $X$. Hence Proved (b).

To prove (c). Let $\mathscr{P}$ be any $g^*\beta$ -connected subset of $X$. If $\mathscr{P} = \Phi$ then $C$ is contained in every $g^*\beta$ -component. If $\mathscr{P} \neq \Phi$ then $\mathscr{P}$ contains a point $a_0 \in X$ and so $\mathscr{P} \subseteq S_{a_0}$ by (a). Hence Proved (c).

Now to prove (d) Let $Q$ be a $g^*\beta$ -connected subset of $X$ which is both $g^*\beta$ -open and $g^*\beta$ -closed. By (c) $Q \subseteq \mathscr{M}$ where $\mathscr{M}$ is a $g^*\beta$ -component of $X$ and hence $\mathscr{M}$ is nonempty. We wish to prove that $Q = \mathscr{M}$. For if $Q \neq \mathscr{M}$, then $Q$ is a proper subset of $\mathscr{M}$ which is both $g^*\ \beta - openand$ $g^*\beta$ -closed in $\mathscr{M}$. Therefore by theorem 3.12,$\mathscr{M}$ is $g^*\beta$ -disconnected which is a contradiction. Hence $Q = \mathscr{M}$ and so $Q$ is $g^*\beta$ -component. □

**Definition 3.23.** *Let $(X, \tau)$ be a topological space. Then $X$ is*

*(i) $g^*\beta$-hyper connected if there exists no disjoint nonempty $g^*\beta$ -open sets*

*(ii)* $g^*\beta$ *-ultra connected if there exists no disjoint nonempty* $g^*\beta$ *-closed sets.*

**Example 3.24.** *i) Consider the topology* $\tau = \{\Phi, X, \{e\}, \{e,f\}, \{e,g\}\}$ *on* $X = \{e,f,g\}$*. Now* $g^*\beta - O(X,\tau) = \{\Phi, X, \{e\}, \{e,f\}, \{e,g\}\}$*. Hence* $(X,\tau)$ *is* $g^*\beta$ *-hyper connected.*

*ii) Consider* $X = (0, 1/n)$*. This space has no disjoint nonempty* $g^*\beta$ *-closed sets. Hence* $X$ *is* $g^*\beta$ *-ultra connected.*

**Proposition 3.25.** *A space* $X$ *is* $g^*\beta$ *- hyper connected if the* $g^*\beta$ *-closure of every* $g^*\beta$ *-open set is the entire space* $X$*.*

*Proof.* Suppose $X$ is not $g^*\beta$ -hyper connected. Let $S$ and $\mathscr{J}$ be the nonempty $g^*\beta$ -open sets of $X$. Then $S \cap \mathscr{T} = \Phi$ that implies $g^*\beta - cl(S) \cap \mathscr{T} = \Phi$. It follows that $X \cap \mathscr{T} = \Phi$. Hence $\mathscr{T} = \Phi$ which is a contradiction to $\mathscr{T} \neq \Phi$. Hence proved. □

**Proposition 3.26.** *A space* $X$ *is* $g^*\beta$ *-ultra connected if the* $g^*\beta$ *-closures of distinct points always intersect.*

*Proof.* Suppose $X$ is not $g^*\beta$ -ultra connected. Let $S$ and $\mathscr{T}$ be the nonempty $g^*\beta$ -closed sets of $X$. Then $S \cap \mathscr{T} = \Phi$. Since $S$ and $\mathscr{T}$ are nonempty there exists $s,t$ with $s \in S$ and $t \in \mathscr{T} \Rightarrow g^*\beta - cl(\{s\}) \cap g^*\beta - cl(\{t\}) = \Phi$ which is a contradiction. Hence proved. □

## References


[1] Atik. A.A, Donia H.M. and Salama A.S., on *b*#-connectedness and *b*#-disconnectedness and their applications, *Journal of the Egyptian Mathematical Society*, 21, 63467, (2013).

[2] Dorsett, C., Semi compactness, semi separation axioms, and product spaces, *Bulletin of the Malaysian Mathematical Sciences Society*, 4 (1)(1981), 21–28.

[3] Goss G. and Viglino G, Some topological properties weaker than compactness, *Pacific J. Math.*, 35(1970), 635–638.

[4] Levine, N. Generalized closed sets in topology, *Rend. Circ. Mat. Palermo*, 2(19)(1970), 89–96.

[5] Mathew P.M, On hyper connected spaces, *Indian J. Pure Appl. Math.*, 19 (12)(1988), 180–184.

[6] Niemenen T, On Ultrapseudo compact and related spaces, *Ann. Acad. Sci. Fenn. Ser. A. I. Math.*, 3(1997), 185–205.

[7] Punitha Tharani, Priscilla Pacifica, pg**-Connected space in Topological Spaces, *International Journal of Mathematical Archive*, 8(3)(2017), 159–164.

[8] Punitha Tharani. A and Sujitha. H, The concept of $g^*\beta$-closed sets in topological spaces, *International Journal of Mathematical Archive*, 11(4)(2020), 14–23.

[9] Veerakumar. M.K.R.S, $g^*$ pre-closed sets, *Acta Ciencia Indica*, XXV(IIIM)(1)(2002), 51–60.



⋆⋆⋆⋆⋆⋆⋆⋆⋆

ISSN(P):2319 − 3786

Malaya Journal of Matematik

ISSN(O):2321 − 5666

⋆⋆⋆⋆⋆⋆⋆⋆⋆